# A new result in game theory
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A presentation of one of the results in *Nash Equilibria of Cauchy-random zero-sum and co-ordination matrix games*, International Journal of Game Theory, 34, pages 167-184 (2006)

Example. The distribution of support sizes k in Cauchy-random zero-sum 50-by-70 matrix games. The most common support sizes are $k = 29$ and $k = 30$, with frequencies 14.8% and 14.6% respectively.
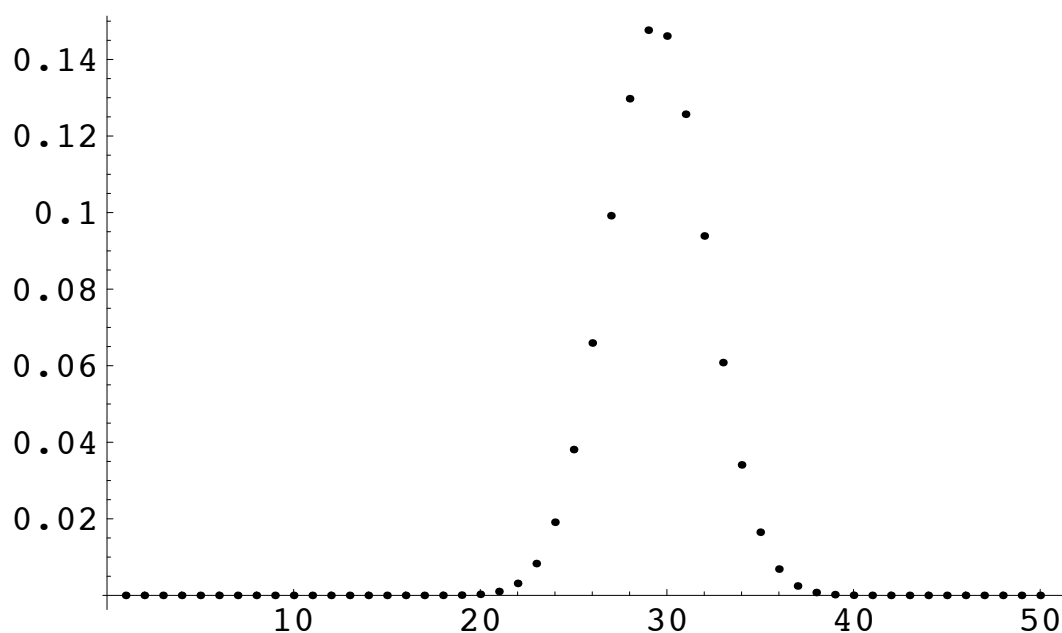
We work always with *zero-sum* games between *two players*.

Familiar Example of Rock-Paper-Scissors.
"Rock smashes scissors, scissors cuts paper, paper suffocates rock." Coded up in a matrix:

		Column Player		
		Rock	Paper	Scissors
Row	Rock	0	-1	1
Player	Paper	1	0	-1
	Scissors	-1	1	0

Kids know that best play is to randomly mix the strategies. In code, row's best strategy is

$$x = (x_1, x_2, x_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

Column's best strategy is similarly

$$y = (y_1, y_2, y_3) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

This is a fair game, in the sense that on average the column player pays the row player $\lambda = 0$.

Example of biased Rock-Paper-Scissors.

		Column Player		
		Rock	Paper	Scissors
	Rock	0	-1	1
Row	Paper	1	c	-1
Player	Scissors	-1	1	0

Unique solution if $0 \le c < 3$:

$$(x_1, x_2, x_3) = \left(\frac{1}{3} + \frac{c}{9}, \frac{1}{3}, \frac{1}{3} - \frac{c}{9} \right)$$

$$(y_1, y_2, y_3) = \left(\frac{1}{3} - \frac{c}{9}, \frac{1}{3}, \frac{1}{3} + \frac{c}{9} \right)$$

$$\lambda = \frac{c}{9}$$

(Each player still mixes all three strategies.)

Unique solution if $c > 3$:

$$(x_1, x_2, x_3) = \left(\frac{2}{3}, \frac{1}{3}, 0 \right)$$

$$(y_1, y_2, y_3) = \left(\frac{2}{3}, 0, \frac{1}{3} \right)$$

$$\lambda = \frac{1}{3}$$

(Each player now uses only two strategies.)

General set-up. A game is given by an m-by-n matrix $A = (a_{ij})$.

The row player can choose to mix his strategies according to any probability vector $x = (x_i)$.

The column player can choose to mix his strategies according to any probability vector $y = (y_j)$.

Theorem (von Neumann (1928)) *For "almost all" games, there is a unique best strategy mix x for the row player and a unique best strategy mix y for the column player.*

Theorem (Shapley & Snow (1950)) *In their mixed best strategies, both players mix the same number k of their pure strategies.*

The number k is the **support size** of the game.

A typical 5-by-7 game:

−18	−54	−70	−93	40	−25	−64
−1	93	49	141	−116	−417	89
173	5	−347	−2716	254	83	58
−652	−84	−66	−119	76	1212	−591
33	−80	53	18	219	47	196

Its solution:

$$k = 4$$
$$x \approx (0, 32\%, 3\%, 7\%, 58\%)$$
$$y \approx (29\%, 50\%, 0, 3\%, 0, 18\%, 0)$$
$$\lambda \approx -21.93$$

The game and its solution again:

−21.93	29%	50%		3%		18%	
	−18	−54	−70	−93	40	−25	−64
32%	−1	93	49	141	−116	−417	89
3%	173	5	−347	−2716	254	83	58
7%	−652	−84	−66	−119	76	1212	−591
58%	33	−80	53	18	219	47	196

Natural Question: *How is the support size k distributed for "random" m-by-n games?*

Progress was made by

Goldman	1957	k=1
Thrall & Falk	1965	Small m, n
Faris & Maier	1987	Small m, n
Berg &Engel	1998	Asymptotics

Goldman's result was that, for any reasonable notion of randomness, $k = 1$ occurs with probability $\dfrac{m!n!}{(m + n - 1)!}$. For large m and n, Goldman's quantity becomes very small.

Berg & Engel used remarkable statistical mechanics techniques to conjecture a formula valid for matrices with any given shape but only in the limit of large size.

Experimental data. I took experimental data in the spirit of Thrall & Falk and Faris & Maier. For example, from 1,000,000 different 5-by-7 games chosen with a_{ij} independent and normally distributed, I found support sizes distributed as

1	2	3	4	5
1.46%	18.26%	45.58%	30.22%	4.48%

1.46% compares well with Goldman's proved $1/66 \cong 1.51\%$.

Guessing a formula. I multiplied all the above percentages by 66, obtaining

1	2	3	4	5
0.96	12.05	30.08	19.95	2.96

These are all within experimental error of being integers. The same phenomenon occurred for all other small (m, n). From the rational numbers obtained, I conjectured a general Goldman-style formula. So, e.g., my formula predicted that a random 5-by-7 game would have support size four with probability $20/66 = 30.\overline{30}\%$.

Finding the right notion of randomness.
My conjectured formula seemed roughly right
for many notions of randomness. But it was
likely to be exactly right for at most one notion
of randomness. Looking at Goldman's proof
led me to choose the a_{ij} independently with
respect to the Cauchy density $\frac{1}{\pi(x^2+1)}$ rather
than the Gaussian density $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$.
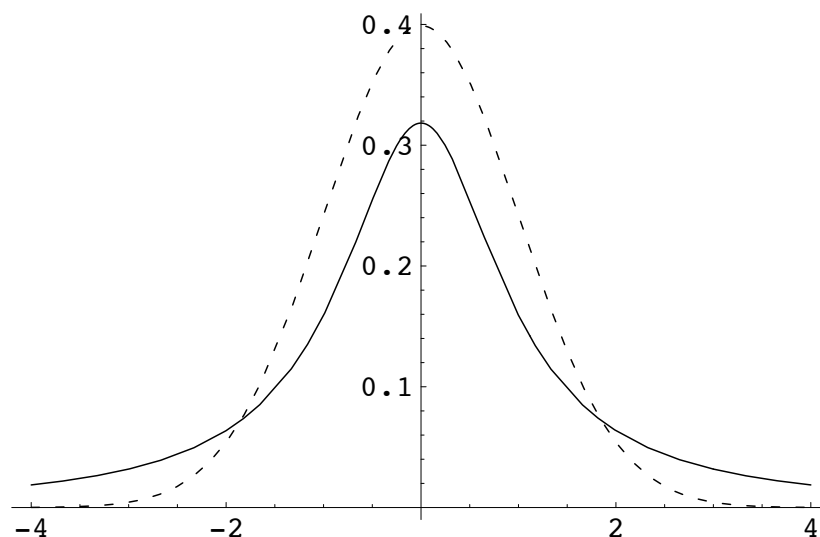
The two densities are qualitatively similar but
the Cauchy density has fatter tails.

Main result. With the right density, I succeeded in establishing my conjectured formula:

Theorem *For Cauchy-random m-by-n zero-sum matrix games, the chance that the support size is k is given by*

$$\frac{m!}{k!(m-k)!} \frac{n!}{k!(n-k)!} \frac{(m-1)!(n-1)!}{(m+n-1)!} \, k.$$

Proof. The proof is a complicated induction. It involves reducing multivariate integrals to univariate integrals by using special properties of the Cauchy density.

Very simple concluding principle. The theorem implies that *The support size k is approximately normally distributed with mean $\dfrac{mn}{m+n}$ and standard deviation $\dfrac{mn}{(m+n)^{3/2}}$.*