Hypergeometric motives and their wild ramification

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Goal of group project: illustrate the theory of motives with a well-organized and broad collection of examples having completely computed L-functions with numerically checked functional equations.

Review of generalities with two examples:

1. **Motives in $M(\mathbb{Q},\mathbb{Q})$**
2. **Galois representations in $M(\mathbb{Q},\mathbb{F}_\ell)$**
3. **Wild ramification at p**

Explicitation for hypergeometric motives:

4. **HGMs in $M(\mathbb{Q},\mathbb{Q})$**
5. **Their reduction to $M(\mathbb{Q},\mathbb{F}_\ell)$**
6. **Their wild ramification at p**
7. **Examples**

1. Motives in $M(\mathbb{Q},\mathbb{Q})$. In the 1990s André modified Grothendieck's original 1960s definitions to get an unconditional and useful theory of pure motives. In particular,

- There is a reductive proalgebraic group $\mathbb{G}$, called the absolute motivic Galois group of $\mathbb{Q}$. It surjects to Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$.

- The category $M(\mathbb{Q},\mathbb{Q})$ of motives "over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$" is the category of representations of $\mathbb{G}$ on finite dimensional $\mathbb{Q}$-vector spaces, thus semisimple.

- For X a smooth projective variety over $\mathbb{Q}$, the cohomology groups $H^w(X(\mathbb{C}),\mathbb{Q})$ are objects in $M(\mathbb{Q},\mathbb{Q})$ and they generate the whole category.

- $\mathbb{C}^\times$ sits naturally in $\mathbb{G}(\mathbb{R})$. On a motive M, it gives a Hodge decomposition $M \otimes \mathbb{C} = \bigoplus M^{p,q}$ with $\mathbb{C}^\times$ acting by $z^p \overline{z}^q$ on $M^{p,q}$.

- For each prime ℓ, there is a section $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{G}(\mathbb{Q}_\ell)$.

To motivically understand a given X, one should

1: Express each cohomology group as a sum of irreducibles, $H^w(X(\mathbb{C}), \mathbb{Q}) = \bigoplus_i M_{w,i}$.

2: Study each appearing M individually, starting with computing its motivic Galois group $G_M := \mathrm{Image}(\mathbb{G}) \subseteq GL_M$.

In Step 2, the original variety X may fade into the background. For example, one may already have encountered M in the study of another variety.

Famous conjectures in arithmetic geometry can be studied for individual motives:

- (Hodge) $\rho_\infty : \mathbb{C}^\times \to G_M(\mathbb{R})$ has $\mathbb{Q}$-Zariski dense image in the identity component G_M^0.

- (Tate) ρ_ℓ : Gal$(\overline{\mathbb{Q}}/\mathbb{Q}) \to G_M(\mathbb{Q}_\ell)$ has open image for each ℓ.

- (Compatibility) L-functions $L(M,s)$ and conductors N defined via ℓ-adic cohomology are independent of ℓ. *[We tacitly assume compatibility to simplify statements]*

- (Automorphy) For $M \subseteq H^w(X(\mathbb{C}),\mathbb{Q})$, the L-function $L(M,s)$ is automorphic and hence satisfies a functional equation with respect to $s \mapsto w+1-s$.

The conjectures are known for many motives, sometimes easily, sometimes by deep theorems.

Examples: Let

$X_1 \quad : \quad y^2 = x(x-1)(x-9) \qquad$ (Elliptic Curve 24.a3),

$X_3 \quad : \quad y^2 = x_1x_2x_3(x_1+x_2)(x_2+x_3)(x_1+1)(x_3+1)$,

and take

$M_1 = H^1(X_1(\mathbb{C}), \mathbb{Q}) \qquad$ (so $\qquad (h^{1,0}, h^{0,1}) = (1,1)$),

$M_3 = H^3(X_3(\mathbb{C}), \mathbb{Q}) \qquad$ (so $(h^{3,0}, \ldots, h^{0,3}) = (1,0,0,1)$).

Put

$$L(M_1, s) \;=\; \sum_{n=1}^{\infty} \frac{a_n}{n^s} \;=\; 1 \cdot \frac{1}{1+3^{-s}} \prod_{p \geq 5} \frac{1}{1 - a_p p^{-s} + p^{1-s}},$$

$$L(M_3, s) \;=\; \sum_{n=1}^{\infty} \frac{b_n}{n^s} \;=\; 1 \cdot \prod_{p \geq 3} \frac{1}{1 - b_p p^{-s} + p^{3-s}}.$$

Then automorphy holds via

$$\sum_{n=1}^{\infty} a_n q^n \;=\; \eta_{12}\eta_6\eta_4\eta_2 \;\in\; S_2(\Gamma_0(24)),$$

$$\sum_{n=1}^{\infty} b_n q^n \;=\; \eta_4^4\eta_2^4 \qquad \in\; S_4(\Gamma_0(8)),$$

where $\eta_k = q^{k/24} \prod_{j=1}^{\infty}(1-q^{kj})$.

2. Galois representations in $M(\mathbb{Q},\mathbb{F}_\ell)$. Let $M(\mathbb{Q},\mathbb{F}_\ell)$ be the category of representations of Gal$(\overline{\mathbb{Q}}/\mathbb{Q})$ on finite-dimensional $\mathbb{F}_\ell$-vector spaces. A motive M in $M(\mathbb{Q},\mathbb{Q})$ determines a semisimple object M/ℓ in $M(\mathbb{Q},\mathbb{F}_\ell)$ up to isomorphism. We write $M \stackrel{\ell}{\equiv} M'$ for $M/\ell \cong M'/\ell$.

Examples. Here $a_p \stackrel{3}{\equiv} b_p$ for all primes and so $M_1 \stackrel{3}{\equiv} M_3$. Via $GL_2(\mathbb{F}_3) \subset S_8$ the common mod 3 Galois representation corresponds to

$$f(x) = x^8 - 6x^4 + 4x^2 - 3.$$

Some data illustrating the connections:

p	2	3	5	7	11	13	17	19	23	29
a_p	0	-1	-2	0	4	-2	2	-4	-8	6
b_p	0	-4	-2	24	-44	22	50	44	-56	198
$\overline{a}_p$	0	2	1	0	1	1	2	2	1	0
$\overline{p}$	2	0	2	1	2	1	2	1	2	2
λ_p			8	44	8	62	8	62	8	$2^3 1^2$

Here λ_p is the factorization partition of $f(x) \in \mathbb{F}_p[x]$. It is correlated with $(\overline{a}_p, \overline{p}) \in \mathbb{F}_3 \times \mathbb{F}_3^\times$:

$\overline{a}_p$	0	1	1	2	2	0	1	2
$\overline{p}$	1	1	1	1	1	2	2	2
λ_p	44	62	2^4	$3^2 1^2$	1^8	$2^3 1^2$	8	8

3. Wild ramification at p. Fix a decomposition group $D = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \subset \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with inertial subgroup I, wild inertia group P, and canonical filtration

$$D \overset{\widehat{\mathbb{Z}}}{\supset} I \overset{\widehat{\mathbb{Z}}^p}{\supset} P \supset \cdots \supset P^{\geq s} \overset{\mathbb{F}_p^\infty}{\supset} P^{>s} \supset \cdots$$

with s running over positive rationals. Let $\mathrm{Fr}_p \in D$ generate $D/I \cong \widehat{\mathbb{Z}}$. For M in $M(\mathbb{Q},\mathbb{Q})$ its local L-factor is

$$L_p(M,s) = \frac{1}{\det(1 - \mathrm{Fr}_p p^{-s}|M_\ell^I)}$$

The *tame exponent* of M is $\tau_p(M) = \dim(M_\ell/M_\ell^I)$.

One has a canonical decomposition into summands indexed by Swan slopes:

$$M_\ell = M_\ell^P \bigoplus \bigoplus_{s>0} M_\ell^s$$

Here $P^{\geq s}$ acts non-trivially and $P^{>s}$ acts trivially on M_ℓ^s. The *Swan exponent* of M is

$$s_p(M) = \sum_{s>0} \dim(M_\ell^s)s$$

The *exponent* of M is $c_p(M) = \tau_p(M) + s_p(M)$.

Write $M \overset{p}{\sim} M'$ if $M_\ell \cong M'_\ell$ as P-representations. Elementary group theory then says an equivalent condition is $M/\ell \cong M'/\ell$ as P-representations. Hence

$$\boxed{M \overset{\ell}{\equiv} M' \overset{\star}{\Longrightarrow} M \overset{p}{\sim} M'}$$

Only the conductor $N = \prod_p p^{c_p}$ appears in the functional equation for $L(M, s)$. However it is good to focus on the s_p part of c_p because of the stability ($\star$). **"Wild ramification is sometimes easier than tame ramification."**

Examples. The splitting field K of

$$x^8 - 6x^4 + 4x^2 - 3$$

has $\mathrm{Gal}(K/\mathbb{Q}) = GL_2(3)$. The quotient filtration is

$$\begin{array}{ccccccccc} D & \overset{2}{\supset} & I & \overset{3}{\supset} & P^{\geq 1/3} & \overset{2^2}{\supset} & P^* & \overset{2}{\supset} & P^{>1/2} \\ \| & & \| & & \| & & \| & & \| \\ GL_2(3) & \supset & SL_2(3) & \supset & Q_8 & \supset & C_2 & \supset & \{e\} \end{array}$$

with $P^* = P^{>1/3} = P^{\geq 1/2}$. For both M_1 and M_3, this forces $s_2 = 1/2 + 1/2 = 1$ and $\tau_2 = 2$ so that $c_2 = 3$.

4. HGMs in $M(\mathbb{Q},\mathbb{Q})$. Indices and matrices.
For

$$f(x) = x^d + c_1 x^{d-1} + \cdots + c_d,$$

let

$$C(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & -c_d \\ 1 & 0 & \cdots & 0 & 0 & -c_{d-1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & -c_3 \\ 0 & 0 & \cdots & 1 & 0 & -c_2 \\ 0 & 0 & \cdots & 0 & 1 & -c_1 \end{pmatrix}$$

be its companion matrix. Let

$$A = [a_1, a_2, \ldots] \text{ and } B = [b_1, b_2, \ldots]$$

be such that

$$f_\infty(x) = \prod_i \Phi_{a_i}(x) \text{ and } f_0(x) = \prod_j \Phi_{b_j}(x)$$

have the same degree d. Put

$$g_\infty = C(f_\infty) \text{ and } g_0 = C(f_0).$$

Assume for several slides that A and B are disjoint. In this case, $< g_\infty, g_0 >$ acts absolutely irreducibly on $\mathbb{Q}^d$.

Monodromy Representations. Define g_1 by $g_0 g_1 g_\infty = 1$. Let $T = \mathbb{P}^1 - \{0, 1, \infty\}$. View (g_0, g_1, g_∞) as giving a representation of the fundamental group

$$\pi_1(T(\mathbb{C}), 1/2) = \langle \gamma_0, \gamma_1, \gamma_\infty | \gamma_0 \gamma_1 \gamma_\infty = 1 \rangle.$$

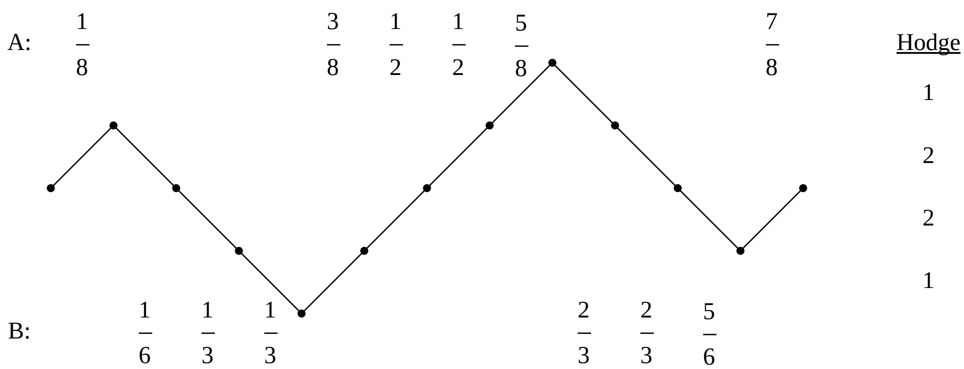
The representation corresponds to an absolutely irreducible local system $H(A, B, t)$ of $\mathbb{Q}$-vector spaces over $T(\mathbb{C})$. (The local system underlies classical hypergeometric functions, e.g.

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{5}\right)_n \left(\frac{2}{5}\right)_n \left(\frac{3}{5}\right)_n \left(\frac{4}{5}\right)_n}{n! n! \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n} t^n$$

for $A = [5]$ and $B = [1, 1, 6]$.)

Hypergeometric Motives. For $t \in T(\mathbb{Q}) = \mathbb{Q}^\times - \{1\}$, the vector space $H(A, B, t)$ is naturally a degree d motive in $M(\mathbb{Q}, \mathbb{Q})$. Also one naturally has a motive $H(A, B, 1)$ (which we won't mention again until §7).

Hodge numbers. Hodge numbers are determined by how the roots of $f_\infty(x)$ and $f_0(x)$ intertwine on the unit circle. For example, for $(A, B) = ([2, 2, 8], [3, 3, 6])$, the diagram



yields the Hodge vector

$$(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1, 2, 2, 1).$$

Both extremes are particularly interesting: complete intertwining yields

$$h^{0,0} = (d).$$

Complete separation yields

$$(h^{d-1,0}, \ldots, h^{0,d-1}) = (1, 1 \ldots, 1, 1).$$

Signatures. Action of Gal($\mathbb{C}/\mathbb{R}$) on $H(A, B, t)$ is known, completing the determination of the ∞-factor $L_\infty(H(A, B, t), s)$.

Monodromy groups. Hodge numbers are always normalized by requiring Hodge vectors of the form $(h^{w,0}, \ldots, h^{0,w})$ with $h^{w,0} > 0$. If $w = 0$, then monodromy groups $\langle g_\infty, g_0 \rangle$ are finite. If $w > 0$ and $\gcd(a_1, \ldots, b_1, \ldots) = 1$, then Zariski closures of monodromy groups are

$$\begin{array}{ll} \text{Symplectic } Sp_d, & \text{if } w \text{ is odd,} \\ \text{Orthogonal } O_d, & \text{if } w \text{ is even.} \end{array}$$

Similar but more complicated statements hold for mod ℓ monodromy groups and motivic Galois groups of specializations.

Types of primes. A prime p is called *very bad* for $(A, B, u/v)$ if it divides an index in A or B. It is called *slightly bad* if it is not very bad, but it divides $uv(u-v)$. It is called *good* otherwise.

Good primes are unramified in $H(A, B, t)$. Slightly bad primes are at most tamely ramified. Very bad primes are typically wildly ramified.

Frobenius traces. Frobenius traces and hence good factors $L_p(H(A,B,t),s)$ are given by an efficient formula. As a special case, for odd prime powers q define functions on $\mathbb{F}_q^\times$:

$$\begin{aligned} m_1(t,q) &= \left(\frac{1-t}{q}\right) \quad \text{(Legendre Symbol)}, \\ m_d(t,q) &= -\sum_{u \in \mathbb{F}_q^\times} m_{d-1}(\tfrac{t}{u},q) m_1(u,q). \end{aligned}$$

Then, for $t \in \mathbb{Q}^\times - \{1\}$ reducing to an element of $\mathbb{F}_p^\times$,

$$\text{Trace}\left(\text{Fr}_q | H([2^d],[1^d],t)\right) = m_d(t,q).$$

Modifications of the general formula work for slightly bad primes and for $t=1$.

Earlier examples.

$$\begin{aligned} M_1 &= H([2,2],[1,1],9), && \text{so } a_p = m_2(9,p), \\ M_3 &= H([2^4],[1^4],1), && \text{so } b_p = m_4(1,p) - p. \end{aligned}$$

Examples from trinomials. For positive integers b and β, put

$$\begin{aligned} a &= b+\beta, \\ g &= \gcd(b,\beta), \\ d &= a-g. \end{aligned}$$

Take

$$\begin{aligned} A &= \mathrm{Divs}(a) - \mathrm{Divs}(g) \\ B &= \mathrm{Divs}(b) + \mathrm{Divs}(\beta) - \mathrm{Divs}(g) \end{aligned}$$

Then $T(b,\beta) := H(A,B)$ is a motivic family with unique Hodge number $h^{0,0} = d$. It arises from trinomial covers of $\mathbb{P}^1$.

Example. $T(4,1) = H([5],[4,2,1])$. Indices really do intertwine:

[5]:		$\frac{1}{5}$		$\frac{2}{5}$		$\frac{3}{5}$		$\frac{4}{5}$
[4, 2, 1]:	$\frac{0}{1}$		$\frac{1}{4}$		$\frac{1}{2}$		$\frac{3}{4}$	

For $t \in \mathbb{Q}^\times - \{1\}$, trinomials enter via

$$\begin{aligned} X_t &= \mathrm{Spec}\left(\mathbb{Q}[x]/(x^5 - 5tx - 4t)\right), \\ H^0(X_t(\mathbb{C}),\mathbb{Q}) &= T(4,1,t) \oplus \mathbb{Q} \quad \text{ in } M(\mathbb{Q},\mathbb{Q}). \end{aligned}$$

More weight zero examples. Beukers and Heckman classified all finite monodromy examples, with Weyl groups figuring prominently:

$$\begin{aligned} W(E_6): &\quad BH45 - BH49, \\ W(E_7): &\quad BH58 - BH62, \\ W(E_8): &\quad BH63 - BH77. \end{aligned}$$

We have equations for almost all these covers.

Example. $BH45 = H([3,12],[1,2,8])$ has indices that really do intertwine:

[3, 12]:		$\frac{1}{12}$		$\frac{1}{3}$		$\frac{5}{12}$		$\frac{7}{12}$		$\frac{2}{3}$		$\frac{11}{12}$
[1, 2, 8]:	$\frac{0}{1}$		$\frac{1}{8}$		$\frac{3}{8}$		$\frac{1}{2}$		$\frac{5}{8}$		$\frac{7}{8}$	

Governing polynomial is

$$\begin{aligned} & f(t,x) = \\ & \quad t 2^4 x^3 \left(x^2-3\right)^{12} \\ & \quad -3^9 (x-2)(x-1)^8 \left(x^2-2x-1\right)^8 . \end{aligned}$$

Uniform normalization. For §5, an alternative normalization is needed, where $h(A,B)$ is the "Tate twist" of $H(A,B)$ which has weight 0 or 1.

Degenerate cases. Also for §5, It is convenient to define $h(A,B,t)$ also when there is overlap between A and B. Write

$$\begin{aligned} A &= A' + [c_1^{m_1}, \ldots, c_k^{m_k}], \\ B &= B' + [c_1^{m_1}, \ldots, c_k^{m_k}]. \end{aligned}$$

Then, by definition,

$$h(A,B,t) = \\ h(A',B',t) \bigoplus \bigoplus_{i=1}^{k} \bigoplus_{j=0}^{m_k-1} H^0_{\mathrm{prim}}(X_{c_i}(\mathbb{C}), \mathbb{Q}(j)),$$

where $X_{c_i} = \mathrm{Spec}(\mathbb{Q}[x]/(x^{c_i} - t))$.

We call $h(A',B')$ the *core* of $h(A,B)$.

5. Reduction of HGMs to $M(\mathbb{Q},\mathbb{F}_\ell)$. Let ℓ be a prime. If $c = u\ell^k$ with u coprime to ℓ then

$$\Phi_c(x) \stackrel{\ell}{\equiv} \Phi_u(x)^{\phi(\ell^k)}.$$

Thus the monodromy representation of $H(A, B, t)$ does not change modulo ℓ when one "kills ℓ" and thereby passes to the associated ℓ-free family $H(A^\ell, B^\ell, t)$. In the uniform normalization, Frobenius traces do not change modulo ℓ either and for $t \in \mathbb{Q}^\times$,

$$h(A, B, t) \stackrel{\ell}{\equiv} h(A^\ell, B^\ell, t).$$

as semisimple Galois representations.

Examples:

$$\begin{aligned} h([5],[1,1,6]) &\stackrel{2}{\equiv} h([5],[1,1,3]), \\ h([5],[1,1,6]) &\stackrel{3}{\equiv} h([5],[1,1,2,2]), \\ h([5],[1,1,6]) &\stackrel{5}{\equiv} h([1,1,1,1],[1,1,6]). \end{aligned}$$

The ℓ-free families on the right are often degenerate, making their analysis reduce to HGMs of lower degree.

The prime ℓ being disallowed in indices, there aren't so many mod ℓ families in low degrees and it is reasonable to tabulate them.

Mod 2 hypergeometric families in rank ≤ 7						
				Ram for §6		
Label	M	A	B	3	5	7
0	1	–	–			
$T(2,1)$	$O_2^-(2)$	3	11	$2a$		
$T(4,2)$	$O_4^+(2)$	33	1111	$4a$		
$T(3,2)$	$O_4^-(2)$	5	311	$2a$	$4a$	
•**T(4,1)**•	$O_4^-(2)$	5	1111		$4a$	
$T(5,1)$	$Sp_4(2)$	5	33	$4a$	$4a$	
$T(6,3)$	$S_3 \wr A_3$	9	3311	$6b$		
$T(5,2)$	S_7	7	511		$4a$	$6a$
$T(6,1)$	S_7	7	3311	$4a$		$6a$
$T(4,3)$	S_7	7	31111	$2a$		$6a$
$T(7,1)$	$O_6^+(2)$	7	111111			$6a$
$T(5,3)$	$O_6^+(2)$	53	111111	$2a$	$4a$	
•**6BH45**•	$O_6^-(2)$	333	111111	$6e$		
$6BH46$	$O_6^-(2)$	333	511	$6e$	$4a$	
$6BH47$	$O_6^-(2)$	9	111111	$6d$		
$6BH48$	$O_6^-(2)$	9	31111	$6c$		
$6BH49$	$O_6^-(2)$	9	511	$6d$	$4a$	
$7BH58$	$Sp_6(2)$	9	333	$6a$		
$7BH59$	$Sp_6(2)$	9	53	$6c$	$4a$	
$7BH60$	$Sp_6(2)$	9	7	$6d$		$6a$
$7BH61$	$Sp_6(2)$	7	333	$6e$		$6a$
$7BH62$	$Sp_6(2)$	7	53	$2a$	$4a$	$6a$

Mod 3 hypergeometric families in ranks ≤ 4					
				Ram for §6	
Label	M	A	B	2	5
1^3A	$O_1(3)$	2	1	$1A$	
$T(4,2)$	$O_2^-(3)$	4	21	$2A$	
$31,31,22$	$Sp_2(3)$	4	11	$2a$	
$31,31,31$	$Sp_2(3)$	22	11	$2b$	
$T(3,1)$	$O_3(3)$	42	111	$3A$	
$T(3,3)$	$O_3^+(3)$	222	111	$3B$	
$T(4,4)$	64	8	421	$4A$	
$\bullet\mathbf{T(4,1)}\bullet$	120	5	421	$3A$	$4a$
$T(3,2)$	120	5	2111	$1A$	$4a$
4^3D	384	8	2111	$4B$	
$4BH37$	576	44	2111	$4C$	
$T(5,1)$	$O_4(3)^-$	10	2111	$3B$	$4a$
4^3a	1152	8	44	$4a$	
4^3b	1152	8	2211	$4d$	
4^3c	1152	44	2211	$4f$	
$4BH24$	$Sp_4(3)$	2222	1111	$4h$	
$4BH25$	$Sp_4(3)$	422	1111	$4g$	
$4BH26$	$Sp_4(3)$	44	1111	$4e$	
$4BH27$	$Sp_4(3)$	8	411	$4b$	
$4BH28$	$Sp_4(3)$	8	1111	$4c$	
$4BH29$	$Sp_4(3)$	10	5	$4h$	
$4BH30$	$Sp_4(3)$	5	44	$4e$	$4a$
$4BH31$	$Sp_4(3)$	8	5	$4c$	$4a$
$4BH32$	$Sp_4(3)$	5	2211	$2b$	$4a$
$4BH33$	$Sp_4(3)$	10	411	$4g$	$4a$
$4BH34$	$Sp_4(3)$	5	411	$2a$	$4a$
$4BH35$	$Sp_4(3)$	5	1111		$4a$
$4BH36$	$Sp_4(3)$	10	1111	$4h$	$4a$

Mod 3 hypergeometric families in rank 5					
				Ram for §6	
Label	M	A	B	2	5
$T(5,5)$	$2^5.5$	$10,2$	51	$5F$	
$T(8,2)$	$2^4.S_5$	82	51	$5C$	$4a$
$T(6,4)$	$2^5.S_5$	$10,2$	4111	$5E$	$4a$
$5BH41$	$O_5(3)^*$	442	11111	$5D$	
$5BH42$	$O_5(3)^*$	442	51	$5D$	$4a$
$5BH43$	$O_5(3)^*$	$10,2$	11111	$5F$	$4a$
$5BH44$	$O_5(3)^*$	22222	11111	$5F$	
•6BH45•	$O_5(3)^+$	82	441	$5A$	
$6BH46$	$O_5(3)^+$	52	441	$4C$	$4a$
$6BH47$	$O_5(3)^+$	4222	11111	$5E$	
$6BH48$	$O_5(3)^+$	82	11111	$5C$	
$6BH49$	$O_5(3)^+$	52	11111	$1A$	$4a$
$N1$	$O_5(3)$	82	4111	$5B$	
$N2$	$O_5(3)$	52	81	$4B$	$4a$
$N3$	$O_5(3)$	52	101	$3B$	
$N4$	$O_5(3)$	52	4111	$2A$	$4a$

We have computed a corresponding cover for almost all of the Galois representations just listed, many having been already seen in characteristic zero.

6. Analysis of HGMs at p. Let p be a prime. One can kill all $\ell \neq p$ in turn to get from a given $H(A,B,t)$ to its associated p-primary $H(A_p,B_p,t)$. The original and new motives have the same wild p-adic ramification:

$$H(A,B,t) \stackrel{p}{\sim} H(A_p,B_p,t).$$

Example with no degree drop at each p:

			Type
$H([5],[12])$	$\stackrel{2}{\sim}$	$H([1,1,1,1],[4,4])$	$4e$
$H([5],[12])$	$\stackrel{3}{\sim}$	$H([1,1,1,1],[3,3])$	$4a$
$H([5],[12])$	$\stackrel{5}{\sim}$	$H([5],[1,1,1,1])$	$4a$

Example with full degree drop at each p:

			Type
$H([3,2,2],[6,1,1])$	$\stackrel{2}{\sim}$	$H([1,1,2,2],[1,1,2,2])$	0
$H([3,2,2],[6,1,1])$	$\stackrel{3}{\sim}$	$H([1,1,3],[1,1,3])$	0

Most examples have an intermediate behavior depending on p.

All examples in low degrees can be studied via explicitly computed covers:

Possibilities for 2-adic ramification in degrees ≤ 5										
L	A	B	1	2	3	4	5	6	7	Mod 3
0	–	–		1		7		26		
1A	2	1	1	1	2	4	10	21	46	
2A	4	21		1	1	3	5	12	24	
2a	4	11		2		10		50		
2b	22	11		3		8		46		
3A	42	111			2	4	8	20	42	
3B	222	111			3	4	7	16	39	
4A	8	421				1	1	3	5	$T(8,4)$
4B	8	2111				2	4	6	16	4^3D
4C	44	2111				4	8	12	32	$4BH37$
4a	8	44				2		4		4^3a
4b	8	411				2		10		$4BH27$
4c	8	1111				4		16		$4BH28$
4d	8	2211				3		8		4^3b
4e	44	1111				8		32		$4BH26$
4f	44	2211				6		16		4^3c
4g	422	1111				8		40		$4BH33$
4h	2222	1111				10		32		$4BH24$
5A	82	441					2	4	6	**•6BH45•**
5B	82	4111					2	4	8	$N1$
5C	82	11111					4	8	12	$T(8,2)$
5D	442	11111					8	16	24	$5BH41$
5E	4222	11111					8	16	32	$T(6,4)$
5F	22222	11111					10	16	26	$T(5,5)$
(Lower case in L: symplectic. Capital: orthogonal.)										

Possibilities for p-adic ramification in degrees ≤ 7

3-adic ramification										
L	A	B	1	2	3	4	5	6	7	Mod 2
0	–	–	1	4	4	30	25	135	102	
$2a$	3	11		4	4	32	28	216	164	$T(2,1)$
$4a$	33	1111				28	24	124	96	$T(4,2)$
$6a$	9	333						6	12	$7BH58$
$6b$	9	3311						16	16	$T(6,3)$
$6c$	9	31111						24	24	$6BH47$
$6d$	9	111111						30	24	$6BH48$
$6e$	333	111111						90	72	$6BH45$

5-adic ramification										
L	A	B	1	2	3	4	5	6	7	Mod 2
0	–	–	1	8	8	68	53	425	326	
$4a$	5	1111				22	24	216	184	$\bullet\mathbf{T(4,1)}\bullet$

7-adic ramification										
L	A	B	1	2	3	4	5	6	7	Mod 2
0	–	–	1	8	8	90	77	565	434	
$6a$	7	111111						76	76	$T(6,1)$

Possibilities for wild p-adic ramification in a given degree d decrease rapidly with p. E.g. in degree seven for $p = 2$, 3, 5, 7 there are 13, 8, 2, 2 possibilities for the p-core.

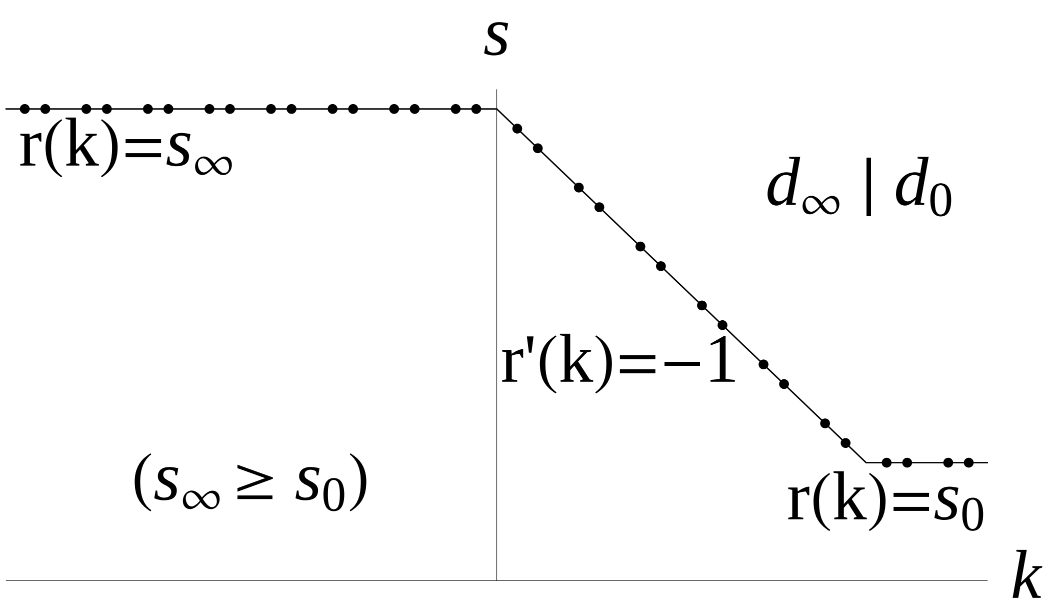
p-adic ramification as a function of t. For a family $H(A,B)$, define

$$\begin{array}{rclrcl} d_\infty & = & \sum_{p|a}\phi(a), & d_0 & = & \sum_{p|b}\phi(b), \\ s_\infty & = & \sum_{p|a}s(a), & s_0 & = & \sum_{p|b}s(b). \end{array}$$

where

$$s(a) = \phi(a)\left(\mathrm{ord}_p(a) + \frac{1}{p-1}\right).$$

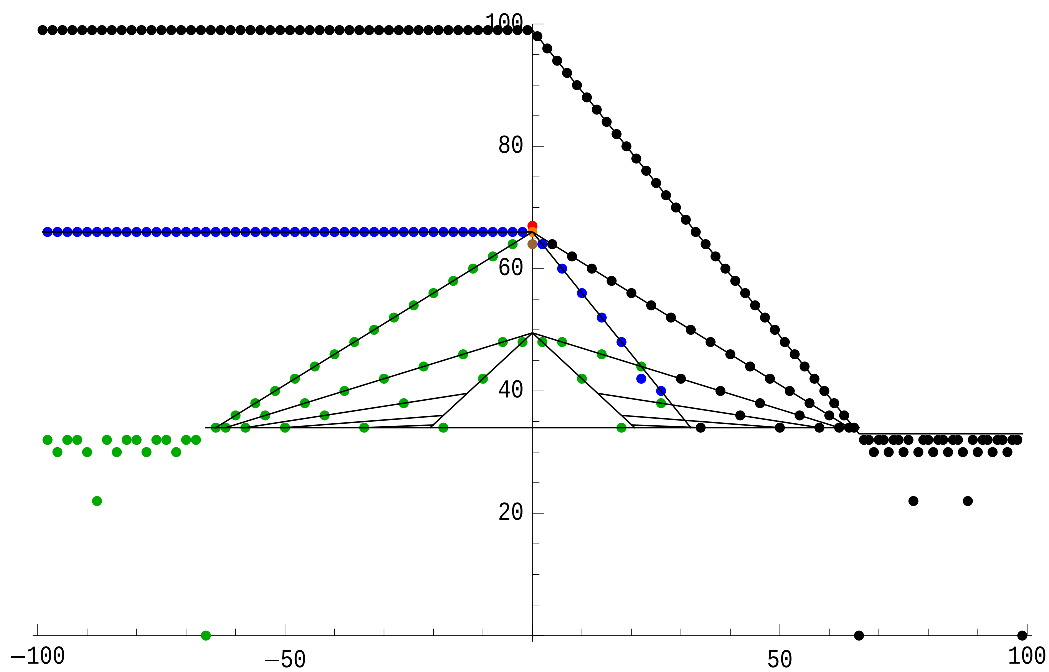
Define a "ramp function" $r(k)$ as indicated:



Conjecture (with FRV). *The Swan conductor of $H(A,B,up^k)$ is at most $r(k)$. If k is coprime to p then one has equality, there being exactly d_∞ or d_0 wild slopes as indicated.*

There are other general patterns, but computation suggests that a universal formula covering all cases would be complicated.

Example. for d odd, $H([2d],[d],t)$ can be analyzed via $2^{2d}x^d(x-1)^d + t = 0$. Then conclusions about 2-wild ramification can be transferred to other motives like $H([2^d],[1^d],t)$. The case $d = 33$, with c_2 as a function of $t = u2^k$:



A black dot is above k if u does not matter. Otherwise a green dot indicates $u \equiv 1$ (4) and a blue dot indicates $u \equiv 3$ (4).

7. Examples. For uniformity: all families are symplectic with Hodge vector $(1, 1, \ldots, 1, 1)$; all specializations have wild L-factors $L_p(M, s) = 1$. All L-functions are numerically checked via `CheckFunctionalEquation` to high precision.

Wild at 3. Some $H([3^{d/2}], [1^d], t)$, all with conductor $N = 2^a 3^b$ with $a \in \{0, 1\}$.

$d \setminus t$	$1/9$	$1/3$	$-1/3$	1	-1	3	-3	9
2	5	$\boxed{5}$	$\boxed{5}$		$\underline{\mathbf{3}}$	4	$\boxed{4}$	3
4	$\boxed{10}$	$\boxed{10}$	$\boxed{10}$	$\boxed{\underline{\mathbf{3}}}$	$\underline{\mathbf{6}}$	9	9	$\boxed{8}$
6	15	$\boxed{15}$	$\boxed{15}$	$\underline{\mathbf{5}}$	$\underline{\mathbf{7}}$	14	$\boxed{14}$	13
8				$\boxed{\boxed{\underline{\mathbf{9}}}}$	$\underline{\mathbf{12}}$			
10				$\boxed{\underline{\mathbf{12}}}$			(exponents b)	

The order of central vanishing is indicated by the number of boxes. The underlined bold entries are not covered by the ramp formula.

Example. The motive $M = H([3^4], [1^8], 1)$ has Hodge vector $(1, 1, 1, 0, 0, 1, 1, 1)$, Galois group CSp_6, conductor 3^9, and rank two with

$$L''(M, 4) \approx 6.49484010081002007804077 2$$

Wild at 2. Specializations of $H([2^d],[1^d],t)$ with conductor 2^a3^b with $b \in \{0,1\}$:

	$-\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	-1	-2	2	4	-8
2	6	**3**	6	6		**5**	5	5	**3**	3
4	12	**7**	12	12	**3**	**9**	11	11	**7**	9
6	18	**11**	18	18	**6**	**13**	17	17	**11**	15
8		**15**			**7**	**17**			**15**	
10					**12**					
12					**13**					

(exponents a)

Here $H([2^d],[1^d],t)$ and $H([2^d],[1^d],1/t)$ are twists of one another, forcing a drop in Galois group at $t=-1$ and a decomposition at $t=1$.

Example. $H([2^8],[1^8],1) = M_2 \oplus M_4$ with

$$\begin{aligned} \text{Hodge}(M_2) &= (1,0,0,0,0,1) \\ \text{Hodge}(M_4) &= (1,0,1,0,0,1,0,1) \\ \text{Conductor}(M_2) &= 2^2 \\ \text{Conductor}(M_4) &= 2^5 \end{aligned}$$

and M_2 corresponding to $\eta_2^{12} \in S_6(\Gamma_0(4))$.

Wild at 2 and 3. When several wild primes are involved, one often knows the L-function completely from congruences. However the range of degrees that can be analytically studied is smaller because conductors are larger.

Example. $M = H([3^3],[2^6],1)$ has Hodge vector $(1,1,0,0,1,1)$, motivic Galois group CSp_4, and conductor $2^6 3^5$. All initial good a_p are negative:

p	5	7	11	13	17	19	23
a_p	-6	-126	-477	-883	-426	-1898	-4692

From 8000 coefficients and two minutes of computation, it has numeric rank two with

$$L''(M,3) \approx 12.6191334778913437117846768$$

Longer run times and less precision make many more $L(M,s)$ in computational reach.

Some reports by other group members available online:

Henri Cohen. *L-functions of Hypergeometric Motives* (slides).

Fernando Rodriguez Villegas. *Hypergeometric Motives* (video).

Mark Watkins. *What I know about Hypergeometric Motives* (text).

Some key references:

Yves André. *Pour une théorie inconditionnelle de motifs.* Inst. Hautes Études Sci. Publ. Math. No. 83 (1996), 5–49.

Frits Beukers and Gert Heckman. *Monodromy for the hypergeometric function* ${}_nF_{n-1}$. Invent. Math. 95 (1989), no. 2, 325–354.

Alessio Corti and Vasily Golyshev. Hypergeometric equations and weighted projective spaces. Sci. China Math. 54 (2011), no. 8, 1577–1590.

Nicholas M. Katz. *Exponential sums and differential equations.* Annals of Mathematics Studies, 124.

A key software resource:

John Cannon, et al. *MAGMA.* Especially the Hypergeometric Motive package (Mark Watkins) and the L-function package (Tim Dokchitser).