Numerical verification of Deligne's conjecture relating L-values and periods for hypergeometric motives

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1. The familiar case of elliptic curves: $L(X,1)/\Omega_{X,+} \in \mathbb{Q}$

2. Deligne's conjecture: $L(M,n)/\Omega_{M,n} \in \mathbb{Q}$ for n critical

3. Hypergeometric motives: $H(\alpha,\beta,t)$

4. Hypergeometric L-values: calculation of $L(H(\alpha,\beta,t),n)$

5. Hypergeometric periods: calculation of $\Omega_{H(\alpha,\beta,t),n}$

6. Numerical verifications: examples going beyond $(h^{0,1},h^{1,0}) = (1,1)$ from elliptic curves to Hodge vectors $(h^{0,w},\ldots,h^{w,0}) = (1,1,1,1)$, $(1,1,1,1,1)$, $(1,1,0,1,1)$ and $(1,1,0,0,1,1)$.

1. Elliptic curves. Let X be an elliptic curve defined by $y^2 = x(x-1)(x-t)$ with $t \in \mathbb{Q}_{>1}$. Associated are two rational vector spaces, each with an extra structure

$$H_1(X(\mathbb{C}),\mathbb{Q}) = H_1(X(\mathbb{C}),\mathbb{Q})^+ \oplus H_1(X(\mathbb{C}),\mathbb{Q})^-,$$
$$H^1_{DR}(X) \supset F^1H^1_{DR}(X).$$

Here complex conjugation acts on $H_1(X(\mathbb{C}),\mathbb{Q})^\epsilon$ with sign ϵ and $F^1H^1_{DR}(X)$ is the subspace represented by everywhere regular differentials.

Choose, as below, the standard bases

$$\sigma_1 \in H_1(X(\mathbb{C}),\mathbb{Q})^+ \text{ and } \sigma_2 \in H_1(X(\mathbb{C}),\mathbb{Q})^-.$$

Let $\omega_1 = \frac{x\,dx}{2y}$ and $\omega_2 = \frac{dx}{2y}$ so that $\{\omega_1, \omega_2\}$ is a basis for $H^1_{DR}(X)$ with ω_2 lying in $F^1H^1_{DR}(X)$. The corresponding period matrix $\left(\int_{\sigma_i} \omega_j\right)$ is

$$P = \begin{pmatrix} \int_0^1 \frac{x\,dx}{\sqrt{x(x-1)(x-t)}} & \int_0^1 \frac{dx}{\sqrt{x(x-1)(x-t)}} \\ \int_1^t \frac{x\,dx}{\sqrt{x(x-1)(x-t)}} & \int_1^t \frac{dx}{\sqrt{x(x-1)(x-t)}} \end{pmatrix}$$

The Legendre relation says $\det(P) = -2\pi i$.

The colored entries $\Omega_{X,+}$ and $\Omega_{X,-}$ are the real and imaginary periods respectively. A proved part of the Birch and Swinnerton-Dyer conjecture is that

$$\frac{L(X,1)}{\Omega_{X,+}} \text{ is rational.}$$

This statement is also a special case of Deligne's conjecture.

Suppose $L(X,s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ and X has conductor N. Then, as a simple case of general analytic continuation techniques,

$$L(X,1) = 2\sum_{n=1}^{\infty} \frac{a_n}{n} e^{-2\pi/\sqrt{N}}.$$

For $t = 3$, the conductor is $N = 96$ and the ratio is

$$\frac{L(X,1)}{\Omega_{X,+}} \approx \frac{1.00107738}{2.00215476} = 0.50000000 = \frac{1}{2}.$$

For the twist $X_D : Dy^2 = x(x-1)(x-t)$, the ratio $L(X_D,1)/(\sqrt{D}\Omega_{X,\text{sign}(D)})$ is rational. So, in a sense, $\Omega_{X,+}$ and $\Omega_{X,-}$ are equally involved.

2A. Period matrices. A motive $M \subseteq H^w(X, \mathbb{Q})$ has two associated rational vector spaces,

$$\check{M}_B \subseteq H_w(X(\mathbb{C}), \mathbb{Q}) \quad \text{and} \quad M_{DR} \subseteq H^w_{DR}(X).$$

These spaces have extra structures, as before:

$$\check{M}_B = \check{M}_B^+ \oplus \check{M}_B^-,$$
$$M_{DR} = F^0 \overset{h^{0,w}}{\supseteq} F^1 \overset{h^{1,w-1}}{\supseteq} \cdots \overset{h^{w-1,1}}{\supseteq} F^w \overset{h^{w,0}}{\supseteq} \{0\}.$$

Integration of forms over cycles again gives a non-degenerate pairing:

$$\check{M}_B \times M_{DR} \to \mathbb{C} : (\sigma, \omega) \mapsto \int_\sigma \omega.$$

Choosing bases $\{\sigma_i\}$ and $\{\omega_j\}$ respecting the structures, one gets a block period matrix P, e.g.

		F^0		F^1	F^2	F^3	F^4	
		ω_1	ω_2		ω_3		ω_4	ω_5
In $\check{M}^+$:	σ_1	$P_{1,1}$	$P_{1,2}$		$P_{1,3}$		$P_{1,4}$	$P_{1,5}$
	σ_2	$P_{2,1}$	$P_{2,2}$		$P_{2,3}$		$P_{2,4}$	$P_{2,5}$
	σ_3	$P_{3,1}$	$P_{3,2}$		$P_{3,3}$		$P_{3,4}$	$P_{3,5}$
In $\check{M}^-$	σ_4	$P_{4,1}$	$P_{4,2}$		$P_{4,3}$		$P_{4,4}$	$P_{4,5}$
	σ_5	$P_{5,1}$	$P_{5,2}$		$P_{5,3}$		$P_{5,4}$	$P_{5,5}$

A pair (p, ϵ) *is called* **critical** *for* M *if* $\dim(F^p) = \dim(\check{M}^\epsilon)$*.* In our example, the critical pairs are $(1,+)$, $(2,+)$, $(3,-)$, and $(4,-)$. *Ongoing notations:* P, P^+,P^- *of size* $d = d_+ + d_-$.

2B. Critical pairs in terms of Hodge vectors. Assuming at least two positive Hodge numbers, there are four situations which can give rise to critical pairs (n, ϵ):

Types with example	Hodge vector $(h^{0,w}, \ldots, h^{w,0})$	
$1, 3, \underline{5}, 7, \ldots$	$(\ldots, *, \bullet; 0; 0; 0; 0; \bullet, *, \ldots)$	
$0, 2, \underline{4}, 6, \ldots$	$(\ldots, *, \bullet; 0; 0; 0; \bullet, *, \ldots)$	
$0^+, 2^+, \underline{4}^+, \ldots$	$(\ldots, *, \bullet, 0, \bullet, 0, \bullet, *, \ldots)$	$h_-^{\frac{w}{2},\frac{w}{2}} = 0$
$0^-, 2^-, \underline{4}^-, \ldots$	$(\ldots, *, \bullet, 0, \bullet, 0, \bullet, *, \ldots)$	$h_+^{\frac{w}{2},\frac{w}{2}} = 0$

Here $\bullet$ indicates a positive Hodge number and $*$ indicates an arbitrary Hodge number. The punctuation marks are indexed by integers n. *The n^{th} mark has red if $(n, +)$ is critical and orange if $(n, -)$ is critical.* Familiar examples:

Example	Type	$(h^{0,w}, \ldots, h^{w,0})$
$h^1(\text{elliptic curve}, \mathbb{Q})$	1	$(1 ; 1)$
$h^2(\text{K3 surface}, \mathbb{Q})$	0	$(1, 20, 1)$
$h^3(\text{big 3-fold}, \mathbb{Q})$	1	$(*, \bullet ; \bullet, *)$
Previous page	4^+	$(2,0,1,0,2)$
Ramanujan motive	11	$(1;0;0;0;0;0;0;0;0;0;0;1)$

2C. Notation. Let $c_\epsilon = \det(P_\epsilon)/\det(P)$. For n an integer and D a square-free integer, let

$$\epsilon(n, D) = (-1)^{n-1}\text{sign}(D).$$

2D. Deligne's conjecture (with twisting incorporated). Let M be a weight w motive. Let n and D be as above with $(n, \epsilon(n, D))$ critical for M. Let M_D be the twist of M by the quadratic character χ_D. Then

$$\frac{L(M_D, n)}{c_{\epsilon(n,D)}(\sqrt{D}(2\pi i)^n)^{d_{-\epsilon(n,D)}}} \in \mathbb{Q}.$$

Note 1: One expects that always

$$\det(P) = (2\pi i)^{wd/2}\sqrt{\delta}$$

for some rational number δ. With this assumption, the conjecture for $L(M_D, n)$ is true if and only if it is true for $L(M_D, w + 1 - n)$.

Note 2: *Deligne's conjecture applies to M itself at odd integers with red and even integers with orange.*

3. Hypergeometric motives. Let $\alpha_1, \ldots, \alpha_d$ and $\beta_1, \ldots, \beta_d$ be in $\mathbb{Q}/\mathbb{Z}$ with always $\alpha_i \neq \beta_j$. Let $t \in \mathbb{Q} - \{0, 1\}$. Suppose the multisets

$$\alpha = \{\alpha_1, \ldots, \alpha_d\} \text{ and } \beta = \{\beta_i, \ldots, \beta_d\}$$

are each stable under multiplication by $\hat{\mathbb{Z}}^\times$. Then there is a corresponding degree d motive

$$H(\alpha_1, \ldots, \alpha_d; \beta_1, \ldots, \beta_d; t) \in M(\mathbb{Q}, \mathbb{Q}).$$

The Hodge numbers depend on how the α_i and the β_j intertwine on circle $\mathbb{R}/\mathbb{Z}$. The two extremes are

$$\begin{aligned} \vec{h} &= (d), && \text{(Complete intertwining)}, \\ \vec{h} &= (1, 1, \ldots, 1, 1), && \text{(Complete separation)}. \end{aligned}$$

In general, each α_i and β_j has an associated Hodge filtration $p \in \{0, \ldots, w\}$. Also, in the case w even, there are formulas giving the decomposition $h^{w/2,w/2} = h_+^{w/2,w/2} + h_-^{w/2,w/2}$.

For this talk, we don't need the $H(\alpha, \beta, t)$ themselves. All we need is procedures to pass from (α, β, t) to L-values and structured period matrices.

4A. Hypergeometric L-functions. In

$$L(H(\alpha, \beta, t), s) = \prod_p \frac{1}{f_p(p^{-s})} \quad \text{and} \quad N = \prod_p p^{c_p},$$

it is essential to distinguish three types of primes:

- Primes dividing the denominator of an α_i or β_j are called **wild** because they are typically wildly ramified.
- Non-wild primes dividing Num(t), Num($t-1$), or Denom(t) are called **tame** because they are at most tamely ramified.
- The remaining primes are **unramified**.

There are general formulas for L-factors and conductors at tame and unramified primes.

Magma has implemented these formulas and makes educated guesses at L-factors and conductors at wild primes p. As time goes on, the contexts where we expect our guesses to be right increases!

4B. Example. $M = H(0,0,0,0;\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-1)$ has Hodge vector $(1,1;1,1)$. Some more local invariants:

p	Type	c_p	$f_p(x)$				
2	Tame	1	$1+$	$x+$	$2 \cdot 3x^2+$	2^4x^3	
3	Unram	0	$1+$	$5x+$	$5 \cdot 3^2x^2+$	$5 \cdot 3^3x^3+$	3^6x^4
5	Wild	*5*	*1*				
7	Unram	0	$1+$	$25x+$	$7 \cdot 50x^2+$	$25 \cdot 7^3x^3+$	7^6x^4

Put, following the standard definitions,

$$\Lambda(M,s) \;=\; 6250^{s/2}\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s-1)L(M,s).$$

One should have the functional equation

$$\Lambda(M,4-s) = \pm\Lambda(M,s)$$

Magma does everything:

```
H := HypergeometricData(
              [0,0,0,0],[1/5,2/5,3/5,4/5]);
L := LSeries(H,-1);
WARNING: Guessing wild prime information
CheckFunctionalEquation(L);
0.000000000000000000000000000000
Evaluate(L,2)
0.417801574320826941827293917960
```

5A. Hypergeometric period matrices. We will work with period matrices $P(t)$ of $H(\alpha, \beta, t)$ which deviate slightly from the previous conventions to exploit particular features of the hypergeometric situation.

Assume first that the β_j are distinct and $t \in (-1, 0)$. For $\{i, c\} \in \{1, \ldots, w\}$, define

$$F_{i,c}(\alpha, \beta, t) = (\epsilon t)^{1-\beta_i} \sum_{k=0}^{\infty} \frac{(\alpha_1 - \beta_i + k)! \cdots (\alpha_n - \beta_i + k)!}{(\beta_1 - \beta_i + k)! \cdots (\beta_n - \beta_i + k)!} t^k.$$

Here $\epsilon = (-1)^{d-1}$ and lifts from $\mathbb{Q}/\mathbb{Z}$ to $\mathbb{Q}$ are chosen so that $\beta_c \in [0, 1)$ and all the α_i's and β_j's are in $(\beta_c - 1, \beta_c]$.

Then $P(t)$ has entries

$$P_{r,c}(t) \;=\; \pi^{\frac{w-1}{2}} \sum_{i=1}^{n} \frac{e^{-2\pi i r \beta_i}}{\prod_{\ell \neq i} \sin(\pi(\beta_i - \beta_\ell))} F_{i,c}(t).$$

For general t, one analytically continues, getting similar formulas. Mellin-Barnes integral representations make the $P_{r,c}(t)$ arise directly, without assuming that the β_j are distinct.

5B. Example. $M = H(0,0,0,0;\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-1)$ has period matrix $P(-1) \approx$

In F^0	In F^1	In F^2	In F^3
$0.44 + 0.35i$	$-0.61 + 1.3i$	$-7.76 - 1.27i$	$-15.72 - 171.89i$
$-0.02 - 0.08i$	$0.09 - 0.43i$	$2.49 - 3.55i$	$125.39 - 75.23i$
$-0.02 + 0.08i$	$0.09 + 0.43i$	$2.49 + 3.55i$	$125.39 + 75.23i$
$0.44 - 0.35i$	$-0.61 - 1.3i$	$-7.76 + 1.27i$	$-15.72 + 171.89i$

5C. Structures on the period matrix. Complex conjugation on Betti cohomology corresponds to reversing the rows.

Each column belongs to F^p where p is the Hodge filtration associated to β_c.

The example illustrates how one picks out matrices P_+ and P_- in general. Here $\det(P) = 16\pi^6/25$ and

$$c_+ \approx -1.51797066368284571002 13,$$
$$c_- \approx -0.83772334921031011851 47.$$

There are many more structures in hypergeometric period matrices $P(t)$!

6. Numerical verifications. *Type 1 example.* Our example $M = H(0,0,0,0;\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5};-1)$ has Hodge vector $(1,1,1,1)$ and hence Type 1. The Deligne ratio is

$$\frac{L(M,2)}{c_-(2\pi i)^4} \approx \frac{0.41780157432082694182729391796}{1305.62991975258419321029349363} \approx 0.00032000000000000000000000000 = 1/5^5.$$

While M has conductor $2\cdot 5^5 = 6250$, the twist M_{-1} has the much larger conductor $2^8 5^5 = 800000$. Computations are still feasible on the L-value side and we use the other submatrix on the period side:

$$\frac{L(M_{-1},2)}{c_+(2\pi i)^4} \approx \frac{2.36582628105003864200091200382}{-2365.82628105003864200091200382} \approx -0.00100000000000000000000000000 = -1/1000.$$

Example with M of type 2^+. Let

$$M = H(0,0,0,0,0;\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8},-1)$$

with period matrix

$-0.1-1.0i$	$1.9-0.2i$	$1.6+6.4i$	$-35.6+22.3i$	$102.1-4716.6i$
$0.0+0.0i$	-0.3	$-1.7-1.6i$	$-5.2-22.7i$	$3284.9-3130.5i$
0.0	0.2	1.8	19.7	4474.3
$0.0+0.0i$	-0.3	$-1.7+1.6i$	$-5.2+22.7i$	$3284.9+3130.5i$
$-0.1+1.0i$	$1.9+0.2i$	$1.6-6.4i$	$-35.6-22.3i$	$102.1+4716.6i$

Here M has $\vec{h}=(1,1,1,1,1)$ and hence no critical points. So we work instead with M_{-1} where again the conductor increases:

$$\begin{aligned} \mathrm{Cond}(M) &= 2^{17}, & \mathrm{Cond}(M_{-1}) &= 2^{19}, \\ f_2(M,x) &= 1-4x, & f_2(M_{-1},x) &= 1. \end{aligned}$$

Deligne's conjecture is again numerically verified:

$$\frac{L(M_{-1},3)}{c_-(i(2\pi i)^3)^3} = \frac{1.8212393432853}{5092554.3083328} \approx \frac{3}{2^{23}}.$$

Example with M of type 2^-. The family

$$H(0,0,0,0,0;\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8};t)$$

has $\vec{h} = (1,1,1,1,1)$ for $t \in (-\infty,0) \cup (1,\infty)$, as on the previous page. But for $t \in (0,1)$, $\vec{h} = (1,1,1,1,1)$ and no twisting is required to have an opportunity to test Deligne's conjecture. We take $t = 1/2$.

Since $t > 0$, the structures on P are different and we need to extract P_+ and P_- slightly differently:

$-0.1-31.2i$	$0.0+13.2i$	$2.2-12.9i$	$22.1+52.6i$	$-1714.3-4021.0i$
$0.1+0.0i$	$0.0+0.4i$	$-2.2+1.4i$	$-22.1-8.2i$	$1714.3-3868.9i$
$0.0+0.0i$	$0.0-0.2i$	$0.5-1.4i$	$10.3-13.3i$	$3832.6-1534.2i$
$0.0+0.0i$	$0.0+0.2i$	$0.5+1.4i$	$10.3+13.3i$	$3832.6+1534.2i$
$0.1+0.0i$	$0.0-0.4i$	$-2.2-1.4i$	$-22.1+8.2i$	$1714.3+3868.9i$

The numerical verification is

$$\frac{L(M,2)}{c_-(2\pi i)^4} = \frac{20.52960471086}{-525.557880598} \approx \frac{-5}{2^7}.$$

Example with M of type 2. Hypergeometric motives at $t = 1$ make sense after modification. The Hodge numbers are the same except $h^{w/2,w/2}$ drops by 1 if w is even and the two middle Hodge numbers drop by 1 if w is odd.

Let $M = H(0,0,0,0,0; \frac{1}{8}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{7}{8}; 1)$. Then both M and M_{-1} have Hodge vector $(1,1;0;1,1)$. Invariants at 2 are $(2^{11}, 1)$ and $(2^7, 1-16x^2)$ respectively. A period matrix for M is obtained from the general formulas by crossing off a row and a column:

x	x	x	x	x
x	$-0.1+0.6i$	$-3.0+1.5i$	$-26.6-11.8i$	$1887.1-4211.9i$
x	$0.0-0.2i$	$0.7-1.8i$	$12.9-16.0i$	$4171.1-1662.5i$
x	$0.0+0.2i$	$0.7+1.8i$	$12.9+16.0i$	$4171.1+1662.5i$
x	$-0.1-0.6i$	$-3.0-1.5i$	$-26.6+11.8i$	$1887.1+4211.9i$

Deligne's conjecture applies to both real and imaginary twists of M. Two independent verifications:

$$\frac{L(M,2)}{c_-((2\pi i)^2)^2} \approx \frac{2.71501421952698}{521.28273014918} \approx \frac{1}{2^6 3},$$

$$\frac{L(M_{-1},2)}{c_+(i(2\pi i)^2)^2} \approx \frac{0.4799512414113}{1474.4102136156} \approx \frac{1}{2^{10} 3}.$$

Example with M of type 3. Consider

$$M_D = H(0,0,0,0,0,0;\frac{1}{4},\frac{1}{4},\frac{1}{2},\frac{1}{2},\frac{3}{4},\frac{3}{4};1)_D$$

for $D \in \{1,2,-1,-2\}$. The Hodge vector is $(1,1;0;0;1,1)$. Conductors, wild L-factors, and signs of functional equations are

D	1	2	-1	-2
$\mathrm{cond}(M_D)$	2^5	2^{12}	2^9	2^{12}
$f_2(M_D,x)$	$1+4x+32x^2$	1	1	1
$\mathrm{sign}(M_D)$	1	1	-1	-1

Deligne's conjecture can be investigated without periods as certain ratios are predicted to be rational. Some numerical verifications:

$$\frac{L(M_1,3)}{L(M_8,3)} \approx \frac{0.50211308435460702833}{2.0084523374184281133} \approx \frac{1}{4},$$

$$\frac{L(M_1,4)}{L(M_8,4)} \approx \frac{0.74305199726313190079}{1.0216964962368063734} \approx \frac{8}{11},$$

$$\frac{L(M_{-1},4)}{L(M_{-8},4)} \approx \frac{0.82594291786513031711}{1.0324286473314128965} \approx \frac{4}{5}.$$

Some Principal References:

Deligne's Conjecture: Pierre Deligne. Valeurs de fonctions L et périodes d'intégrales. 1979. (Deligne's c^{ϵ} is our $c_{-\epsilon}$.)

Hodge numbers for hypergeometric motives: Alessio Corti and Vasily Golyshev. Hypergeometric equations and weighted projective spaces. 2011.

Good Euler factors for Hypergeometric L-functions: Nicholas Katz. Exponential Sums and Differential Equations. 1990. (Fast p-adic implementation due to Henri Cohen)

Bad Euler factors and conductors of Hypergeometric L-functions: Ongoing Work with Fernando Rodriguez Villegas and Mark Watkins.

Analytic Computations with L-functions: Tim Dokchitser. Computing special values of motivic L-functions. 2004. (Fast implementation in *Magma*)

Hypergeometric period matrices: Frits Beukers. Notes on differential equations and hypergeometric functions. 2009.

Vasily Golyshev and Anton Mellit. Gamma structures and Gauss's contiguity. 2014.

Hossein Movasati. Modular-type functions attached to mirror quintic Calabi-Yau varieties. 2012.