Division polynomials with Galois group $\mathbf{SU_3(\mathbb{F}_3).2 = G_2(\mathbb{F}_2)}$

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General Inverse Galois Problem. Given a finite group G, find number fields with Galois group G, preferably of small discriminant.

Our case today. $G = SU_3(\mathbb{F}_3).2 = G_2(\mathbb{F}_2)$ of order $12096 = 2^6 \cdot 3^3 \cdot 7$. We'll produce two related two-parameter polynomials:

$$f_1(p,q;x) = x^{28} + \cdots \in \mathbb{Q}(p,q)[x],$$
$$f_2(a,b;x) = x^{28} + \cdots \in \mathbb{Q}(a,b)[x].$$

Connections with:

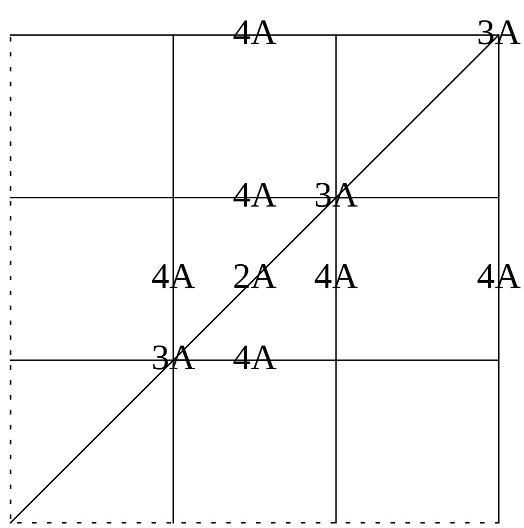
1. Rigid four-tuples in G
2. Motives with Galois group U_3, Sp_6, G_2
3. Three-point covers with Galois group G
4. Number fields with Galois group G

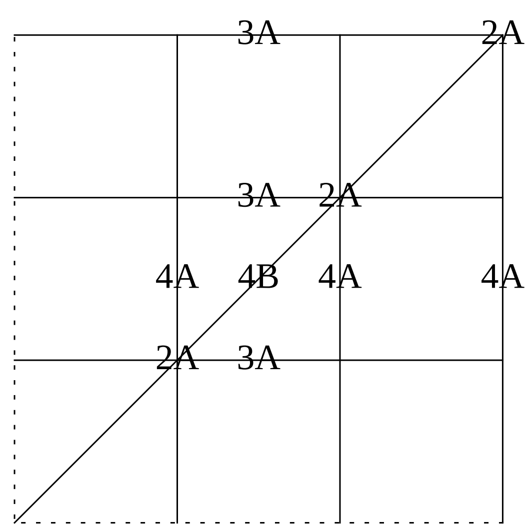
1. Rigid four-point covers. Mass formulas give five four-tuples of conjugacy classes in G' giving rigid four-point covers of $\mathbb{P}^1(\mathbb{C})$:

$$\begin{array}{ll} (4A, 4A, 4A, 2A), & (3A, 3A, 3A, 4B), \\ & (4A, 4A, 4A, 4B), \\ (4A, 4A, 3A, 3A), & (2A, 2A, 3A, 4A). \end{array}$$

All other quadruples are far from rigid.

Let $M_{0,5}$ be the moduli space of five labeled points in the projective line. The left two four-tuples give the same cover of $M_{0,5}$ and this cover has $S_3 \times S_2$ symmetry. The right three give a cover of $M_{0,5}$ having S_3 symmetry:

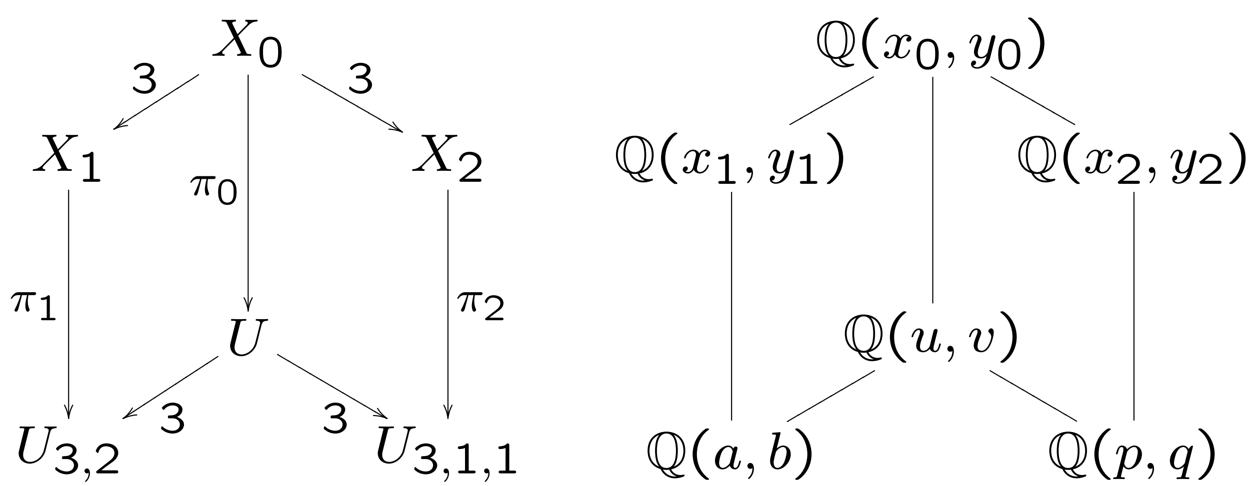




Our covers descend to covers of bases

$$U_{3,2} := M_{0,5}/(S_3 \times S_2),$$
$$U_{3,1,1} := M_{0,5}/S_3.$$

They are correlated by a cubic correspondence:



It is remarkable that the three fields upstairs are also rational.

We seek to algebraically describe π_1 and π_2 by polynomial relations

$$f_1(a, b, x_1) = x_1^{28} + \cdots = 0,$$
$$f_2(p, q, x_2) = x_2^{28} + \cdots = 0.$$

2A. Motives with Galois group U_3. Deligne and Mostow studied families of covers

$$y^d = f(u_1, \ldots, u_n; x)$$

of the x-line. Two of their first examples are

$$y^4 = \left(x^2 + 2x + 1 - 4u\right)^2 \left(x^2 - 2x + 1 - 4v\right) \quad \text{(genus 3)},$$

$$y^4 = (x-1)^3 x^2 \left(x^2 + ux - vx - x + v\right) \quad \text{(genus 4)}.$$

They prove that the Jacobian J_1 of the first is a factor of the Jacobian J_2 of the second.

The 3-torsion points of either cover correspond to our $\pi_0 : X_0 \to U$. There are natural descents to families of curves

$$\Pi_1 : C_1 \to U_{3,2}, \quad \Pi_2 : C_2 \to U_{3,1,1}.$$

On 3-torsion, these become our

$$\pi_1 : X_1 \to U_{3,2}, \quad \pi_2 : X_2 \to U_{3,1,1}.$$

We get explicit polynomials for the π_i via this connection; hundreds of terms in each case.

2B. Motives with Galois group Sp_6. Shioda studied the family of degree four plane curves

$$x^3+(y^3+cy+e)x+(ay^4+by^3+dy^2+fy+g)=0$$

in the x-y plane.

He obtained an explicit 1784-term polynomial with Galois group $Sp_6(\mathbb{F}_2)$ corresponding to their 2-torsion:

$$S(a,b,c,d,e,f,g;z)=z^{28}-8az^{27}+72bz^{25}+\cdots$$

This polynomial is universal for $Sp_6(\mathbb{F}_2)$ and so, via $G=G_2(\mathbb{F}_2)\subset Sp_6(\mathbb{F}_2)$, our polynomials must be specializations.

In fact, our π_0 is given via $w=u-v+1$ by

$$S(1,w,-3u,0,-uw,-uw,-u^2;z)=0.$$

Our π_1 and π_2 are given by much more complicated formulas.

2C. Motives with Galois group G_2. Define matrices a, b, c, and d:

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ -3 & 1 & & 1 & & & \\ 3 & -1 & & & 1 & & \\ 9 & -3 & & & & 1 & \\ -1 & & 3 & -1 & 2 & -1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & & & & 3 & -1 & \\ & 1 & & & 9 & -3 & \\ & & -2 & 1 & & & \\ & & -9 & 4 & & & \\ & & & & 1 & & \\ & & & & & 1 & \\ & & -3 & 1 & & & 1 \end{pmatrix}$$

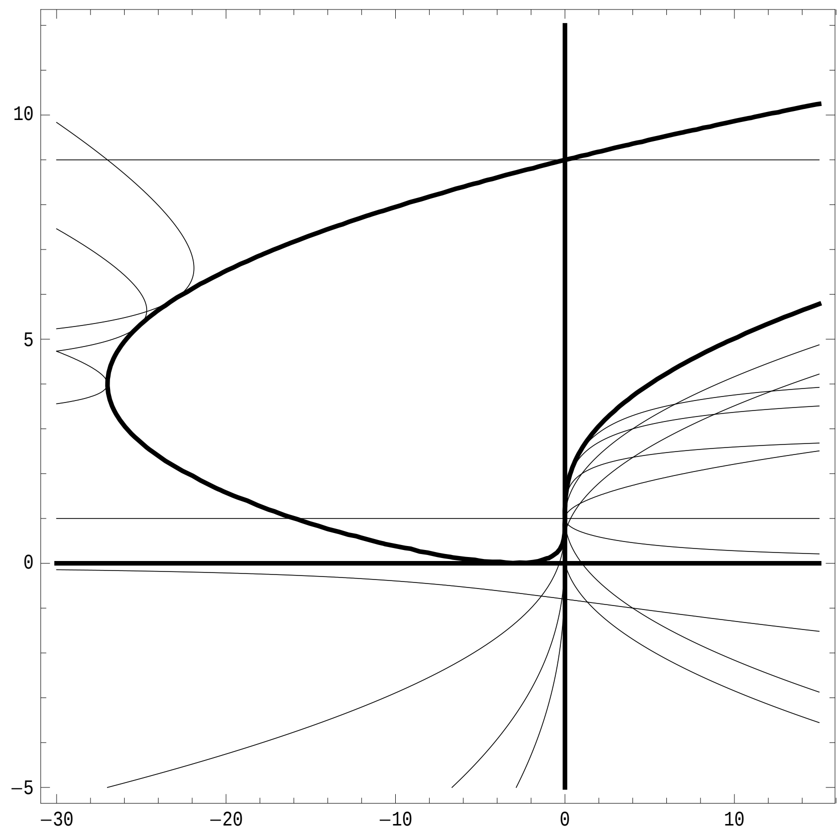
$$\begin{pmatrix} 1 & -1 & & & & & -3 \\ 3 & -2 & & & & & \\ & & 1 & -1 & & & 3 \\ & & 3 & -2 & & & 6 \\ & & & & 1 & -1 & -3 \\ & & & & 3 & -2 & \\ & & & & & & 1 \end{pmatrix} \quad \begin{pmatrix} 10 & -5 & & & 9 & -5 & -6 \\ 15 & -8 & & & 18 & -9 & -9 \\ & & 1 & & & & \\ -3 & 2 & -3 & 1 & -6 & 3 & 3 \\ 9 & -5 & & & 10 & -5 & -6 \\ 18 & -9 & & & 15 & -8 & -9 \\ -2 & 1 & & & -2 & 1 & 1 \end{pmatrix}$$

Then $abcd = 1$ and the Zariski-closure of the group $\langle a, b, c, d \rangle$ is the algebraic group G_2. This monodromy representation underlies a family of G_2 motives appearing in a classification of similar families by Dettweiler and Reiter.

In $GL_7(\mathbb{F}_2)$, the matrices generate $G_2(\mathbb{F}_2)'$ and (a, b, c, d) is in our rigid class $(2A, 2A, 3A, 4A)$. Hence $\pi_2 : X_2 \to U_{3,1,1}$ also functions as a division polynomial for a family of G_2 motives.

In all three cases, our explicit division polynomials aid in studying the source motives.

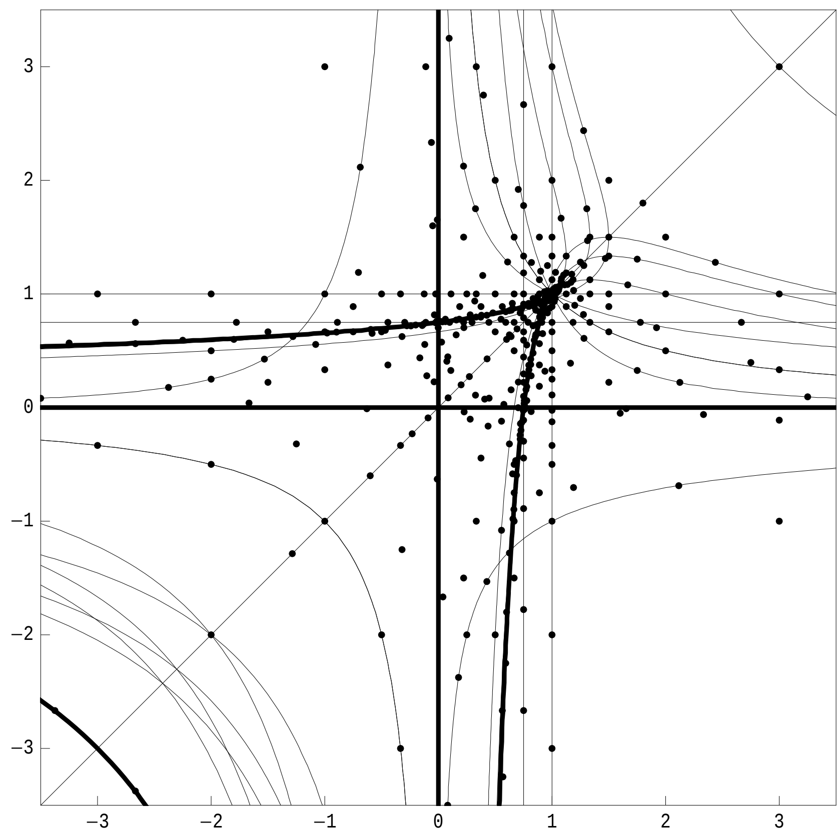
3. Specialization to three-point covers. A picture of $U_{3,2}(\mathbb{R})$ inside the a-b plane and its complementary discriminant locus (thick):



To review, the drawn space is the base of our degree twenty-eight cover $\pi_1 : X_1 \to U_{3,2}$.

Preimages of the thin curves are three-point covers, all of positive genus. It would be hard to construct these three-point covers directly.

4. Specialization to number fields. A similar picture of $U_{3,1,1}(\mathbb{R})$ inside the p-q plane, with some specialization points now added:



The points $(p_0, q_0) \in U_{3,1,1}(\mathbb{Q}) \subset \mathbb{Q}^2$ are chosen so that $K = \mathbb{Q}[x]/f_2(p_0, q_0; x)$ has discriminant of the form $2^a 3^b$. More than 300 such fields with Galois group $SU_3(\mathbb{F}_3).2 = G_2(\mathbb{F}_2)$ are obtained. It would be hard to construct these fields by purely number-theoretic methods.

A particular specialization:

The point $(p_0, q_0) = (1, 1/2)$ gives a number field with Galois group $SU_3(\mathbb{F}_3).2 = G_2(\mathbb{F}_2)$ and the very small field discriminant $2^{66}3^{46}$. A defining polynomial is

$$x^{28} - 4x^{27} + 18x^{26} - 60x^{25} + 165x^{24} - 420x^{23}$$
$$+798x^{22} - 1440x^{21} + 2040x^{20} - 2292x^{19}$$
$$+2478x^{18} - 756x^{17} - 657x^{16} + 1464x^{15}$$
$$-4920x^{14} + 3072x^{13} - 1068x^{12} + 3768x^{11}$$
$$+1752x^{10} - 4680x^{9} - 1116x^{8} + 672x^{7} + 1800x^{6}$$
$$-240x^{5} - 216x^{4} - 192x^{3} + 24x^{2} + 32x + 4.$$

Close 2- and 3-adic analysis says that the root discriminant of the Galois closure is

$$2^{43/16}3^{125/72} \approx 43.39$$

For comparison, extensive searches have been done on the smaller group S_7 and the larger group S_8, with smallest known Galois root discriminants being 40.49 and 43.99, respectively.

A paper corresponding to the talk is in preparation.

References for the three parts of §2:

A. Pierre Deligne and George Daniel Mostow. *Commensurabilities among lattices in PU(1,n).* Annals of Mathematics Studies, 132. Princeton University Press, 1993. viii+183 pp.

B. Tetsuji Shioda. *Plane quartics and Mordell-Weil lattices of type E7.* Comment. Math. Univ. St. Paul. 42 (1993), no. 1, 61–79.

C. Michael Dettweiler and Stefan Reiter. *The classification of orthogonally rigid G2-local systems.* Arxiv: 1103.5878v2. To appear in Trans. of the AMS. (Relevant family is P5.1 in §6.4. Matrices from e-mail from Reiter)