Number fields from covers of $M_{0,5}$

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1. Context of rigid local systems

2. The base $M_{0,5}$: (5 points=$\{0,1,\infty,s,t\}$)

3. Background on three groups

4. $X_{27} \to M_{0,5}^{\alpha}$ with Gal $= W(E_6)$ and $\alpha = (st)$ symmetry

5. $X_{28a} \to M_{0,5}^{\beta}$ with Gal $= W(E_7)^+$ and $\beta = (01)(st)$ symmetry

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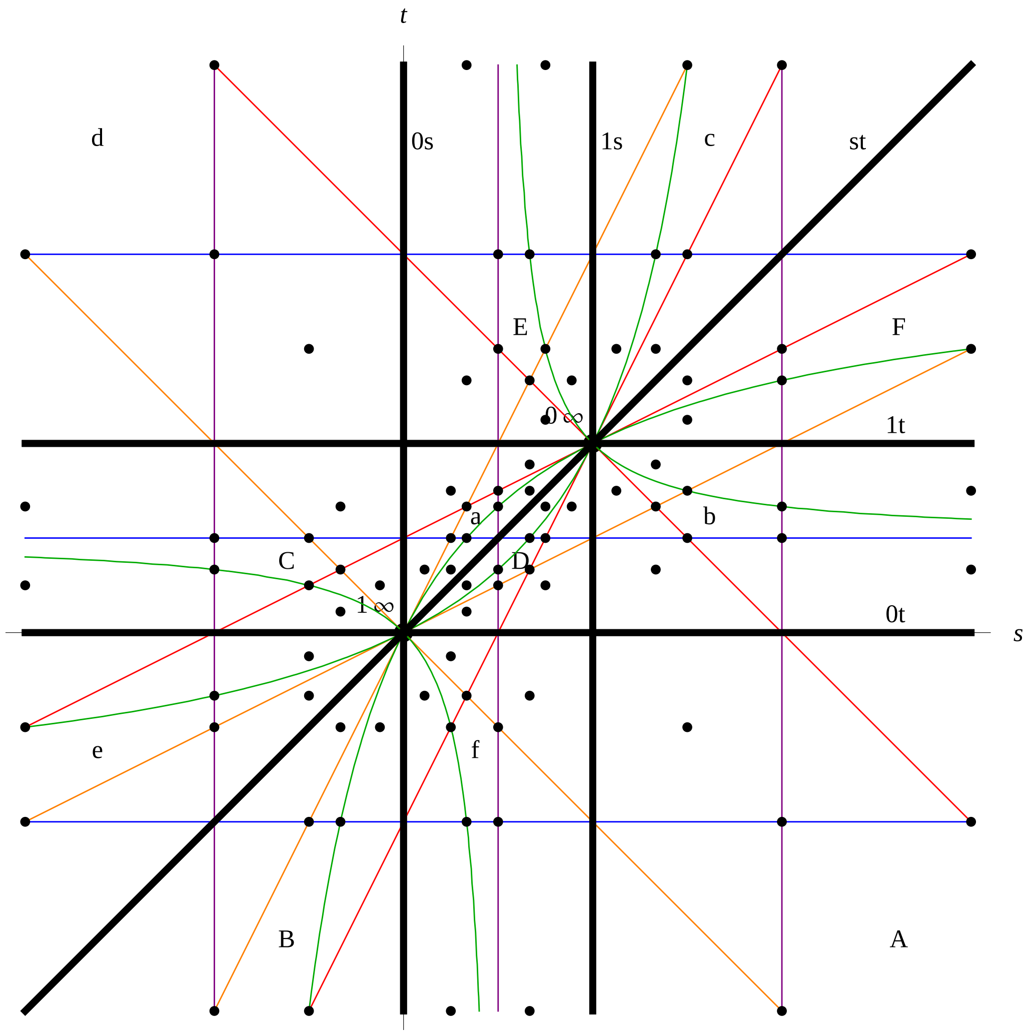
1. Context of rigid local systems. Katz's remarkable theory of rigid local systems can be viewed as a method of constructing number fields:

$$\begin{array}{cl}
\text{Rigid Jordan } z\text{-tuples} & \\
\downarrow & \\
\text{Rigid matrix } z\text{-tuples} & \\
\downarrow & \\
\text{Rigid motivic local systems} & \\
\downarrow 2 \;\; \downarrow 3 \;\; \downarrow 5 \;\; \cdots & \text{Reduction modulo } \ell \\
\text{Covers of } M_{0,z+1} & \\
\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \cdots & \text{Specialization } M_{0,z+1}(Z[\frac{1}{S}]) \\
\text{Number fields} &
\end{array}$$

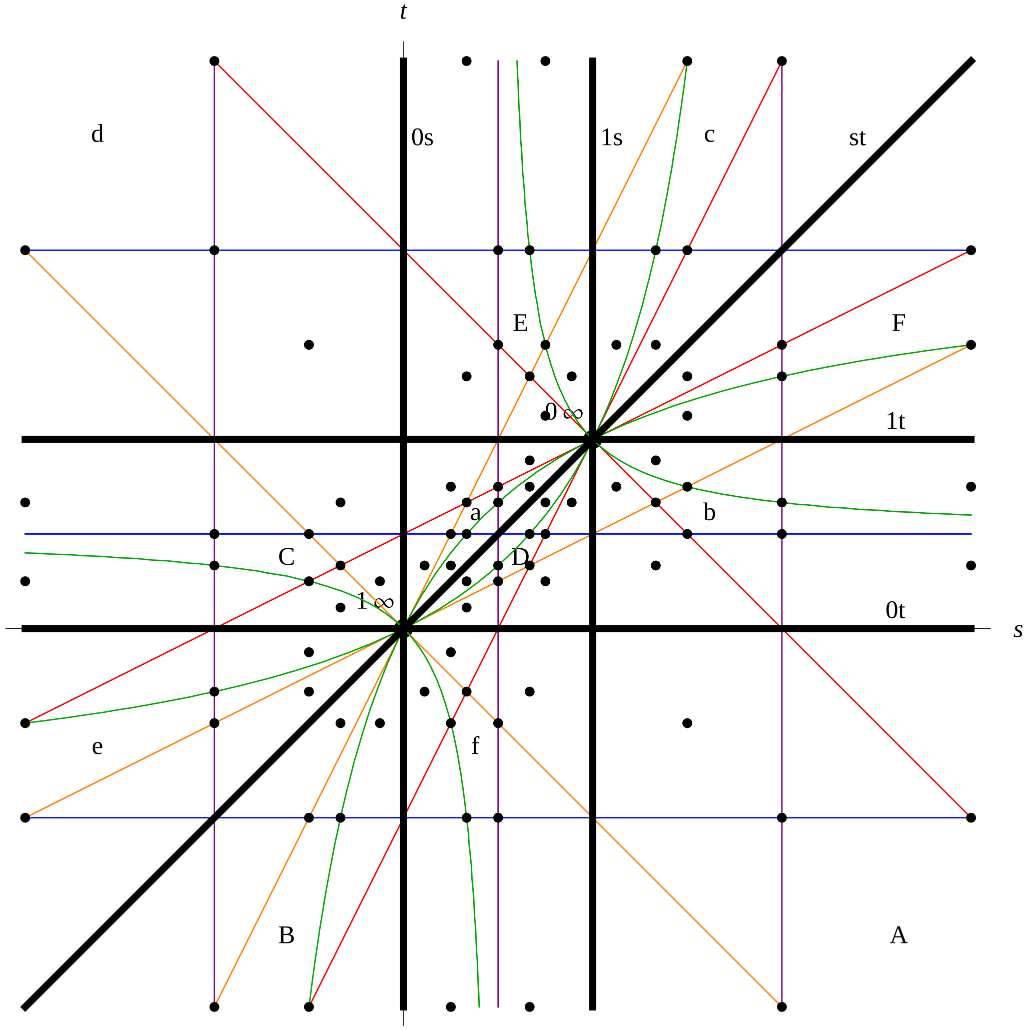
The number fields obtained are **classified via parameters**, have **Lie-type Galois groups**, and have **controlled bad reduction**. We want to understand this more explicitly, especially in the little-explored $z \geq 4$ case.

2. The base $M_{0,5}$: (5 points$=\{0,1,\infty,s,t\}$).



$M_{0,5}(\mathbb{R})$ with its discriminant locus and
0: red lines through $(1,1)$,
1: orange lines through $(0,0)$,
∞: green hyperbolas,
s: blue horizontal lines,
t: violet vertical lines.

$$M_{0,5} = \text{Spec } \mathbb{Z}[s, t, \frac{1}{st(s-1)(t-1)(s-t)}].$$



$$\alpha = (st) \quad = (eB)(aD)(cF)(Eb)(Ad)(Cf)$$

$$\beta = (01)(st) \quad = (EC)(bf)(ec)(BF)$$

$$\gamma = (s\infty t10) \quad = (bcaef)(BCAEF)$$

$$\delta = (01\infty) \quad = (BDF)(bdf)(EAC)(eac)$$

3. Background on $W(E_6) \subset W(E_7) \supset G_{12096}$.

Char 0:
$$W(E_6) = W(E_6)^+.2 \quad \subset O_6(\mathbb{R})$$
$$W(E_7) = W(E_7)^+ \times 2 \subset O_7(\mathbb{R})$$

Char 2:
$$W(E_6) \cong O_6^-(2)$$
$$W(E_7)^+ \cong Sp_6(2)$$
$$G_{12096} = G_2(2)'.2$$

Char 3:
$$W(E_6) \cong SO_5(3)$$
$$G_{12096} = U_3(3).2$$

Perm:
$$W(E_6) \subset A_{27} \quad \text{“27 lines”}$$
$$W(E_7)^+ \subset A_{28} \quad \text{“28 bitangents”}$$

Orders:
$$|W(E_6)| = 2^7 \cdot 3^4 \cdot 5 \qquad = 51,840$$
$$|W(E_7)^+| = 2^9 \cdot 3^4 \cdot 5 \cdot 7 = 1,451,520$$
$$|G_{12096}| = 2^6 \cdot 3^3 \cdot 7 \qquad = 12,096$$

Conjugacy classes:

$W(E_6)$	$W(E_7)$	j	μ_6	λ_{28}
$1a$	$1A$	$\widehat{000000}/1$	6	1^{28}
$2c$	$2A$	$\widehat{00000}0/1$	51	$2^{6}1^{16}$
$2a$	$2B$	$\widehat{0000}\widehat{00}/1$	42	$2^{12}1^{4}$
$2b$	$2C$	$\widehat{0000}\widehat{00}/1$	42	$2^{10}1^{8}$
$2d$	$2D$	$\widehat{000}\widehat{000}/1$	33	$2^{12}1^{4}$
$3c$	$3A$	$\widehat{0000}12/3$	411	$3^{6}1^{10}$
$3ab$	$3B$	$\widehat{111}\widehat{222}/3$	33	$3^{9}1$
$3d$	$3C$	$\widehat{00}\widehat{11}\widehat{22}/3$	222	$3^{9}1$
$4a$	$4A$	$\widehat{00}\widehat{00}\widehat{00}/1$	222	$4^{6}1^{4}$
$4c.$	$4B$	$\widehat{000}000/1$	3111	$4^{5}2^{3}1^{2}$
$4d.$	$4C$	$\widehat{000}000/1$	3111	$4^{5}2^{1}1^{6}$
	$4D$			$4^{6}2^{2}$
$4b$	$4E$	$\widehat{00}\widehat{00}00/1$	2211	$4^{5}2^{3}1^{2}$
$5a$	$5A$	$\widehat{00}1234/5$	21111	$5^{5}1^{3}$
$6g.$	$6A$	$\widehat{000}012/3$	3111	$6^{1}3^{4}2^{3}1^{4}$
$6cd$	$6B$	$\widehat{00}\widehat{00}12/3$	2211	$6^{3}2^{3}1^{4}$
$6ab$	$6C$	$\widehat{11}1\widehat{22}2/3$	2211	$6^{4}31$
$6f$	$6D$	$\widehat{00}\widehat{00}12/3$	2211	$6^{2}3^{2}2^{4}1^{2}$
$6h.$	$6E$	$00\widehat{11}\widehat{22}/3$	2211	$6^{2}3^{5}1$
$6e$	$6F$	$\widehat{00}1122/3$	21111	$6^{4}31$
$6i.$	$6G$	$001122/3$	111111	$6^{4}31$
	$7A$	$123456/7$	111111	7^{4}
$8a.$	$8A$	$000000/8$	111111	$8^{3}2^{1}1^{2}$
	$8B$			$8^{3}4$
$9ab$	$9A$	$124578/9$	111111	$9^{3}1$
$10a.$	$10A$	$001234/5$	111111	$10^{1}5^{3}21$
$12c.$	$12A$	$000012/3$	111111	$12^{1}6^{1}4^{2}1^{2}$
	$12B$			$12\,4^{2}3^{2}2$
$12ab$	$12C$	$111222/3$	111111	$12^{2}31$
	$15A$			$15\,5^{2}3$

4. $X_{27} \to M_{0,5}^{\alpha}$ **with** $\alpha = (st)$**.**
Gal $= W(E_6)$.

The starting point:

X^*	$t0$	$t1$	$t\infty$ & ts	
C	$6ab$	$4a$	$2c$	
λ	$6^4 3$	$4^6 1^3$	$2^6 1^{15}$	$g = 0$
j	$\widehat{11}1\widehat{22}2/3$	$\widehat{00}\widehat{00}\widehat{00}/2$	$\widehat{000000}/1$	
μ	2211	222	51	$m = 0$
X	1∞	0∞	$s\infty$ & $t\infty$	

Genus computation:

$$\begin{aligned} \sum_{i=1}^{z} |\lambda_i| &= (z-2)n + 2 - 2\boxed{\text{g}} \\ 5 + 9 + 2(21) &= 2 \cdot 27 + 2 - 2\boxed{\mathbf{0}} \end{aligned}$$

Mobility computation:

$$\begin{aligned} \sum_{i=1}^{z} ||\mu_i|| &= (z-2)n^2 + 2 - 2\boxed{\text{m}} \\ 10 + 12 + 2(26) &= 2 \cdot 36 + 2 - 2\boxed{\mathbf{0}} \end{aligned}$$

In the family J_6.

An equation for $X_{27}^* \to M_{0,5}$ is

$$tb_6(s,x)^4 b_3(s,x) + 2^3 3^2 s(1-t) a_4(s,x)^6 \;=\; 0$$

where

$$\begin{aligned} a_4(s,x) &= x^4 - 2x^3 + 2sx - s, \\ b_3(s,x) &= 2x^3 - 3x^2 + s, \\ b_6(s,x) &= x^6 - 3x^5 + 10sx^3 - 15sx^2 + 9sx - 2s^2. \end{aligned}$$

It was found by first finding it modulo 5 and then lifting.

The discriminant of

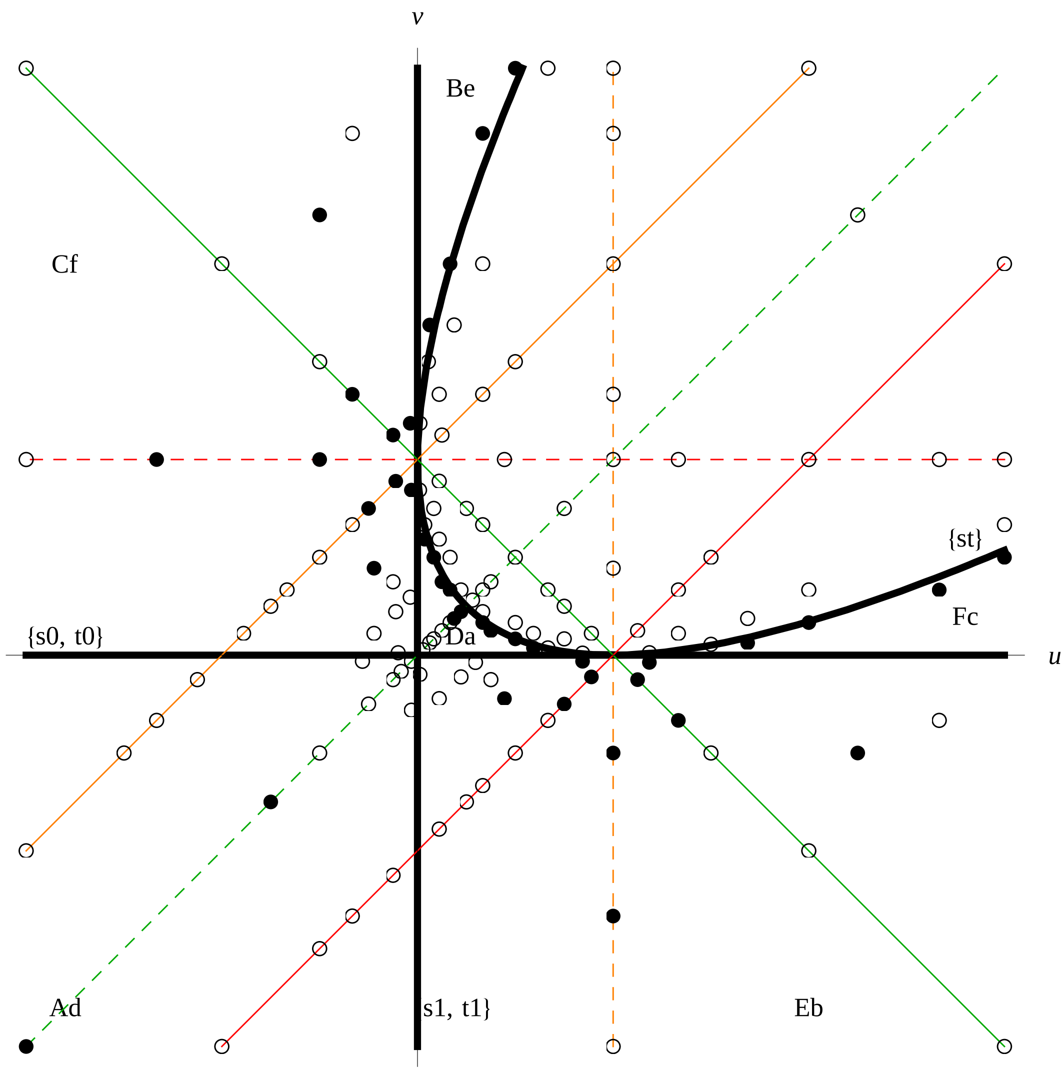
$$f_{27}^*(s,t,x) = tb_6(s,x)^4 b_3(s,x) + 2^3 3^2 s(1-t) a_4(s,x)^6$$

is

$$D_{27}^*(s,t) = 2^{190} 3^{270} s^{126} (s-1)^{102} t^{22} (t-1)^{18} (t-s)^6.$$

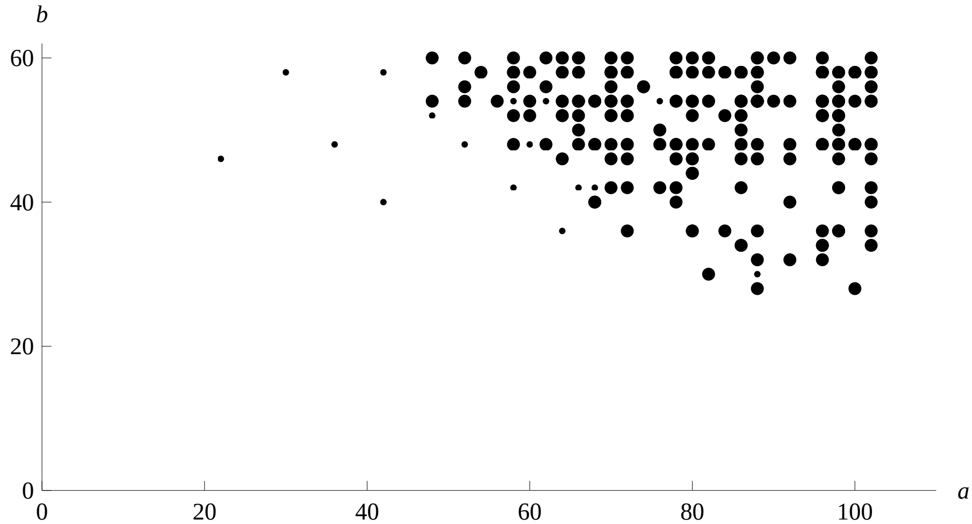
The desired $X_{27} \to M_{0,5}^{\alpha}$ then comes from simultaneously quotienting by $\tilde{\alpha}$ and α.

We now specialize at the universal set $M^{\alpha}_{0,5}(\mathbb{Z}[1/6])$.



The black dots come from $M_{0,5}(\mathbb{Z}[1/6])$ while the white dots are new.

Discriminants $2^a 3^b$ arising from the 229 points of $M^{\alpha}_{0,5}(\mathbb{Z}[1/6])$ ($\bullet = W(E_6)^*$ and $\cdot =$ smaller):



More details:

λ_{27}	$\|G\|$		(u,v)	#	GRD	r	R
$12,6,6,3$	48	$S_4 \times S_2$	$(1/3,1/3)$	1	21.20	4	25
27	54	$3^{1+2}_{+}.2$	$(-1/32,27/32)$	1	19.94	2	4
27	216	$3^{1+2}_{+}.D_4$	$(1/3,1/6)$	1	37.92		
27	324	$(S_3 \wr A_3)^+$	$(1/4,-3/4)$	1	29.96	1	1
$15,6,6$	360	A_6	$(-32,-27)$	1	31.66	1	1
$24,3$	576	$2.A_4^2.2$	$(-4/3,-1/3)$	1	54.41		
27	648	$S_3 \wr A_3$	$(27/32,-1/32)$	1	51.41	4	
27	648	$(S_3 \wr S_3)^+$	$(1/16,1/16)$	2	42.95	4	29
$15,6,6$	720	S_6	$(-9/2,-2)$	1	72.71	13	
27	1296	$S_3^3 \wr S_3$	$(8/3,1/3)$	6	37.92	2	3
27	1296	$3^{1+2}_{+}.\tilde{S}_4$	$(1,2)$	1	44.45		
$15,12$	1440	$S_6 \times S_2$	$(1/8,3/8)$	2	66.35		
$16,10,1$	1920	$2^4.S_5$	$(3,8)$	1	59.95	2	
27	25920	$W(E_6)^+$	$(4/3,-1/3)$	15	48.12		
27	51840	$W(E_6)$	$(27/8,-1/8)$	194			

plus 300+ more

The least ramified $W(E_6)^+$ field, as measured by the root discriminant of the Galois closure, is given by

$$f(x) =$$
$$x^{27} - 90x^{19} - 336x^{18} + 1080x^{15} - 1215x^{11}$$
$$+3312x^{10} - 2544x^9 + 1944x^7$$
$$-1728x^6 - 324x^3 - 144x - 64$$

Its Galois slope content is $[7/2, 3, 3, 2, 2, 2]^3$ at 2 and $[5/3, 3/2, 3/2]_2^3$ at 3.

Its Galois root discriminant is therefore

$$GRD(f) = 2^{99/32} 3^{85/54} \approx 48.12$$

This somewhat above the Serre-Odlyzko constant $\Omega = 8\pi e^\gamma \approx 44.7632$. For comparison, the largest Galois number field in the literature with root discriminant less than Ω is an S_7 field with $|S_7| = 5040$.

5. $X_{28a} \to M_{0,5}^{\beta}$ **with** $\beta = (01)(st)$**.**
$\mathrm{Gal}^{\mathrm{geom}} = \mathrm{Gal} = W(E_7)^+$.

The starting point:

	$t0$	$t1$	$t\infty$	ts	
C	$3A$	$4A$	$3B$	$2A$	
λ	$3^6 1^{10}$	$4^6 1^4$	$3^9 1$	$2^6 1^{16}$	$g = 0$
j	$\widehat{000}012/3$	$\widehat{00}\widehat{00}\widehat{00}/1$	$\widehat{111}\widehat{222}/3$	$\widehat{000000}/1$	
μ_2	411	222	33	51	$m = 0$

Genus computation:

$$\sum_{i=1}^{z} |\lambda_i| = (z-2)n + 2 - 2\boxed{g}$$

$$16 + 10 + 10 + 22 = 2 \cdot 28 + 2 - 2\boxed{\mathbf{0}}$$

Mobility computation:

$$\sum_{i=1}^{z} ||\mu_i|| = (z-2)n^2 + 2 - 2\boxed{m}$$

$$18 + 12 + 18 + 26 = 2 \cdot 36 + 2 - 2\boxed{\mathbf{0}}$$

In the family u_6.

An equation for $X^*_{28a} \to M_{0,5}$ is

$$sb_6(s,x)^4 b_4(s,x) + (1-t)c_9(s,x)^3 x = 0$$

where

$$\begin{aligned} b_6(s,x) &= x^6 - 3x^5 + 10sx^3 - 15sx^2 + 9sx - 2s^2, \\ b_4(s,x) &= x^4 - 4sx + 3s, \\ c_9(s,x) &= x^9 - 12sx^7 + 24sx^6 - 18sx^5 + 24s^2x^4 \\ &\quad -60s^2x^3 + 72s^2x^2 - (12s^3 + 27s^2)x + 8s^3 \end{aligned}$$

It was found by a massive "Lego" computation using the total *a priori* control of the monodromy, assisted by agreement of some cuspidal specializations between X^*_{27} and X^*_{28a}.

The discriminant of

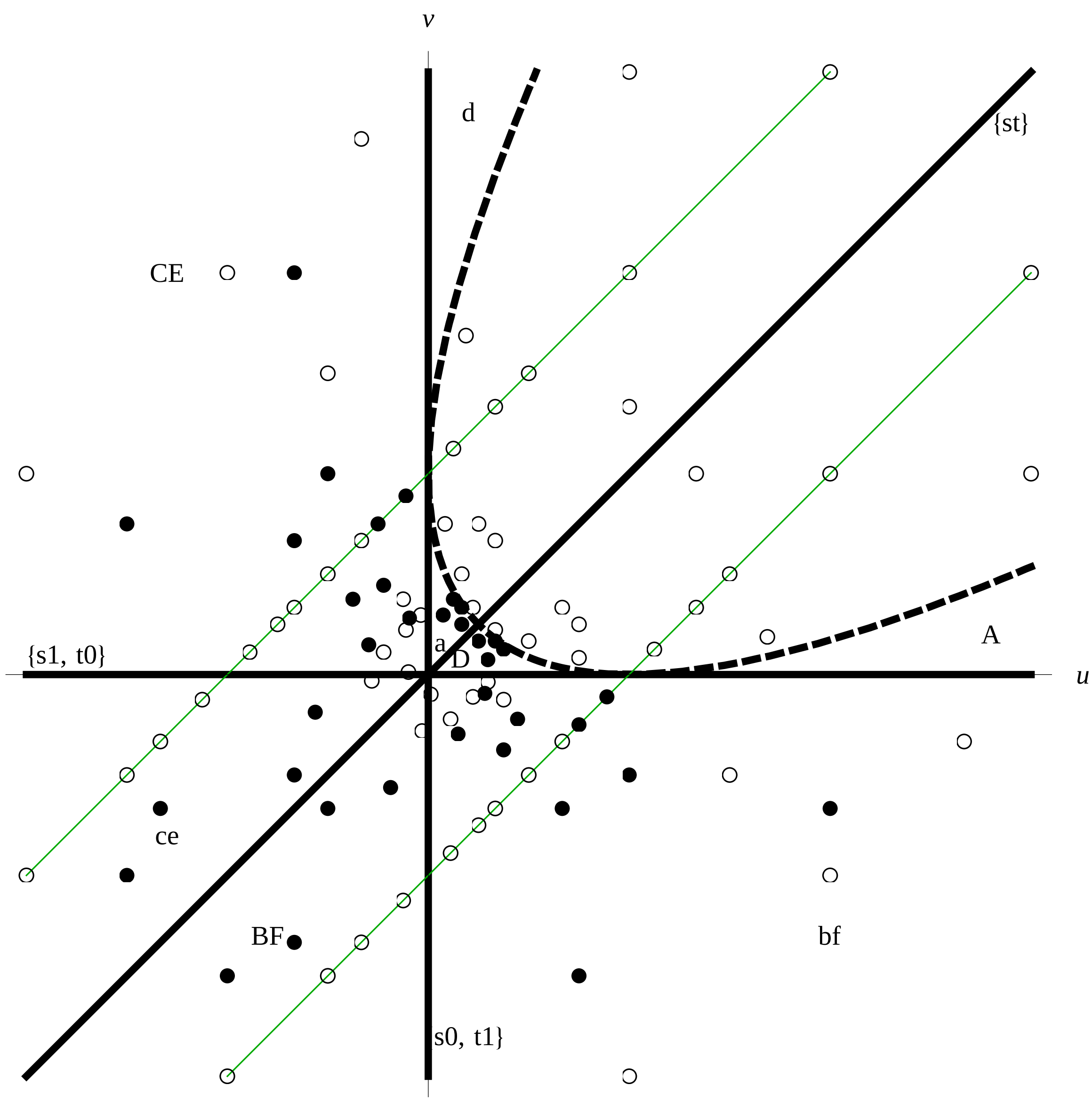
$$f^*_{28a}(s,t,x) = sb_6(s,x)^4 b_4(s,x) + (1-t)c_9(s,x)^3 x$$

is

$$D^*_{28a}(s,t) = 2^{228}3^{270}s^{198}(s-1)^{102}t^{12}(t-1)^{18}(t-s)^6.$$

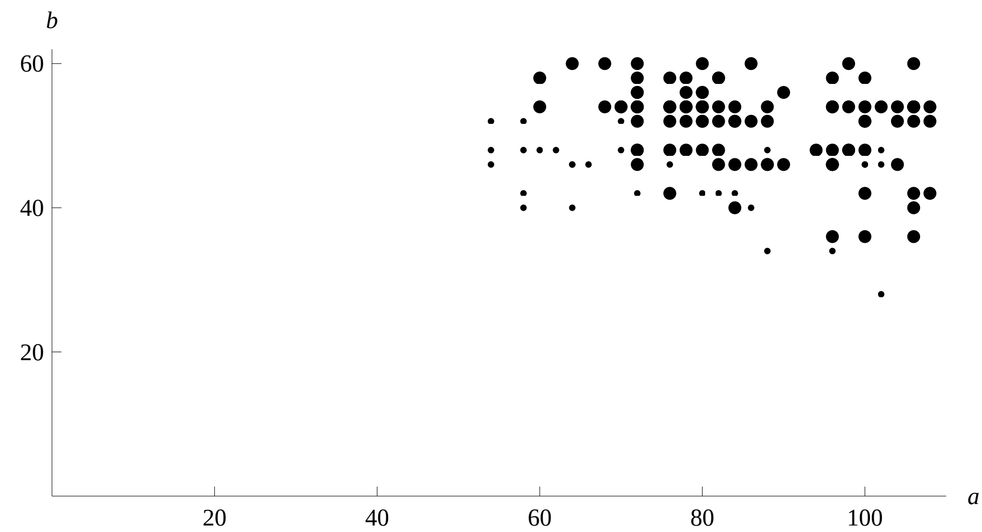
The desired $X_{28a} \to M^{\beta}_{0,5}$ is obtained by taking the $(\tilde{\beta}, \beta)$ quotient.

We now specialize at the universal set $M^{\beta}_{0,5}(\mathbb{Z}[1/6])$:



Again, the black dots come from $M_{0,5}(\mathbb{Z}[1/6])$ and the white dots are new.

Discriminants $2^a 3^b$ arising from the 138 points of $M_{0,5}^{\beta}(\mathbb{Z}[1/6])$ ($\bullet = W(E_7)^+$ and $\cdot =$ smaller):



More details:

μ_{28}	$\lvert G \rvert$	G	(u,v)	#	GRD	r	R
16, 12	384	$12T138$	$(1/4, -3/4)$	1	35.65		
27, 1	432	$3_+^{1+2}.\tilde{D}_4$	$(-3/4, 1/4)$	1	32.04		
18, 10	720	$S_5 \times S_3$	$(-4/3, -4)$	1	50.41		
16, 12	1152	$S_4^2 \wr S_2$	$(36, 24)$	1	47.43	11	
27, 1	1296	$S_3 \wr S_3$	$(-9, -8)$	1	37.92	2	3
28	1512	$P\Gamma L_2(8)$	$(9/32, -3/32)$	1	36.12		
21, 7	5040	S_7	$(-18, -24)$	2	47.43	1	
28	12096	$G_2(2)$	$(3/4, 1/4)$	2			
28	20160	A_8	$(9, 12)$	1	76.38		
16, 12	23040	$2^5.S_6$	$(1, 3)$	1	79.29		
27, 1	25920	$W(E_6)^+$	$(1, 2)$	1	70.10		
28	40320	S_8	$(-4, -6)$	6	73.44		
27, 1	51840	$W(E_6)$	$(3, 4)$	18	61.36		
28	1451520	$W(E_7)^+$	$(-1/2, 1)$	101	61.43		

plus 8 more

6. $X_{28b} \to M_{0,5}^{S_3}$.
$\mathrm{Gal}^{\mathrm{geom}} = G_2(2)'$, $\mathrm{Gal} = G_2(2)$,
$\mathbb{Q}(i)$, corresponding to $\mathrm{Gal}/\mathrm{Gal}^{\mathrm{geom}}$, is present in all specializations.

In this case the best initial 4-tuple is $2A$, $2A$, $3A$, $4A$, which gives a unique cover. One has an obvious S_2 symmetry from $2A = 2A$, and in fact the total symmetry is S_3.

The mobility with respect to any representation is $\boxed{\text{positive}}$, so one is not *a priori* guaranteed at all that the bad reduction set is $\{2, 3\}$.

The genus is $\boxed{3}$, obstructing calculations of the style of the previous two examples.

To find a defining equation for the cover, we used Shioda's universal $W(E_7)$ family and its defining polynomial $S(a, b, c, d, e, f, g; x)$.

Specializing via $f^*(s, t, x) =$

$S(1, s+t, -3st, 0, -st(s+t), -st(s+t), -s^2t^2; x)$

gives the desired cover over $M_{0,5}$.

The discriminant is

$D^*(s, t) = 2^{326}3^{156}s^{42}(s-1)^{24}t^{42}(t-1)^{24}(s-t)^{84}$

times the square of a large-degree irreducible polynomial in $\mathbb{Z}[s, t]$. So the bad reduction set is in fact just $\{2, 3\}$, rather than the other possibility $\{2, 3, 7\}$.

Descending via the S_3 symmetry gives the final cover $X_{28b} \to M_{0,5}^{S_3}$.

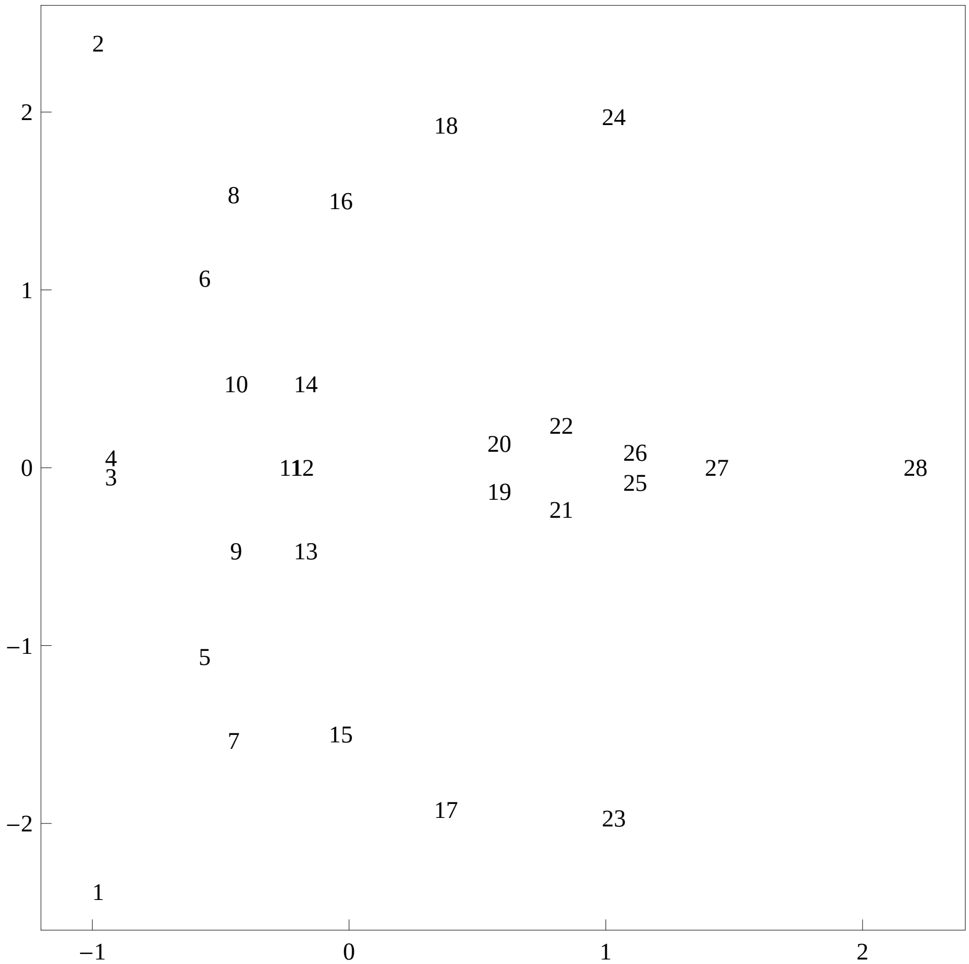
Specialization behaves similarly to the other the cases, giving here about four hundred $G_2(2)$ fields with discriminant $2^a 3^b$. (Compare: there are exactly ten S_7 fields with discrim. $\pm 2^a 3^b$.)

One field K with small discriminant comes from

$$
\begin{aligned}
f(x) = \\
& x^{28} - 4x^{27} + 18x^{26} - 60x^{25} + 165x^{24} \\
& -420x^{23} + 798x^{22} - 1440x^{21} + 2040x^{20} \\
& -2292x^{19} + 2478x^{18} - 756x^{17} - 657x^{16} \\
& +1464x^{15} - 4920x^{14} + 3072x^{13} - 1068x^{12} \\
& +3768x^{11} + 1752x^{10} - 4680x^{9} - 1116x^{8} \\
& +672x^{7} + 1800x^{6} - 240x^{5} - 216x^{4} \\
& -192x^{3} + 24x^{2} + 32x + 4.
\end{aligned}
$$

The discriminant is $D = 2^{66}3^{46}$ and so the root discriminant is $D^{1/28} \approx 31.15$.

The twenty-eight roots of $f(x)$ are:



The Galois group is $\text{Gal}(K^{\text{gal}}/\mathbb{Q}) = \langle c, d \rangle$ with

$$\begin{aligned} c &= (1,2)(3,4)(5,6)(7,8)(9,10)(13,14)(15,16)(17,18) \\ &\quad (19,20)(21,22)(23,24)(25,26)(11)(12)(27)(28) \\ d &= (1,8,9,18,25,20)(2,22,24,19,3,6)(4,27,28,11,21,14) \\ &\quad (5,23,26,10,17,15)(7,16,12)(13) \end{aligned}$$

Much of the ramification in K^{gal} can be seen already from the degree 28 polynomial $f(x)$. However some ambiguities remain and to fully describe ramification we use a degree 63 resolvent built using the description $\text{Gal}(K^{\text{gal}}/\mathbb{Q}) = \langle c, d \rangle$.

The slope content at 2 and 3 work out to

$$SC_2 = [3,3,3,2,2]^3, \qquad SC_3 = [\frac{11}{6}, \frac{13}{8}, \frac{13}{8}]_8^2.$$

The Galois root discriminant

$$2^{43/16} 3^{125/72} \approx 43.3864$$

At present K^{gal} is the largest known Galois field with almost simple Galois group and root discriminant less than the Serre-Odlyzko constant $\Omega \approx 44.7632$.