Wild Partitions and Number Theory

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1. **Wild Partitions**

2. **Analytic Number Theory**

3. **Local algebraic number theory**

4. **Global algebraic number theory**

For more, see my paper of the same title!

1. Wild partitions. A typical partition of 20:

$$\mu_{\text{ordinary}} = 9+3+3+2+2+1.$$

Fix a prime p and positive integers e_0 and f_0. We define a new notion of (p, e_0, f_0)-wild partition, with definitions incorporating the Krasner mass formula from local algebraic number theory. A typical $(3, 2, 1)$-wild partition of 20:

$$\mu_{\text{wild}} = 9_{13;1,1,0,2,2} + 3_{1;i} + 3_{1;-i} + 2 + 2 + 1.$$

The first subscript on a part is a discrete invariant, its wild conductor c_w. The second subscript is a continuous invariant, a vector in $\overline{\mathbb{F}}_p^{d(c_w)}$, where $d(c_w)$ weakly increases with c_w.

For e a positive integer, put $w = \text{ord}_p(e)$ and accordingly factor e into its tame and wild parts tp^w. If $w = 0$, the only possible subscript pair on e is $0; 0$ and these are omitted. For $w > 0$, the allowed first subscripts run over a set $\text{Ore}(p, e_0, e) \subset \{1, \ldots, we_0e\}$. The final part of the definition of (p, e_0, f_0)-wild partition is stability under the Frobenius operator $v \mapsto v^{p^{f_0}}$, acting on second subscripts.

As an example, the set of Ore numbers for $(p, e_0, e) = (3, 2, 9)$ is

$$
\begin{array}{ccccccccc}
8 & 7 & . & 5 & 4 & . & 2 & 1 & . \\
17 & 16 & . & 14 & 13 & . & 11 & 10 & . \\
\hline
26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & . \\
35 & 34 & 33 & 32 & 31 & 30 & 29 & 28 & . \\
\hline
 & & & & & & & & 36
\end{array}
$$

In general, Ore(p, e_0, e) consists of w full blocks, each with e_0 rows and e columns, followed by the single entry we_0e. The j^{th} block omits every $p^{j\,\text{th}}$ number.

The dimension $d(c_w)$ is the number of omitted integers less than c_w.

Second subscripts are required to have first coordinate non-zero except in the maximal case $c_w = we_0e$.

The tame conductor of any part e is by definition $e-1$. So wild partitions have three additive discrete invariants, the degree n, the tame conductor c_t, and the wild conductor c_w. For our example above, $(n, c_t, c_w) = (20, 14, 15)$.

Let $\lambda_{p,e_0,f_0;n,c_t,c_w}$ be the number of (p, e_0, e)-wild partitions with invariants (n, c_t, c_w). Let

$$\Lambda_{p,e_0,f_0}(x,y,z) = \sum_{n=0}^{\infty} \sum_{c_t=0}^{\infty} \sum_{c_w=0}^{\infty} \lambda_{p,e_0,f_0;n,c_t,c_w} x^n y^{c_t} z^{c_w}.$$

The definitions yield

$$\Lambda_{p,e_0,f_0}(x,y,z) = \prod_{e=1}^{\infty} \prod_{c_w \in \mathrm{Ore}(p,e_0,e)} \Lambda_{p,e_0,f_0;e,c_w}(x,y,z).$$

A remarkable specialization is $(y, z) = (1, p^{-f_0})$. Then the e-factor becomes $1/(1-x^e)$ so that

$$\Lambda_{p,e_0,f_0}(x, 1, p^{-f_0}) = \prod_{e=1}^{\infty} \frac{1}{1-x^e} =: \Lambda(x).$$

This is the Serre mass formula translated to our abstract context.

We are interested instead in the specialization $(y, z) = (1, 1)$ which gives an *unweighted* count of wild partitions.

Theorem. *The e-factor of $\Lambda_{p,e_0,f_0}(x, 1, 1)$ depends on e_0 and f_0 only through $n_0 = e_0 f_0$. Putting $Q = p^{n_0}$, it is*

$$\Lambda_{Q;e}(x) = \frac{\prod_{j=0}^{w-1}(1 - Q^{(p^w - p^{w-j})t/(p-1)} x^e)^{(p-1)p^j}}{(1 - Q^{(p^w-1)t/(p-1)} x^e)^{p^w}}.$$

Explicitly,

$$\begin{aligned}
\Lambda_{Q;t}(x) &= \frac{1}{1-x}, \\
\Lambda_{Q;pt}(x) &= \frac{(1-x^e)^{p-1}}{(1-Q^t x^e)^p}, \\
\Lambda_{Q;p^2t}(x) &= \frac{(1-x^e)^{p-1}(1-Q^{pt}x^e)^{p^2-p}}{(1-Q^{(p+1)t}x^e)^{p^2}}.
\end{aligned}$$

The theorem allows us to regard Q as a variable running over $[1, \infty)$, independent from p. In the resulting product $\Lambda_{p,Q}(x) = \prod_e \Lambda_{p,Q;e}(x)$, the case $Q = 1$ reduces to $\Lambda(x)$ independent of p.

2. Analytic number theory. Hardy and Ramanujan used the generating function

$$\Lambda(x) = \sum_{n=0}^{\infty} \lambda_n x^n = \prod_{e=1}^{\infty} \frac{1}{1-x^e}$$
$$= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + \cdots$$

to prove the asymptotic formula

$$\lambda_n \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

One can ask about the asymptotics of our $\lambda_{p,Q,n}$, expecting use the corresponding generating functions, e.g.

$$\Lambda_{2,2}(x) = \frac{1}{1-x} \cdot \frac{1-x^2}{\left(1-2x^2\right)^2} \cdot \frac{1}{1-x^3} \cdot \frac{\left(1-x^4\right)\left(1-4\right.}{\left(1-8x^4\right)^4}$$
$$= 1 + x + 4x^2 + 5x^3 + 36x^4 + 40x^5 + 145x^6$$

$$\Lambda_{3,3}(x) = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{\left(1-x^3\right)^2}{\left(1-3x^3\right)^3} \cdot \frac{1}{1-x^4} \cdot \frac{1}{1-x^5}$$
$$= 1 + x + 2x^2 + 9x^3 + 11x^4 + 19x^5 + 83x^6$$

Let $\Theta_p(x) = \Lambda(x)/\Lambda(x^p)$. Then $\Theta_p(x)$ is the theta-function whose coefficients count "p-cores." Its radius of convergence is 1. Examples:

$$\begin{aligned}
\Theta_2(x) &= \sum_{j=-\infty}^{\infty} x^{2j^2-j} \\
&= 1 + x + x^3 + x^6 + x^{10} + x^{15} + \cdots
\end{aligned}$$

$$\begin{aligned}
\Theta_3(x) &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x^{3(j^2+jk+k^2)-j-2k} \\
&= 1 + x + 2x^2 + 2x^4 + x^5 + 2x^6 + \cdots
\end{aligned}$$

From the theorem giving the e-factors of $\Lambda_{p,Q}(x)$, one gets

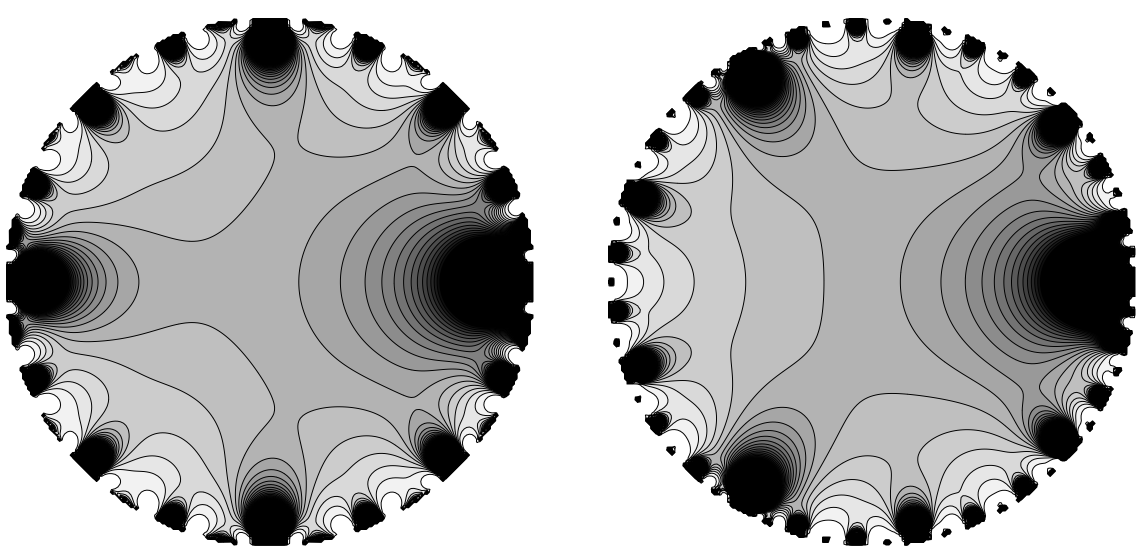
Corollary.

$$\Lambda_{p,Q}(x) = \prod_{j=0}^{\infty} \Theta_p(Q^{(p^j-1)/(p-1)} x^{p^j})^{p^j}.$$

From this corollary, one gets that the radius of convergence of $\Lambda_{p,Q}(x)$ is $Q^{-1/(p-1)}$. Thus $\limsup \lambda_{p,Q,n}^{1/n} = Q^{1/(p-1)}$.

In contrast with the number λ_n of ordinary partitions, the number $\lambda_{p,Q,n}$ of wild partitions grows irregularly. Evidence overwhelmingly suggests that $\lambda_{p,Q,n}/\lambda_{p,Q,n-1}$ does not tend to the growth factor $Q^{1/(n-1)}$. Instead there are contributions to the ratio $\lambda_{p,Q,n}/\lambda_{p,Q,n-1}$ of period p, smaller ones of period p^2, etc. The conjectural Fourier analysis on the next slide gives all these contributions in terms of the singularities of $\Lambda_{p,Q}(x)$ on its boundary circle.

Contour plots of $|\Lambda_{2,2}(x)|$ and $|\Lambda_{3,3}(x)|$ on their disks of convergence:



Define

$$\hat{c}_{p,Q}(y) = \prod_{j=0}^{\infty} \left(\frac{\Theta_p(Q^{-1/(p-1)} y^{p^j})}{\Theta_p(Q^{-1/(p-1)})} \right)^{p^j}$$

for y on the unit circle. It is supported on p-power roots of unity.

Define

$$c_{p,Q}(n) = \sum_y y^{-n} \hat{c}_{p,Q}(y),$$

the sum being over p-power roots of unity. This real-valued function on $\mathbb{Z}$ extends to a continuous function on $\mathbb{Z}_p$ and is designed to capture the oscillatory aspects of the $\lambda_{p,Q,n}$.

Conjecture. *There are functions $A_p(Q)$, $B_p(Q)$, and $C_p(Q)$ such that*

$$\lambda_{p,Q,n} \sim c_{p,Q}(n) C_p(Q) n^{B_p(Q)} e^{A_p(Q)\sqrt{n}} Q^{n/(p-1)}.$$

The conjecture fits well with calculations in the range $0 \leq n \leq 4000$.

Local Algebraic Number Theory. Let F be a finite extension of one of the local fields $\mathbb{R}$, $\mathbb{Q}_2$, $\mathbb{Q}_3$, $\mathbb{Q}_5$, ... If K/F is an algebra extension, then its mass is $1/|\mathrm{Aut}(K/F)|$. The total mass of degree n field extensions is denoted $\phi_{F,n}$. The total mass of degree n algebra extensions is $\lambda_{F,n}$. Let

$$\Phi_F(x) = \sum_{n=1}^{\infty} \phi_{F,n} x^n, \qquad \Lambda_F(x) = \sum_{n=0}^{\infty} \lambda_{F,n} x^n.$$

Because of the way algebras are built from fields, always $\Lambda_F(x) = \exp(\Phi_F(x))$.

Archimedean cases. The only algebra extensions of $\mathbb{C}$ are $\mathbb{C}^n$, with mass $1/n!$. The only algebra extensions of $\mathbb{R}$ are $\mathbb{R}^r\mathbb{C}^s$ with mass $1/(r!s!2^s)$. One has

$$\begin{aligned} \Lambda_{\mathbb{C}}(x) &= e^x &&= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots \\ \Lambda_{\mathbb{R}}(x) &= e^{x+x^2/2} &&= 1 + x + x^2 + \frac{4}{6}x^3 + \cdots \end{aligned}$$

Note both the super-exponential decay and the non-integrality of the coefficients.

Ultrametric cases. Now suppose F is a degree n_0 extension of $\mathbb{Q}_p$ with ramification index e_0 and inertial degree f_0. An algebra extension of F has not only a degree n, but also a tame conductor c_t, and a wild conductor c_w. Let

$$\Lambda_F(x, y, z) = \sum_{n=0}^{\infty} \sum_{c_t=0}^{\infty} \sum_{c_w=0}^{\infty} \lambda_{F;n,c_t,c_w} x^n y^{c_t} z^{c_w}$$

where $\lambda_{F;n,c_t,c_w}$ is the total mass of algebras with the indicated invariants.

Wild partitions are defined exactly so that

$$\Lambda_F(x, y, z) = \Lambda_{p,e_0,f_0}(x, y, z).$$

In particular, taking $(y, z) = (1, 1)$, one has $\Lambda_F(x) = \Lambda_Q(x)$. The previous asymptotics apply and so $\lambda_{F,n}$ grows roughly geometrically as $Q^{n/(p-1)}$. Working with geometric packets of local fields, all of which have total mass one, gives a bijective correspondance with wild partitions and explains integrality.

4. Global Algebraic Number Theory. Let F be a number field and let S be a finite set of places. If S is at all large, then various techniques allow one to construct extension fields K/F ramified only within S. For example, take $F = \mathbb{Q}$ and $S = \{\infty, p, \ell\}$. Then one can get

- infinitely many nilpotent fields by iterated p^{th} or ℓ^{th} roots.

- infinitely many solvable fields by mixing p^{th} and ℓ^{th} roots.

- infinitely many extensions involving $PSL_2(p^f)$ or $PSL_2(\ell^f)$ via modular forms

- many extensions involving groups like $Sp_{2k}(\ell)$ or $Sp_{2k}(p)$ from the ABC construction.

Ironically, there seems to be no systematic way to get A_n or S_n extensions!

Conjecture. *There is a largest* $n_{F,S}$ *such that* F *has no* A_n *or* S_n *extensions ramified within* S *for* $n > n_{F,S}$.

P.S. While the arguments below are reasonable, the conjecture is now to be regarded as highly implausible because of Hurwitz Number Fields as described in later talks.

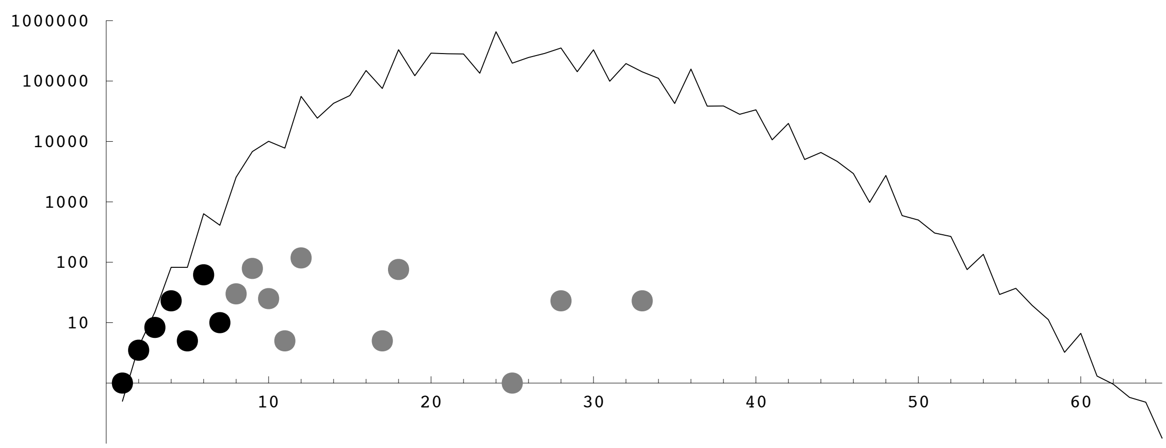
Strong support. A local-global heuristic of Bhargava says that the "expected mass" of such fields, taking all archimedean places in S, is

$$\frac{1}{2}\prod_{v \in S} \lambda_{F_v,n},$$

and this quantity decays super-exponentially to zero. $\square$

The analogous conjecture for F a function field fails extremely badly, but for two understandable reasons. First if n is at least the characteristic, each $\lambda_{F_v,n}$ is already infinite. Second, there are no archimedean places contributing super-exponentially decaying factors.

For $F = \mathbb{Q}$ and $S = \{\infty, 2, 3\}$, complete collections of fields (with Jones) and incomplete collection of fields (with Malle) are compared with Bhargava's heuristic:



Comparison with complete lists suggests that Bhargava's heuristic is too high when discriminants are very low, as in these cases. Comparison with incomplete lists suggests that moreover these lists are very incomplete indeed.

Similar comparisons with Driver's complete lists of fields for F quadratic show again that Bhargava's heuristic captures relative magnitudes well but is too large for small discriminants.