Lightly ramified number fields

(with an eye towards automorphic forms)

David P. Roberts

University of Minnesota, Morris

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1. The problem of identifying $NF(G, D)$. Let $G \subseteq S_n$ a transitive permutation group and D be a positive integer.

Definition. *$NF(G, D)$ is the set of isomorphism classes of degree n number fields with Galois group G and absolute discriminant D.*

Guiding problem. *Identify all $NF(G, D)$.*

The problem is solved for abelian G, but seems unfeasible for any other G. Various subproblems for fixed nonabelian G are of interest:

- For easier G, identify as many $NF(G, D)$ as possible (§2).
- For harder G, find small or otherwise interesting D with $NF(G, D)$ nonempty (§3-7).
- Determine the average of $|NF(G, D)|$ as $D \to \infty$ (Bhargava; Malle)

2. Databases identifying some $NF(G, D)$.

The Klueners-Malle database gives many fields for almost all G in degree $n \leq 19$, often identifying fields which have minimal discriminant for their G and signature.

The Jones-Roberts database covers fewer groups and is also less comprehensive on signatures. However it identifies many $NF(G, D)$ completely. Examples:

There are 1353 fields with $G = S_5$ and $D = 2^*3^*5^*7^*$.

There are 5568 fields with $G = C_2 \wr C_2 \wr C_2 = 8T35$ and $D = 2^*3^*5^*$.

Extending a large computation of Malle, there are 15184 totally real A_5 fields with discriminant $\leq 2^{38}$.

Root discriminants. It is often good to renormalize to root discriminants $\delta = D^{1/n}$. Let

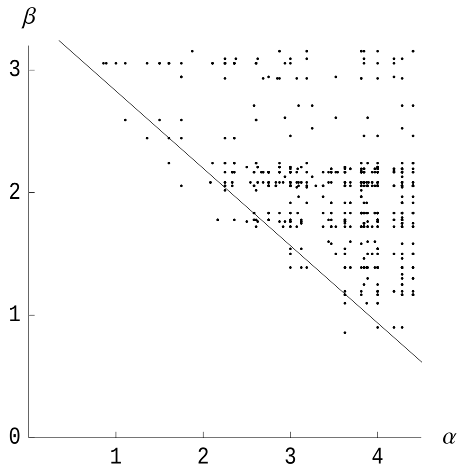
$$\Omega = 8\pi e^{\gamma} \approx 44.76$$

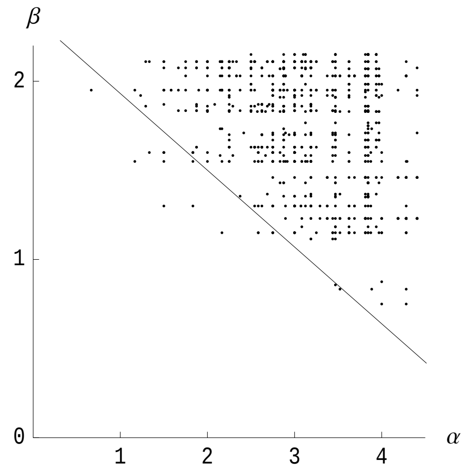
be the Odlyzko-Serre constant. Then under GRH a field with $\delta < \Omega$ has finite Hilbert class field tower. In contrast, Hoelscher recently proved that $\mathbb{Q}(e^{2\pi i/81})$ with $d = 3^{3.5} \approx 47.77$ has infinite Hilbert class field tower.

Galois Root Discriminants. For a field K with Galois closure K^{gal}, the respective root discriminants satisfy $\delta leq \Delta$, with equality if and only if K^{gal}/K is unramified. Computing Δ requires a thorough understanding of p-adic ramification.

Example: there are five twin pairs of sextic fields with $\Delta < \Omega$. The minimum is $\Delta_1 = 2^{13/6} 3^{13/9} \approx 31.66$.

Non-solvable fields in the database with $\Delta = 2^\alpha 3^\beta$ (left) and $\Delta = 2^\alpha 5^\beta$ right, each compared with their line $2^\alpha p^\beta = \Omega$:





There are currently 386 known minimal non-solvable fields with $\Delta < \Omega$:

#	$\lvert H\rvert$	$G = H$	#	$G = H.Q$	#
1	60	A_5	**78**	S_5	**192**
2	168	$SL_3(2)$	*18*	$PGL_2(7)$	*23*
3	360	A_6	**5**	$S_6,\ PGL_2(9),\ M_{10},\ P\Gamma L_2(9)$	**13**, *6*, *0*, *15*
4	504	$SL_2(8)$	*15*	$\Sigma L_2(8)$	*15*
5	660	$PSL_2(11)$	*1*	$PGL_2(11)$	*0*
8	2520	A_7	*1*	S_7	*1*
1^2	3600	A_5^2		$A_5^2.2,\ A_5^2.V,\ A_5^2.C_4,\ A_5^2.D_4$	*1*, *1*, *0*, *0*
10	4080	$SL_2(16)$	*1*	$SL_2(16).2, SL_2(16).4$	*0*, *0*
12	6048	$G_2(2)'$	*0*	$G_2(2)$	*1*
19	20160	A_8	*0*	S_8	*1*

Ω serves as a clarifying reference point in the study of ramification of larger degree fields.

3. NFs from classical modular forms. There are many connections between number fields on the database and classical modular forms.

Example: All 15 known fields with $G = 9T26 = SL_2(8)$ and $\Delta < \Omega$ were found by working backwards from modular forms, five of weight one (Wiese) and ten of weight two.

Example: Bosman has very efficient techniques for working backwards from modular forms, getting many spectacular explicit polynomials. One, with $G = SL_2(16)$ gives a field with $\Delta < \Omega$, namely $\Delta = 2^{15/8} 137^{1/2} \approx 42.93$.

Example: Malle has a remarkable polynomial $f_t(x) \in \mathbb{Z}[t, x]$ with generic Galois group $M_{22}.2$ and discriminant $-2^{484} 11^{253} (t-1)^7 t^{15}$. At

$$\tau = \frac{7^4}{2^6 3} = 1 + \frac{47^2}{2^6 3}$$

there is a group drop from $M_{22}.2$ to $PGL_2(11)$.

The splitting field at τ is also given by

$$x^{12} - 4x^{11} - 4x^{10} + 16x^9 + 24x^8 - 30x^7 - 78x^6$$
$$-18x^5 + 72x^4 + 86x^3 + 52x^2 + 16x + 2.$$

The Galois root discriminant is

$$\Delta = 2^{7/6} 3^{10/11} 11^{9/10} \approx 52.7475.$$

A lift to a Galois representation into $SL_2^{\pm}(11)$ has conductor 24. By Khare-Wintenberger, it corresponds to a modular form of level 24. Because ramification at 11 is tame, Gross's theory of companion forms applies. The corresponding modular forms in $S_4(24)$ and $S_8(24)$ are expressible in closed form in terms of theta functions.

Classical modular forms have the potential to prove explicit completeness results not currently present on the database.

4. NFs related to Hilbert modular forms In work with Dembélé and Diamond, I have been finding polynomials with varied nonsolvable Galois groups and small field discriminants which numerically match Hilbert modular forms modulo p. As an example with $p = 2$,

$$\begin{aligned}&x^{17} - 8x^{16} + 28x^{15} - 48x^{14} + 8x^{13} + 160x^{12} - 376x^{11}\\&+312x^{10} + 316x^9 - 1184x^8 + 1240x^7 + 352x^6\\&-2272x^5 + 1600x^4 + 1152x^3 - 1712x^2 + 12x + 608\end{aligned}$$

has $G = PGL_2(16).4$ and $\Delta = 2^{161/60}3^{3/4}5^{3/4} \approx 48.96$. Similarly for $p = 3$,

$$\begin{aligned}&x^{28} - 4x^{27} + 12x^{26} - 33x^{25} + 63x^{24} - 156x^{23} + 345x^{22}\\&-402x^{21} + 1521x^{20} + 1695x^{19} + 5403x^{18} + 17787x^{17}\\&+19860x^{16} + 73674x^{15} + 61638x^{14} + 182679x^{13}\\&+121506x^{12} + 261114x^{11} + 114639x^{10} + 195027x^9\\&+57960x^8 + 77151x^7 + 17946x^6 - 4257x^5 - 10716x^4\\&+2163x^3 + 9771x^2 + 1471x - 577\end{aligned}$$

has Galois group $G = PGL_2(27).3$ and $\Delta = 2^{6/7}3^{25/18}7^{8/9} \approx 46.98$. An interesting aspect of these matches is that subtle details of ramification at the residual prime p correspond to patterns of Serre weights of associated modular forms.

An earlier example: Dembélé, Greenberg, and Voight computed with Hilbert modular forms over the C_5 field

$$F = \mathbb{Q}[\pi]/(\pi^5 + 5\pi^4 - 25\pi^2 - 25\pi - 5)$$

of discriminant 5^8. They proved the existence of an S_5 extension ramified only at the unique prime π over 5. We found that this extension is given by specializing $x^5 + 5x^4 + 40x^3 - 1728j$ at

$$\begin{aligned} j \;=\; & \frac{-1}{2^6 3^3 5^1 7^{11}} \big(1686352437277476\pi^4 \\ & +8854036993798358\pi^3 - 1124791466055321 5\pi^2 \\ & -4643993605154835 72\pi - 3535058667383836 80\big). \end{aligned}$$

Removing π gives a field defined by

$$\begin{aligned} & x^{25} - 25x^{22} + 25x^{21} + 110x^{20} - 625x^{19} + 1250x^{18} \\ & -3625x^{17} + 21750x^{16} - 57200x^{15} + 112500x^{14} \\ & -240625x^{13} + 448125x^{12} - 1126250x^{11} + 1744825x^{10} \\ & -1006875x^9 - 705000x^8 + 4269125x^7 - 3551000x^6 \\ & +949625x^5 - 792500x^4 + 1303750x^3 \\ & -899750x^2 + 291625x - 36535, \end{aligned}$$

with Galois group $A_5^5.10$ and $\Delta = 5^{3-1/12500} \approx 124.98$.

5. NFs related to hypergeometric motives.

Let $q_\infty(x)$ and $q_0(x)$ be products of cyclotomic polynomials. Suppose $q_\infty(x)$ and $q_0(x)$ are relatively prime and have the same degree d.

Then $(q_\infty(x), q_0(x))$ determines a rank d motivic local system M_t over $\mathbb{Q}(t)$ with coefficients in $\mathbb{Q}$, degenerating only at $t \in \{0, 1, \infty\}$. These motives have classical ${}_dF_{d-1}(t)$ as period integrals.

Example: $(q_\infty(x), q_0(x)) = ((x-1)^2, x^2+x+1)$ yields the elliptic curve $H^1(E_t, x)$, where the elliptic curve $E_t : y^2 = 4(t-1)x^3 - 3tx + t$ has j-invariant $j = 1728t$.

In general, these motives are highly analyzable (. . . , Katz, . . .), all the way to complete motivic L-functions $L(M_t, s)$ which via *Magma*'s `CheckFunctionalEquation` seem to have the expected analytic properties.

Sample explicit polynomials. On the Beukers-Heckman list of HGMs with finite monodromy, five have monodromy group $W(E_7) \approx Sp_6(2)$. Remarkably, polynomials can be obtained from Shioda's universal $W(E_7)$ polynomial

$$S(a_1, b_6, c_8, d_{10}, e_{12}, f_{14}, g_{18}; x)$$

by setting all but two parameters equal to zero:

BH	q_∞	q_0	Kept		Bad Primes
58	$\Phi_2\Phi_{18}$	$\Phi_1\Phi_3\Phi_{12}$	a_2	g_{18}	$\{2,3\}$
59		$\Phi_1\Phi_3\Phi_5$	d_{10}	g_{18}	$\{2,3,5\}$
60		$\Phi_1\Phi_7$	f_{14}	g_{18}	$\{2,3,5,7\}$
61	$\Phi_2\Phi_{14}$	$\Phi_1\Phi_3\Phi_{12}$	e_{12}	f_{14}	$\{2,3,7\}$
62		$\Phi_1\Phi_3\Phi_5$	d_{10}	f_{14}	$\{2,3,5,7\}$

Remarkably, all cases are genus zero and can be given by $f_0(x) + tf_\infty(x) = 0$:

BH	$f_\infty(x)$	$f_0(x)$
58	$-2^{18}\left(x^3+3x^2-3\right)^9$	$3^6x^3(3x+4)\left(x^2+6x+6\right)^{12}$
59	$5^5(x-1)\left(x^3+9x^2+15x-1\right)^9$	$2^{14}3^{12}x^3\left(x^2+7x+1\right)^5$
60	$7^7\left(8x^3+36x^2+12x+1\right)^9$	$2^{14}3^{15}x^7\left(x^3-20x^2-9x-1\right)^7$
61	$2^{18}3^9\left(x^3-7x+7\right)^7$	$7^7(x-1)(x+3)^3\left(x^2-3\right)^{12}$
62	$3^35^5(x+1)^7\left(x^3-x^2-9x+1\right)^7$	$2^{14}7^7x^3\left(x^2+3x+1\right)^5$

6. NFs related to other rigid local systems.

The one-parameter hypergeometric families sit in an extreme position in Katz's theory of rigid local systems. So does the multiparameter Pochhammer family.

An example of the Pochhammer family comes from genus two curves

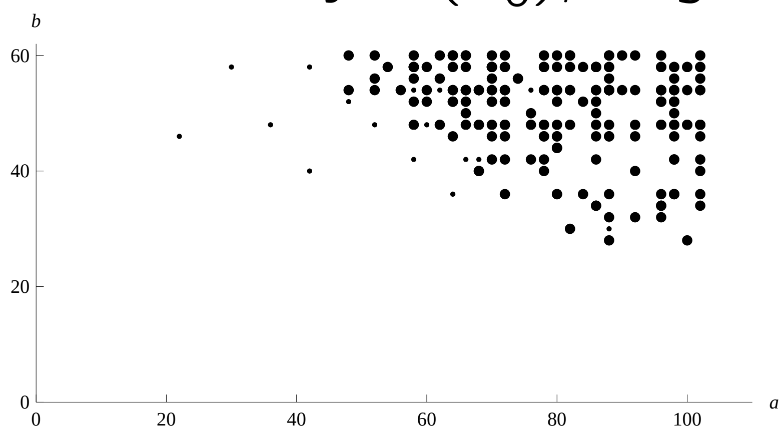
$$X_{a,b,c,d} : y^2 = x^5 + ax^3 + bx^2 + cx + d.$$

From the family $H^1(X_{a,b,c,d}, \mathbb{Q})$, one has degree 40 polynomials $f(a,b,c,d;x)$ with $f(a,b,c,d;x^2)$ having Galois group $G = 2.PSp_4(\mathbb{F}_3).2$. The family is universal subject to the restriction that the .2 corresponds to $\sqrt{-3}$.

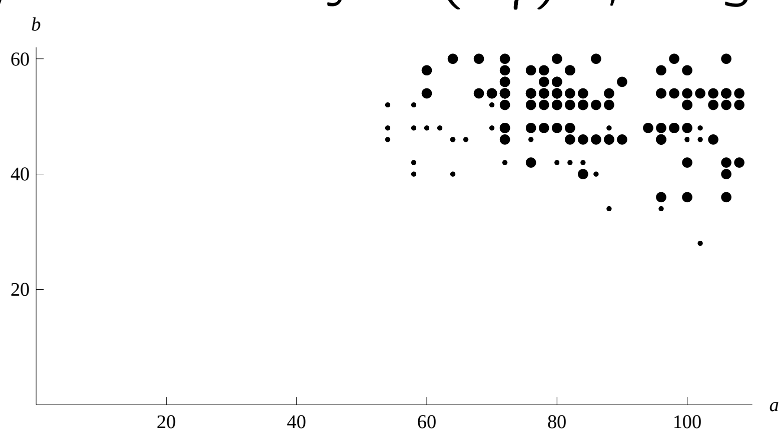
Specializing this 3-parameter family gives about 1000 NFs with $G = 2.PSp_4(\mathbb{F}_3).2$ and $D = 2^*3^*$. Minimal Δ found in this family is slightly above $\Omega \approx 44.7632$.

It is harder to work out equations for other families. Two 2-parameter families and discriminants $D = 2^a 3^b$ of specialized fields:

Type J_6, Monodromy $W(E_6)$, Degree 27:



Type u_6, Monodromy $W(E_7)^+$, Degree 28:



All full-group specializations ($\bullet$) have $\Delta > \Omega$. Some specializations with group-drop ($\cdot$) have $\Delta < \Omega$.

7. NFs with Galois Group $G_2(2) = U_3(3).2$. The twelfth smallest simple group is $G_2(2)' = U_3(3)$ of order $6048 = 2^5 3^3 7$. The extension $G_2(2) = U_3(3).2$ embeds into $W(E_7)^+$.

A genus zero three-point cover with monodromy group $G_2(2)$ is in Malle-Matzat. This cover is a specialization of the Shioda family:

$$S(0, -3t^2, -3^4t^2, 3^5t^3, 3^5t^3, -3^6t^4, 3^6t^5) = 0.$$

A two-parameter specialization with Galois group $G_2(2)$ and .2 corresponding to $\mathbb{Q}(i)$ is

$$S(1, s+t, -3st, 0, -st(+t), -st(s+t), -s^2t^2; x).$$

Its discriminant is

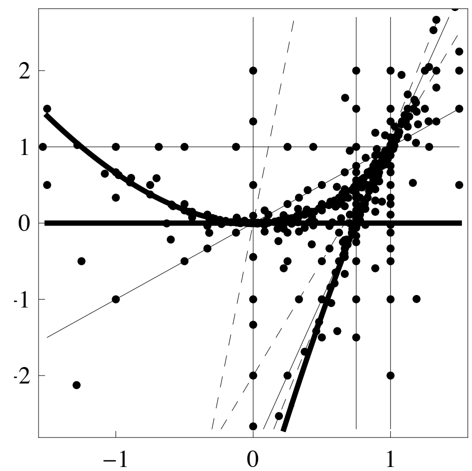
$$D(s,t) = 2^{326}3^{156}s^{42}(s-1)^{24}t^{42}(t-1)^{24}(s-t)^{84}$$

times a square not contributing to field discriminants. Let

$$M_{0,5} = \mathrm{Spec}(\mathbb{Z}[s, t, \frac{1}{st(s-1)(t-1)(s-t)}]).$$

The resulting cover $X \to M_{0,5}$ can be descended to $X/S_3 \to M_{0,5}/S_3$.

Let $S = M_{0,5}/S_3$ be the base scheme, so that $S(\mathbb{R})$ is the complement of the drawn heavy discriminant locus.



Inside $S(\mathbb{R})$, lines serve as bases of 3-point covers and points in $S(\mathbb{Z}[1/6])$ keep ramification within $\{2,3\}$. One specialization point yields

$$x^{28} - 4x^{27} + 18x^{26} - 60x^{25} + 165x^{24} - 420x^{23}$$
$$+798x^{22} - 1440x^{21} + 2040x^{20} - 2292x^{19}$$
$$+2478x^{18} - 756x^{17} - 657x^{16} + 1464x^{15}$$
$$-4920x^{14} + 3072x^{13} - 1068x^{12} + 3768x^{11}$$
$$+1752x^{10} - 4680x^{9} - 1116x^{8} + 672x^{7} + 1800x^{6}$$
$$-240x^{5} - 216x^{4} - 192x^{3} + 24x^{2} + 32x + 4.$$

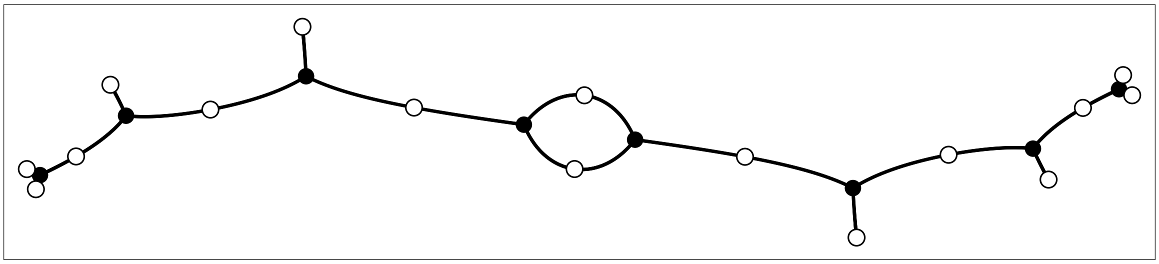
Here $GRD = \Delta = 2^{43/16}3^{125/72} \approx 43.39 < \Omega$.

8. NFs far (?!) from automorphic forms. One can also seek lightly ramified number fields for groups like S_n which do not admit low degree linear representations. An example involving the exceptional group M_{12}:

Let $u = \sqrt{-11}$. A three-point cover defined over $\mathbb{Q}(u)$ with monodromy group M_{12} is

$$f(t,x) = -11^2 u \Big(1188ux^3 + 198ux^2 - 1346ux - 27u + 594x^4 - 7920x^2 - 1474x + 135\Big)^3 - 2^8 3^{13}(253u - 67)tx.$$

The coordinate x has been carefully chosen so that $f(t,x^2)$ has monodromy group the double cover $2.M_{12}$. Its dessin:



Specializing $f(t, x^2)\overline{f}(t, x^2) \in \mathbb{Q}[x]$ at carefully chosen t gives at least 394 different $2.M_{12}.2$ fields with discriminant $2^*3^*11^*$.

One of these specialization points is

$$\tau = \frac{9090072503}{10101630528} = \frac{2087^3}{2^6 3^{15} 11} = 1 - \frac{31805^2}{2^6 3^{15} 11}.$$

It yields an $2.M_{12}.2$ field with discriminant 11^{88} and $GRD = \Delta = 11^{219/110} \approx 118.39$.

Writing $e = 11$, a polynomial defining this field with just fifteen terms is

$$\begin{aligned}
&x^{48} + 2e^3x^{42} + 69e^5x^{36} \\
&+868e^7x^{30} - 4174e^7x^{26} + 11287e^9x^{24} \\
&-4174e^{10}x^{20} + 5340e^{12}x^{18} + 131481e^{12}x^{14} \\
&+17599e^{14}x^{12} + 530098e^{14}x^8 + 3910e^{16}x^6 \\
&+4355569e^{14}x^4 + 20870e^{16}x^2 + 729e^{18}.
\end{aligned}$$

Remark of Gross: perhaps this field can be related to an automorphic form via $M_{12} \subset E_7$.