Motivic Computations in Magma

David P. Roberts
University of Minnesota, Morris

1. Invariants of algebraic varieties

2. Familiar examples

3. Hypergeometric motives

4. L-factors of Hypergeometric motives at wild primes, an example

1. Invariants of algebraic varieties. Let X be a smooth projective variety over $\mathbb{Q}$ of dimension d. The $2d$-dimensional real manifold $X(\mathbb{C})$ has cohomology groups $H^w(X(\mathbb{C}), \mathbb{Q})$. Because of the arithmetic source, these cohomology groups have extra structure.

Three types of extra structures which are local in the sense that they are associated with a place $v \in \{\infty, 2, 3, 5, 7, \dots\}$ of $\mathbb{Q}$:

From the infinite place $v = \infty$. There is a Hodge decomposition

$$H^w(X(\mathbb{C}), \mathbb{C}) = \bigoplus_{p=0}^{w} H^{p,w-p}.$$

with $H^{p,q} = \overline{H}^{q,p}$. In particular the w^{th} Betti number decomposes $b = \sum h^{p,w-p}$ with

$$h^{p,q} = h^{q,p}.$$

Complex conjugation $\mathrm{Fr}_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ interchanges $H^{p,q}$ and $H^{q,p}$ so that moreover

$$h^{w/2,w/2} \;=\; h_+^{w/2,w/2} + h_-^{w/2,w/2} \quad \text{(for } w \text{ even).}$$

From a good place p. From the integers $|X(\mathbb{F}_{p^f})|$, one gets a Frobenius operator Fr_p on $H^w(X(\mathbb{C}),\mathbb{Q})$, well-defined up to conjugation (Dwork, Grothendieck, Deligne). The eigenvalues α_i have absolute value $p^{w/2}$. It is best to package them into a polynomial

$$F_p(H^w(X(\mathbb{C}),\mathbb{Q}),T) = \prod_{i=1}^{b}(1-\alpha_i T) \in \mathbb{Z}[T].$$

From a bad place p. The case is similar to the case of good places, but much more complicated, even still depending on resolution-of-singularities type conjectures for some basic statements. One still has a polynomial $F_p(H^w(X(\mathbb{C}),\mathbb{Q}),T)$, now of degree $< b$. One also has a conductor p^{n_p}, with $n_p \in \mathbb{Z}_{\geq 1}$ measuring the badness of the mod p reduction, from a cohomological viewpoint.

L-functions. The local invariants discussed so far are naturally combined into a global L-function

$$L(H^w(X(\mathbb{C}),\mathbb{Q}),s) = \Gamma(h,s)\prod_p \frac{1}{F_p(H^w(X(\mathbb{C}),\mathbb{Q}),p^{-s})}.$$

This function conjecturally satisfies a functional equation $s \leftrightarrow w+1-s$ in which the conductor $N = \prod_p p^{n_p}$ enters fundamentally and measures computational complexity.

One can also work with suitable direct summands $M \subseteq H^w(X(\mathbb{C}),\mathbb{Q})$ called motives, where all the above invariants are defined. The motivic L-functions $L(M,s)$ are very well implemented in *Magma* (Dokchitser, Watkins). They are the subject of many conjectures: connections to automorphic forms, special values at integers, Riemann hypothesis and further properties of zeros, ...

A natural problem is to produce motives M with small conductor N for given Hodge numbers h (and given "motivic Galois group")

2. Familiar examples. Our examples are in increasing order of $w = \dim(X)$ and focus on $M \subseteq H^w(X(\mathbb{C}), \mathbb{Q})$.

A. Number Fields. Taking X of dimension zero, the study of $M = H^0(X(\mathbb{C}), \mathbb{Q})$ is the study of number fields K in a fancy language. If $K \otimes \mathbb{R} = \mathbb{R}^r \times \mathbb{C}^s$ then signature translates to Hodge numbers:

$$(r, s) = (h_+^{0,0} - h_-^{0,0}, 2h_-^{0,0}).$$

The absolute discriminant translates to conductor: $D = N$. The Dedekind zeta function is an example of an L-function: $\zeta(K, s) = L(M, s)$.

B. Curves. Taking X of dimension one, the study of $M = H^1(X(\mathbb{C}), \mathbb{Q})$ is essentially the study of Jacobians in a fancy language. An example of *Magma* functionality:

```
>ZT<T> := PolynomialRing(Integers());
>P2<x,y,z> := ProjectiveSpace(FiniteField(5),2);
>Factorization(LPolynomial(Curve(P2,x^4+y^4+z^4)));
[<5*T^2 - 2*T + 1, 3>]
```

C. Surfaces and their Hodge numbers.

```
>P3<w,x,y,z> := ProjectiveSpace(Rationals(),3);
>fermat := func<n|Surface(P3,w^n+x^n+y^n+z^n)>;
>hodgesquare := func<S|
>     [[HodgeNumber(S,i,j):i in [0..2]] :
>     j in [2..0 by -1]]>;
>[hodgesquare(fermat(n)) : n in [1..5]];
```

```
[[[ 0, 0, 1 ],
  [ 0, 1, 0 ],
  [ 1, 0, 0 ]],
```

The projective plane $\mathbb{P}^2$

```
 [[ 0, 0, 1 ],
  [ 0, 2, 0 ],
  [ 1, 0, 0 ]],
```

The quadric $\mathbb{P}^1 \times \mathbb{P}^1$

```
 [[ 0, 0, 1 ],
  [ 0, 7, 0 ],
  [ 1, 0, 0 ]],
```

$\mathbb{P}^2$ with six points blown up

```
 [[ 1, 0, 1 ],
  [ 0, 20,0 ],
  [ 1, 0, 1 ]],
```

K3 surface

```
 [[ 4, 0, 1 ],
  [ 0, 45,0 ],
  [ 1, 0, 4 ]]]
```

General type surface

3. Hypergeometric Motives. Let d be an integer. Let $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d \in \mathbb{Q}/\mathbb{Z}$ with always $\alpha_i \neq \beta_j$. Suppose the number of times a rational number occurs depends only on its denominator. Then for every $t \in \mathbb{Q} - \{0, 1\}$ one has a rank d *motive*

$$M_t = H(\alpha_1, \dots, \alpha_d; \beta_1, \dots, \beta_d; t).$$

One would like to compute the L-series $L(M_t, s)$ completely.

The Hodge numbers depend only on how the α's and the β's intertwine in the circle $\mathbb{R}/\mathbb{Z}$. At one extreme, if the α's and the β's are separated, then

$$(h^{d-1,0}, \dots, h^{0,d-1}) = (1, 1, \dots, 1, 1).$$

At the other extreme,

$$(h^{0,0}) = (d)$$

if there is complete intertwining.

Magma's HGM package (Watkins) implements much of what is known about HGMs:

- It computes Hodge numbers and their parity split, hence $L_\infty(M,s) = \Gamma(h,s)$.

- For p not wild, meaning not dividing a denominator of an α_i or a β_j, it very efficiently computes (Katz, Cohen, ...) the L-factor $L_p(M,s) = 1/F_p(M,p^{-s})$ and the conductor p^{n_p}.

- For (α,β) "sufficiently classical," it identifies M as coming from a specific variety X and sometimes thereby computes $L_p(M,s)$ for wild primes p too.

The HGM package feeds nicely into the L-Series package, but the ambiguity at wild primes limits its usefulness.

4. Wild L-factors of HGMs. Example 1 (works perfectly!):

```
>M1 := HypergeometricData([1/6,1/3,2/3,5/6],
>      [0,1/4,1/2,3/4]);
>L1 := LSeries(M1,2);
```
($t = 2$ as an example)

```
>EulerFactor(L1,2);
1-T
>EulerFactor(L1,3);
1
>Factorization(Conductor(L1));
[ <2, 10>, <3, 6> ]
>CheckFunctionalEquation(L1);
1.3126645462877636442051163326 3E-27
```
≈ 0 ✓

Here the wild primes are 2 and 3. The α_i's and β_j's completely intertwine, so the Hodge vector is $(h^{0,0}) = (4)$. *Magma* identifies

$$L(M1_t, s) = \frac{\zeta(\mathbb{Q}[x]/(729x^2(x-1)^4t - 16), s)}{\zeta(\mathbb{Q}[x]/(x^2 - t), s)}.$$

Because of this identification, the local factors $L_p(M1_2, s)$ and local conductors p^{n_p} have been correctly calculated.

Example 2 (problem at 3):

```
>M2 := HypergeometricData([1/3,1/3,2/3,2/3],
>      [0,0,0,0]);
>L2 := LSeries(M2,2);     (t = 2 as an example)
WARNING: Guessing wild prime information
>EulerFactor(L2,2);
1                          (2 is tame, so right)
>EulerFactor(L2,3);
1              (3 is wild, so dubious(but right))
>Factorization(Conductor(L2));
[<2, 4>]  (right at 2, dubious at 3 (and wrong))
>CheckFunctionalEquation(L2);
|Sign| is nowhere near 1,
wrong functional equation?
-0.307498051262530109394761826467 1 ≠ 0   X
```

Here the only wild prime is 3. The α_i's and β_j's are completely separated, so the Hodge vector is $(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1,1,1,1)$. Any underlying variety has to have dimension at least three. *Magma* cannot appeal to its more classical parts to get the correct invariants at 3.

Connection between the two examples. An element of $\mathbb{Q}/\mathbb{Z}$ is the sum of its p-primary parts. E.g. $1/6 = 1/2 + 2/3$ has 2-primary part $1/2$ and 3-primary part $2/3$. In our case, taking 3-primary parts of the indices of

$$M1 = H([1/6, 1/3, 2/3, 5/6], [0, 1/4, 1/2, 3/4])$$

gives the indices of

$$M2 = H([1/3, 1/3, 2/3, 2/3], [0, 0, 0, 0]).$$

Via this connection, as an instance of a general theorem, one has the following fact: If $\mathrm{ord}_3(N1_t) \geq 6$ then $F_3(M1_t, T) = F_3(M2_t, T) = 1$ and $c_3(M1) = c_3(M2)$.

In other words, using the ability of the L-series package to input specified bad factors,

```
LSeries(M2,t : BadPrimes :=
[<3,Valuation(Conductor(LSeries(M1,t))),3),1>])
```

gives the right L-series when $\mathrm{ord}_3(N1_t) \geq 6$.

As numerical examples, all double-checked with `CheckFunctionalEquation`, some conductors are as follows.

t	$N1_t$		$N2_t$		
-1	2^9	3^6	2^1	3^6	
$1/2$	2^{11}	3^6	2^3	3^6	
2	2^{10}	3^6	2^4	3^6	←was 0, now fixed!
-8	2^8	3^6	2^2	3^6	
$-1/8$	2^{11}	3^6	2^3	3^6	
-2	2^{10}	3^7	2^4	3^7	
$8/9$	2^8	3^8	2^2	3^8	
$-1/3$	2^6	3^9	2^1	3^9	
$1/3$	2^9	3^9	2^1	3^9	
$2/3$	2^{10}	3^9	2^4	3^9	
$4/3$	2^6	3^9	2^4	3^9	
$9/8$	2^{11}	3^{10}	2^3	3^{10}	
-3	2^6	3^{10}	2^1	3^{10}	
$3/4$	2^4	3^{10}	2^3	3^{10}	
$3/2$	2^{11}	3^{10}	2^3	3^{10}	
3	2^9	3^{10}	2^1	3^{10}	

We hope to implement the general theorem so as to allow *Magma* to compute most low degree Hypergeometric L-series completely.