MIXED DEGREE NUMBER FIELD COMPUTATIONS

JOHN W. JONES AND DAVID P. ROBERTS

Abstract. We present a method for computing complete lists of number fields in cases where the Galois group, as an abstract group, appears as a Galois group in smaller degree. We apply this method to find the twenty-five octic fields with Galois group $\mathrm{PSL}_2(7)$ and smallest absolute discriminant. We carry out a number of related computations, including determining the octic field with Galois group $2^3{:}\,\mathrm{GL}_3(2)$ of smallest absolute discriminant.

1. Introduction

1.1. **Overview.** Number theorists have computed number fields with minimal absolute discriminants for each of the thirty possible Galois groups in degrees at most 7. In degrees 8 and 9, the minimal fields are known for the seventy-five solvable Galois groups. All these minimal fields are available, together with references to sources, on the Klüners-Malle database [KM01]. The minimal fields are also available, typically as first elements on long complete lists, at our database [JR14a].

The Klüners-Malle paper [KM01] also gives smallest known absolute discriminants for the five nonsolvable octic groups and the four nonsolvable nonic groups. Despite the fifteen years that have passed since its publication, rigorous minima have not been established for these nine groups. In this paper, we address two of the nine cases, proving that the absolute discriminants $3^8 7^8$ and 5717^2 presented in [KM01] for the octic groups $\mathrm{PSL}_2(7)$ and $2^3{:}\,\mathrm{GL}_3(2)$ are indeed minimal. These cases are related through the exceptional isomorphism $\mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$.

One element of our approach for finding the $\mathrm{PSL}_2(7)$ minimum was suggested already in [KM01]: any octic $\mathrm{PSL}_2(7)$ field K_8 has the same Galois closure as two septic $\mathrm{GL}_3(2)$ fields K_{7a} and K_{7b}. As the discriminants satisfy $D_{7a} = D_{7b} \mid D_8$, one can in principle establish minimality of the octic discriminant 21^8 by conducting a search of all septic fields with absolute discriminant $\leq 21^8$. We combine this with the method of targeted Hunter searches, which requires us to analyze, on a prime-by-prime basis, how discriminants either stay the same or increase when one passes from septic to octic fields. This targeting based on discriminants makes the computation feasible. We add several smaller refinements to make the computation run even faster.

Our title refers to the general method of carrying out a carefully targeted search in one degree to obtain a complete list of fields in a larger degree. Section 2 gives background and then Section 3 describes the general method, using our case where the two degrees are 7 and 8 as an illustration. Section 4 presents our minimality result for $\mathrm{PSL}_2(7)$, improved in Theorem 1 to the complete list of twenty-five octic

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$\mathrm{PSL}_2(7)$ fields with discriminant $\leq 30^8$. This section also presents corollaries giving minimal absolute discriminants for certain related groups in degrees 16, 24, and 32.

Section 5 gives a second illustration of the mixed degree method, now with degrees 5 and 6. Here we use the exceptional isomorphisms $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$ and Theorem 2 considerably extends the known list of sextic $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$ fields. We also explain in this section potential connections with asymptotic mass formulas and Artin representations.

Our final section returns to groups related to the septic group $\mathrm{GL}_3(2)$. Theorem 3 finds all alternating septics with discriminant $\leq 12^7$. The long runtime of this search makes clear the importance of targeting for Theorem 1. However just the bound 12^7 is sufficient for our last corollary, which confirms minimality of the $2^3{:}\,\mathrm{GL}_3(2)$ field with discriminant 5717^2.

1.2. **Notation and conventions.**

We denote the cyclic group of order n by C_n. We use $N{:}H$ to denote a semi-direct product with normal subgroup N and complement H.

A number field is a finite extension of $\mathbb{Q}$, which we consider up to isomorphism. If $K/\mathbb{Q}$ is such an extension with degree n, then its normal closure, K^g, is Galois over $\mathbb{Q}$. Moreover, $\mathrm{Gal}(K^g/\mathbb{Q})$ comes with a natural embedding into S_n, which is well-defined up to conjugation. We denote the image of such an embedding, which is a transitive subgroup of S_n, by simply $\mathrm{Gal}(K)$.

Each possibility for $\mathrm{Gal}(K)$ has a standard notation nTj introduced in [CHM98]. In §4.2 we use the classification of nearly 3 million transitive subgroups of S_{32} which was completed more recently in [CH08].

When several non-isomorphic fields have the same splitting field, we refer to them as siblings. For example, fields K_{7a}, K_{7b}, and K_8 as in the overview are siblings. When choosing notations for groups, we choose the name which reflects the group's natural transitive action. For example, $\mathrm{GL}_3(2) = 7T5$ and $\mathrm{PSL}_2(7) = 8T37$ act on the projective spaces $\mathbb{P}^2(\mathbb{F}_2)$ and $\mathbb{P}^1(\mathbb{F}_7)$ of orders 7 and 8 respectively. The third group mentioned in the overview is the group of affine transformations of $\mathbb{F}_2^3$. Our notation emphasizes its semidirect product structure: $2^3{:}\,\mathrm{GL}_3(2) = 8T48$.

It is often enlightening to shift the focus from an absolute discriminant $|D|$ of a degree n number field K to the corresponding root discriminant $\mathrm{rd}(K) = \delta = |D|^{1/n}$. We generally try to indicate both, as in the numbers 21^8 and 30^8 of the overview.

2. Background

Our method of mixed degree targeted Hunter searches is built on well-established methods of searching for number fields, which we now briefly explain.

2.1. **Hunter searches.**

A Hunter search is a standard technique for computing all primitive number fields of a given degree with absolute discriminant less than a given bound [Coh00]. Here a number field K is primitive if it has exactly two subfields, itself and $\mathbb{Q}$. When the degree n is prime, as in our cases $n = 7$ and $n = 5$ here, the primitivity assumption is vacuous.

The only two inputs for a standard Hunter search are the degree and the discriminant bound. Some implementations optimize for a particular signature which can then be thought of as a third input, but here we search all signatures simultaneously. The computation itself is an exhaustive search for polynomials with integer coefficients bounded by various inequalities.

2.2. **Targeted Hunter searches.** Targeting, introduced in [JR99], and refined in [JR03], allows one to search for fields with particular large discriminants. One carries out a Hunter search, but only for fields which match a given combination of local targets. The targets, described below, determine both the discriminant and the local behavior of the field at ramifying primes p. This latter information forces a defining polynomial to satisfy congruences modulo several prime powers, and these congruences greatly reduce the number of polynomials one needs to inspect.

To describe the targets more precisely, let p be a prime number and let K be a degree n number field. Then $K \otimes \mathbb{Q}_p \cong \prod_{i=1}^{g} K_{p,i}$ where each $K_{p,i}$ is a finite extension of $\mathbb{Q}_p$. At its most refined level, a local target may be a single p-adic algebra, up to isomorphism. In a few situations, we do work at this level. However typically, one wants to treat natural collections of p-adic algebras as a single target.

Let $\mathbb{Q}_p^{\mathrm{unr}}$ be the unramified closure of $\mathbb{Q}_p$. Then, similar to the decomposition above we have

$$K \otimes \mathbb{Q}_p^{\mathrm{unr}} \cong \prod_{j=1}^{t} L_{p,j}, \tag{1}$$

where each $L_{p,j}$ is a finite extension of $\mathbb{Q}_p^{\mathrm{unr}}$. Let e_j be the ramification index of the field $L_{p,j}$, and $(p)^{c_j}$ its discriminant ideal. We note that the e_j give the sizes of the orbits of the p-inertia subgroup acting on the roots of an irreducible defining polynomial for K. A typical local target is then a pair $((e_1, \ldots, e_t), \sum_{j=1}^{t} c_j)$ with the list of e_j weakly decreasing. The ramification indices $(e_1, \ldots, e_t)$ give a partition of $[K : \mathbb{Q}]$, and the discriminant of the local algebra, which equals the p-part of the discriminant of K, is $(p)^{\sum_j c_j}$.

A local target at p determines a list of congruences modulo some fixed power of p. For tamely ramified primes we work modulo p, while we use higher powers for wildly ramified primes. When there is more than one ramifying prime, the lists of congruences are simply combined via the Chinese remainder theorem.

3. Mixed degree targeted Hunter searches

In a mixed degree targeted search, one has a Galois group G and transitive permutation representations of two different degrees $n < m$. Each degree n field K_n with Galois group $G \hookrightarrow S_n$ determines a degree m field K_m with Galois group $G \hookrightarrow S_m$. Targets are triples $((e_1, \ldots, e_t), c_n, c_m)$ where c_n is the local discriminant exponent for the small degree fields searched, while c_m is the local discriminant exponent of the larger degree fields actually sought. One uses the values p^{c_m} to decide which combinations of targets to search, and $((e_1, \ldots, e_t), c_n)$ to carry out the actual search in degree n. We describe how one deals with the two degrees here, often by using our first case with $n = 7$ and $m = 8$ as an example. Once we have the degree n polynomials in hand, we compute the corresponding degree m polynomials as resolvents using *Magma* [BCP97].

3.1. **Tame ramification.** The behavior of tame ramification under degree changes is straightforward. Let K be a degree n number field, G its Galois group, and p a tamely ramified prime. The inertia subgroup I for a prime above p is cyclic; let σ

be a generator. Via the given inclusion $G \subseteq S_n$, we let $e_1, e_2, \ldots, e_t$ be the cycle type of σ. These match the e_j of the local target described above. The exponent of p in the discriminant of K is then given by

$$c_n = \sum_{j=1}^{t}(e_j - 1) = n - t. \tag{2}$$

When one is considering also a second degree m, one just runs through the above procedure a second time. In our first case, $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, each row on Table 1 represents a candidate for σ. The row then gives the corresponding pair of partitions (λ_7, λ_8) and pair of discriminant exponents (c_7, c_8). These discriminant exponents are computed from the partitions via formula (2).

TABLE 1. Cycle types and discriminant exponents for $\mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$ in degrees 7 and 8.

λ_7	λ_8	c_7	c_8
7	71	6	6
421	44	4	6
331	3311	4	4
22111	2222	2	4

Note that if a prime p is tamely ramified in our pair of fields (K_7, K_8), then its minimal contribution to the discriminant of the octic is p^4. Thus, when searching for octic fields with absolute discriminant $\leq B$, we need only consider primes $p \leq \sqrt[4]{B}$. Our largest search used $B = 30^8$, so $p \leq 900$.

The relation $D_7 \mid D_8$ mentioned in the introduction is due to the fact that we always have $c_7 \leq c_8$, and that this inequality also holds for wildly ramified primes. In two of the tame cases, one has equality, but in the other two tame cases one has strict inequality. Our method using targeted searches makes use of the strictness of these latter inequalities.

3.2. **Wild ramification.** An explicit description of the behavior of wild p-adic ramification under degree changes becomes rapidly more complicated as $\mathrm{ord}_p(G)$ increases. We describe just our case $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$ here, as this case represents the basic nature of the general case well.

Since $|G| = 2^3{\cdot}3{\cdot}7$, the only primes which can be wildly ramified in a G extension are 2, 3, and 7. For a subgroup of G to be an inertia group for a wildly ramified prime p, it must be an extension of a cyclic group of order prime to p by a non-trivial p-group. The candidates for a $G = \mathrm{PSL}_2(7)$ extension are given in Table 2. They run over all of the non-trivial proper subgroups of $\mathrm{PSL}_2(7)$ up to conjugation, with the exception of two conjugacy classes of subgroups isomorphic to S_4.

Each subgroup in the table is a candidate for being the inertia group for a wild prime for only one prime. The horizontal lines separate the subgroups according to this prime. The second column gives the isomorphism type of the candidate for inertia, and the third column gives corresponding candidates for the decomposition group. Over other 2-adic ground fields, $A_4 = I = D$ is possible, but not over $\mathbb{Q}_2$ since there is no ramified C_3 extension of $\mathbb{Q}_2$.

TABLE 2. Wild ramification data for $\mathrm{PSL}_2(7)$.

p	I	D	λ_7	λ_8	(c_7, c_8)
7	$C_7{:}C_3$	$C_7{:}C_3$	7	71	$(8,8), (10,10)$
	C_7	$C_7, C_7{:}C_3$	7	71	$(12,12)$
3	S_3	S_3	331	62	$(6,8), (10,12)$
	C_3	C_3, S_3	331	3311	$(8,8)$
2	A_4	S_4	43 61	44	$(6,8), (10,16)$
	D_4	D_4	421	8	$(12,22), (14,24)$
	C_4	C_4, D_4	421	44	$(14,22)$
	V	V, D_4, A_4	2221 4111	44	$(6,12), (8,16)$
	C_2	C_2, V, C_4	22111	2222	$(4,8), (6,12)$

The columns labeled λ_7 and λ_8 show the orbit sizes of the actions of I in the degree seven and eight representations respectively. There are two possibilities for A_4 and V in degree 7, so we give both. The orbit sizes are helpful in determining the data c_7 and c_8 in each case, and λ_7 is the partition of 7 needed for carrying out the targeted Hunter search.

In most cases, it is clear from Galois theory how to interpret the orbit sizes. For example, inside a Galois A_4 field, there are unique subfields of degrees 3 and 4 up to isomorphism. So the 4 in the first A_4 entry is for the usual quartic representation, and the 3 is its resolvent cubic. More detailed computations with the groups allow us to resolve the two ambiguities, which are as follows.

- A Galois D_4 field has three quadratic subfields and three quartic subfields (up to isomorphism). In the degree 7 partition 421, the 4 represents a quartic stem field, say defined by a polynomial f, and then the 2 represents the field obtained from a root of $x^2 - \mathrm{Disc}(f)$.
- A V field has three quadratic subfields. In the line for V, the 2221 represents the product of these quadratic fields and $\mathbb{Q}_2$.

The last column gives a list of candidate pairs (c_7, c_8), coming by analyzing the corresponding local extensions. Some cases can be done using just Galois theory and general properties of extensions of local fields. A simple approach, however, which applies to all cases is to make use of the complete lists of the relevant local fields [JR06, LMF15]. For example, suppose 2 is wildly ramified with inertia subgroup isomorphic to A_4. Then, the decomposition group is S_4 and there are three possibilities for a Galois S_4 field with A_4 as its inertia subgroup. It is then a matter of checking the discriminants of the relevant fields to complete the A_4 line.

The list of targets for each prime is fairly straightforward to read off Table 2. For example, for $p = 3$ we have only $((3,3,1),6,8)$, $((3,3,1),8,8)$, and $((3,3),10,12)$. With $p = 2$, one has $((4,2,1),14,24)$ from $I = D_4$ and $((4,2,1),14,22)$ from $I = C_4$; however, only the latter gets used since it has the same partition and c_7 and a smaller value of c_8.

3.3. **Further savings.** Various techniques can reduce the number of polynomials that it is necessary to search. Again we illustrate these reductions by our first case with $G \cong \mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$. The first and fourth reductions below simply eliminate some local targets $((e_1, \ldots, e_t), c_7, c_8)$ from consideration. The second and third let us reduce the size of some of the local targets.

3.3.1. *Savings from evenness at half the tame primes.* Our first savings comes from $\mathrm{GL}_3(2)$ being an even subgroup of S_7, i.e., from the inclusion $\mathrm{GL}_3(2) \subset A_7$. To obtain this savings, we make use of the following general lemma.

Lemma 1. *Suppose n and p are distinct primes, K is a degree n number field whose Galois group G is contained in A_n, and p is totally ramified in K. Then p is a quadratic residue modulo n.*

Proof. Let D be the decomposition group for a prime above p. Tame Galois groups over $\mathbb{Q}_p$ are 2-generated by σ and τ where $\sigma\tau\sigma^{-1} = \tau^p$ (see [Iwa55]). Here, τ is a generator of the inertia subgroup and σ is a lift of Frobenius. Thus D isomorphic to $(\mathbb{Z}/n){:}(\mathbb{Z}/f)$ where f is the order σ, and the action of σ on $\mathbb{Z}/n$ is multiplication by p. The Galois group locally is a subgroup of A_n which forces σ to be an even permutation, which in turn implies that p is a square modulo n. □

In our situation, the lemma says that if $p \neq 7$ is totally ramified, then p must be congruent to 1, 2, or 4 modulo 7, eliminating "totally ramified" as a target for approximately half of the primes.

3.3.2. *Savings from evenness at $p = 7$.* We can achieve a savings from the fact that $\mathrm{GL}_3(2) \subset A_7$ at $p = 7$ as well. It is evident from the complete lists of degree 7 extensions of $\mathbb{Q}_7$ in [JR06, LMF15] that having discriminant $(7)^c$ with c even is not sufficient to ensure that the Galois group is even. In fact, for each even value of c, only half of the fields, counted by mass, have even Galois group. We computed lists of congruences for each 7-adic septic field with even Galois group and then merged the lists of congruences.

To target a specific 7-adic field, we start with a defining polynomial such that a power basis formed from a root α will generate the ring of integers over $\mathbb{Z}_7$. We then consider a generic element $\beta = a_0 + a_1\alpha + \cdots + a_6\alpha^6$ and compute its characteristic polynomial in $\mathbb{Z}_7[a_0, \ldots, a_6][x]$. Working modulo 7^2 we then enumerate all possibilities for the polynomial.

3.3.3. *Savings from* $\mathrm{ord}_3(G) = 1$ *at $p = 3$.* Cases when 3 is wildly ramified also offer an opportunity to reduce the search time by more refined targeting. As can be seen from the two relevant lines of Table 2, the decomposition subgroup is isomorphic to C_3 or S_3. In either case, the orbit partition for the decomposition group is $(3, 3, 1)$. Thus, a defining polynomial factors as the product of two cubics times a linear polynomial over $\mathbb{Z}_3$.

The savings comes from the fact that the two cubics have to define the same 3-adic field. So, the procedure here starts with computing possible polynomials for each ramified cubic extension of $\mathbb{Q}_3$ modulo some 3^r. We take all products of the form $(x+a)g_1g_2$ where the g_i come from the list for a given field and a runs through all possibilities in $\mathbb{Z}/3^t$. In our actual search, we worked modulo 3^2.

The resulting local targets are considerably smaller. For example, a target $((3, 3, 1), 8)$ from our general method includes cases where the two cubic factors define non-isomorphic fields with discriminant ideal $(3)^4$ and also cases where the cubics have discriminant ideals $(3)^3$ and $(3)^5$. All these possibilities are not searched in our refinement.

3.3.4. *Exploiting arithmetic equivalence at* $p = 2$. The final refinement we use exploits the fact that for each octic field sought, we need to find just one of its two siblings in degree 7. These pairs of septic fields are examples of arithmetically equivalent fields. The two fields K_{7a} and K_{7b} have the same Dedekind zeta function, the same discriminant, and the same ramification partition at all odd primes. However, at $p = 2$ one can have $\lambda_{7a} \neq \lambda_{7b}$.

In Table 2, there are two orbit partitions for the inertia group A_4, and again two orbit partitions for V. For each of these cases, if a septic $\mathrm{GL}_3(2)$ field has inertia subgroup I and one orbit partition, its sibling has the other orbit partition for I. We save by targeting 43 and 4111, but not their transforms 61 and 2221.

3.3.5. *Savings from global root numbers being* 1. As we mentioned in §2.1, our code does not distinguish signatures. If it did, there would be an opportunity for yet further savings as follows. A separable algebra K_v over $\mathbb{Q}_v$ has a local root number $\epsilon(K_v) \in \{1, i, -1, -i\}$. For $v = \infty$, one has $\epsilon(\mathbb{R}^r\mathbb{C}^s) = (-i)^s$. For v a prime p, one has $\epsilon(K_p) = 1$ unless the inertia group I_p has even order. Further information about local root numbers is at [JR06, §3.3], with many root numbers calculated on the associated database. The savings comes from the reciprocity relation $\prod_v \epsilon(K_v) = 1$, so that the signature is restricted by the behavior at ramifying primes.

While we are not using local root numbers in our searches, we are using them in our interpretation of the output of our first case. Interesting facts here include the general formulas $\epsilon(K_{7a,v}) = \epsilon(K_{7b,v})$ and $\epsilon(K_{8,v}) = 1$. Also, from Tables 1 and 2, one has equality of discriminant exponents $c_7 = c_8$ at a prime p if and only if $|I_p|$ is odd; so in this case the septic sign $\epsilon(K_{7a,p}) = \epsilon(K_{7b,p})$ is 1.

4. Results for $\mathrm{PSL}_2(7)$ and related groups

4.1. **A complete list of $\mathrm{PSL}_2(7)$ octics.** Our search for $\mathrm{PSL}_2(7)$ fields with $\mathrm{rd}(K) \leq 21$ took 41 CPU-hours and confirmed that the discriminant 21^8 given in [KM01] is indeed the smallest. The extended search through $\mathrm{rd}(K) \leq 30$ took approximately four CPU-months. In this extended search, we combined targets for a given prime in a subsearch whenever the contribution to the octic field discriminant is the same. In this sense, the computation consisted of 1471 subsearches of varying difficulty. The fastest 380 cases took at most 10 seconds each, the median length case took 6.5 minutes, and the slowest ten cases took from 20 to 35 hours each. The slowest cases all involved searches where $c_7 = c_8$ for every ramifying prime. This larger search found twenty-five fields.

Theorem 1. *There are exactly* 25 *octic fields with Galois group* $\mathrm{PSL}_2(7)$ *and discriminant* $\leq 30^8$. *The smallest discriminant of such a field is* 21^8 *and the field is given by*

$$x^8 - 4x^7 + 7x^6 - 7x^5 + 7x^4 - 7x^3 + 7x^2 + 5x + 1.$$

The full list of fields is available in a computer-readable format by searching the website [JR14a].

Figure 1 gives a visualization of how complete lists of $\mathrm{PSL}_2(7)$ octics and complete lists of $\mathrm{GL}_3(2)$ septics have little to do with one another. The 25 $\mathrm{PSL}_2(7)$ octics of Theorem 1 give the 24 points beneath the $\delta_8 = 30$ line, the drop of one coming from the fact that the point $(\delta_7, \delta_8) \approx (24.88, 28.00)$ comes from two fields. Theorem 3 likewise says that there are only 23 points to the left of the $\delta_7 = 12$ line. The rectangle where both $\delta_7 \leq 12$ and $\delta_8 \leq 30$ contains just a single point.

Another way of making clear how the complete lists differ sharply is to consider first fields. The first octic field, highlighted in Theorem 1, is a sibling of the famous Trinks field $\mathbb{Q}[x]/(x^7-7x+3)$ [Tri68]. As noted earlier, septic $\mathrm{GL}_3(2)$ fields come in sibling pairs with the same discriminant. In the list of septic $\mathrm{GL}_3(2)$ fields ordered by root discriminant at [JR14a], the Trinks field is currently in the 1009th pair, with root discriminant about 23.70. Conversely, the octic sibling of the first septic $\mathrm{GL}_3(2)$ pair currently ranks 66th in the corresponding list of octic fields at [JR14a].

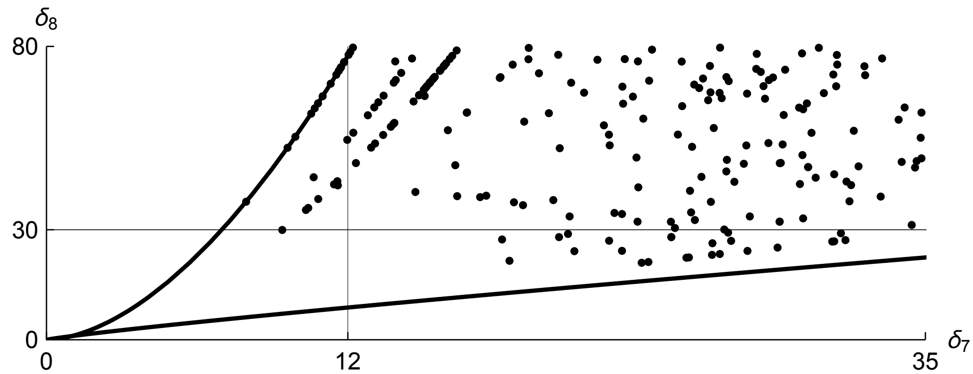

FIGURE 1. Root discriminant pairs (δ_7, δ_8) associated to $\mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, including all pairs with $\delta_7 \leq 12$ from Theorem 3 and all pairs with $\delta_8 \leq 30$ from Theorem 1.

Figure 1 also gives some sense of the geography of sibling triples (K_{7a}, K_{7b}, K_8). The upper bound corresponds to the extreme case $D_8 = D_7^2$, graphed as $\delta_8 = \delta_7^{7/4}$. The lower bound likewise corresponds to the extreme case $D_8 = D_7$, graphed as $\delta_8 = \delta_7^{7/8}$.

As one example of the details visible in this geography, note that the figure shows no point on the lower bound. This is because a necessary and sufficient condition to be on the lower bound is that all inertia groups have odd size, by Tables 1 and 2. As mentioned in §3.3.5, this condition forces the septic p-adic local signs to all be 1. Reciprocity then forces the infinite local sign to be 1 as well, which forces the fields to be totally real. We expect that the first instance of such a totally real sibling triple is well outside the window of Figure 1. There is just one such sibling triple currently on the Klüners-Malle database [KM01], with $D_7 = D_8 = 2^4 13^4 131^4$, and thus $(\delta_7, \delta_8) = (104.33, 58.36)$. On the other hand, the figure shows twelve points seemingly on a curve just above the lower bound. These points are the ones with $D_8/D_7 = 9$, and thus indeed lie on the curve $\delta_8 = 3^{1/4}\delta_7^{7/8}$. They come from the twelve fields having the prime 3 in bold on Table 3.

4.2. **Class groups and class fields.** The class groups of all twenty-five fields in Theorem 1 can be computed unconditionally by either *Magma* [BCP97] or *Pari* [PAR15]. They are all cyclic and Table 3 gives their orders. The fact that all but three of these class groups are non-trivial is already remarkable. By way of contrast, the first 620 S_5 quintic fields ordered by absolute discriminant all have trivial class group.

Using Theorem 1, one can get complete lists of fields with root discriminant ≤ 30 for many other groups via class field theory. We restrict ourselves to three corollaries, chosen because they interact interestingly with the class groups on Table 3. The Galois groups involved in these corollaries are all even, and so absolute discriminants coincide with discriminants.

$\mathrm{rd}(K_8)$	$\mathrm{rd}(K_7)$	D_8	D_7	h	$=$	a	ℓ	c	e
21.00	23.70	$\mathbf{3}^8 7^8$	$\mathbf{3}^6 7^8$	4	$=$	2	2		
21.21	23.97	$2^6 \mathbf{3}^4 53^4$	$2^6 \mathbf{3}^2 53^4$	4	$=$	2	2		
21.54	18.44	$2^{16} 29^4$	$\mathbf{2}^{10} 29^4$	1					
22.37	25.48	$2^6 \mathbf{3}^4 59^4$	$2^6 \mathbf{3}^2 59^4$	4	$=$	2	2		
22.45	25.58	$\mathbf{3}^6 97^4$	$\mathbf{3}^4 97^4$	2	$=$	2			
23.16	26.50	$2^6 \mathbf{3}^6 11^6$	$2^6 \mathbf{3}^4 11^6$	6	$=$	2		3	
23.39	26.81	$\mathbf{3}^4 5^4 11^6$	$\mathbf{3}^2 5^4 11^6$	4	$=$	2	2		
24.16	21.02	$2^{16} 11^6$	$\mathbf{2}^{10} 11^6$	1					
24.23	27.91	$\mathbf{3}^6 113^4$	$\mathbf{3}^4 113^4$	2	$=$	2			
24.25	22.92	$2^8 \mathbf{3}^4 7^8$	$2^6 \mathbf{3}^2 7^8$	2	$=$	2			
25.14	29.11	$2^6 \mathbf{3}^6 43^4$	$2^6 \mathbf{3}^4 43^4$	2	$=$	2			
26.32	26.52	$2^6 \mathbf{5}^4 7^8$	$2^6 \mathbf{5}^2 7^8$	2	$=$	2			
26.78	31.29	$\mathbf{3}^4 239^4$	$\mathbf{3}^2 239^4$	4	$=$	2	2		
26.84	31.37	$2^6 \mathbf{3}^6 7^8$	$2^6 \mathbf{3}^4 7^8$	2	$=$	2			
26.97	27.26	$2^6 \mathbf{5}^6 23^4$	$2^6 \mathbf{5}^4 23^4$	8	$=$	4	2		
27.01	22.41	$2^8 \mathbf{5}^4 11^6$	$2^6 \mathbf{5}^2 11^6$	2	$=$	2			
27.17	31.82	$2^6 \mathbf{3}^8 29^4$	$2^6 \mathbf{3}^6 29^4$	4	$=$	2	2		
27.35	18.14	$2^8 \mathbf{11}^4 17^4$	$2^6 \mathbf{11}^2 17^4$	2	$=$	2			
28.00	20.41	$\mathbf{2}^{16} 7^8$	$\mathbf{2}^8 7^8$	2	$=$	2			
28.00	24.88	$2^{16} 7^8$	$\mathbf{2}^{10} 7^8$	1					
28.00	24.88	$2^{16} 7^8$	$\mathbf{2}^{10} 7^8$	2	$=$				2
28.86	20.77	$7^8 \mathbf{17}^4$	$7^8 \mathbf{17}^2$	2	$=$	2			
29.05	31.64	$\mathbf{2}^8 211^4$	$\mathbf{2}^4 211^4$	4	$=$	2	2		
29.22	27.14	$2^4 \mathbf{7}^4 61^4$	$2^4 \mathbf{7}^2 61^4$	4	$=$	2	2		
29.94	9.39	$2^6 \mathbf{317}^4$	$2^6 \mathbf{317}^2$	6	$=$	2		3	

TABLE 3. Discriminants of the octic $\mathrm{PSL}_2(7)$ fields of Theorem 1 and their septic siblings, and also the class number h of each octic. The boldface conventions and the factorization $h = a\ell ce$ are explained in the paragraph containing equation (3).

To start, consider the unramified tower $K_{32}/K_{16}/K_8$ coming from the first field K_8 and its cyclic class group of order four. Defining equations can be computed using *Magma* [BCP97] or *Pari*'s [PAR15] class field theory commands. Following the conventions of §1.2, let $G_{16} = \mathrm{Gal}(K_{16})$ and $G_{32} = \mathrm{Gal}(K_{32})$ be the corresponding Galois groups. Then K_{16} and K_{32} are the unique fields with smallest discriminant for these Galois groups. In fact, G_{16} is just the Cartesian product $\mathrm{PSL}_2(7) \times C_2 = 16T714$. More interestingly, $G_{32} = 32T34620$ is a non-split double cover of G_{16}, having $\mathrm{SL}_2(7) = 16T715$ as a subgroup with quotient group C_2. Similar analysis for all twenty-five base fields, allowing ramified towers $K_{32}/K_{16}/K_8$ as well, gives the following consequence of Theorem 1.

Corollary 1. *There are exactly* 25 *number fields with Galois group* $\mathrm{PSL}_2(7)\times C_2 = 16T714$ *and discriminant* $\leq 30^{16}$*. The smallest discriminant of such a field is* 21^{16} *and the field has defining polynomial*

$$x^{16} - 4x^{15} + 9x^{14} - 14x^{13} + 14x^{12} - 14x^{10} + 8x^9 + 45x^8 - 82x^7 + 49x^6 + 63x^5 - 112x^4 + 49x^3 + 99x^2 - 130x + 100.$$

There are exactly 14 *number fields with Galois group* $32T34620$ *and discriminant* $\leq 30^{32}$*. The smallest discriminant of such a field is* 21^{32} *and the field has defining polynomial*

$$x^{32} - x^{31} + 2x^{30} + x^{29} + 8x^{28} - 7x^{27} + 21x^{26} - 9x^{25} - 12x^{24} + 248x^{23} - 548x^{22} - 65x^{21} + 2653x^{20} - 4879x^{19} + 2564x^{18} + 4198x^{17} - 7780x^{16} + 3593x^{15} + 4020x^{14} - 7014x^{13} + 4935x^{12} - 2042x^{11} + 929x^{10} - 787x^9 + 695x^8 - 215x^7 + 70x^6 - 42x^5 + 15x^4 - 15x^3 + 2x^2 + x + 1.$$

As a next example, note that the sixth field K_8 has class number divisible by 3, yielding a tower K_{24}/K_8. Let $G_{24} = \mathrm{Gal}(K_{24})$. To find the field with smallest discriminant and Galois group G_{24}, we only have to look at perhaps-ramified abelian cubic extensions for the first six fields on the list.

The Galois group G_{24} is in fact $24T284$, which is $\mathrm{PSL}_2(7)$ itself, but now in its action on cosets of C_7. So $\mathrm{PSL}_2(7)$ octics are in bijection with $24T284$ fields via an elementary resolvent construction. Inspecting the twenty-five $24T284$ fields coming from the twenty-five octics gives the following result.

Corollary 2. *There are exactly three number fields with Galois group* $24T284$ *and discriminant* $\leq 30^{24}$*. The smallest discriminant of such a field is* $(66^{3/4})^{22} \approx 23.16^{22}$ *and the field has defining polynomial*

$$x^{24} - 6x^{23} + 14x^{22} - 8x^{21} - 26x^{20} + 34x^{19} + 72x^{18} - 204x^{17} + 109x^{16} + 162x^{15} - 148x^{14} - 260x^{13} + 496x^{12} - 248x^{11} + 18x^{10} - 216x^9 + 484x^8 - 402x^7 + 156x^6 - 74x^5 + 102x^4 - 76x^3 + 22x^2 - 2x + 1.$$

The fields in question are respectively unramified, ramified, and unramified extensions of the sixth, eighth, and twenty-fifth field on Table 3.

Our discussion so far has exhibited instances of three systematic contributors to class groups of $\mathrm{PSL}_2(7)$ octics $K = K_8$. We call these the abelian, lifting, and cubic contributions, and they are the sources of some of the numbers on Table 3:

$$(3) \qquad a = [AK : K], \qquad \ell \in \{1, 2\}, \qquad c \in \{1, 3\}.$$

For the abelian contribution, A is the largest cyclotomic field such that AK is unramified over K. On the table, the primes for which $A/\mathbb{Q}$ is ramified are put in bold in the D_8 column of Table 3. To identify the lifting contribution, we use the septic local signs ϵ_v of §3.3.5. A prime p is put in bold in the D_7 column exactly when $\epsilon_p = -1$. If, for every such odd p, the inertia group I_p has order divisible by $2^{\mathrm{ord}_2(p-1)}$, and if also an analogous condition at 2 holds if 2 is in bold, then KA has an unramified quadratic extension $\widetilde{KA}$ with $\widetilde{KA}/A$ having Galois group $\mathrm{SL}_2(7)$. The lifting contribution is $\ell = 2$ in this case and otherwise $\ell = 1$. The cubic contribution is $c = 3$ if the canonical extension K_{24}/K_8 as above is unramified, and $c = 1$ if it is ramified.

In the general analysis of $\mathrm{PSL}_2(7)$ number fields K, let e denote the rest of the class number h, meaning $e = h/(a\ell c)$. On the table, $e > 1$ only once. In this case, computation shows that the Hilbert class field K_{16} has defining polynomial

$$x^{16} + 40x^{14} + 588x^{12} + 3808x^{10} + 12236x^8 + 9856x^6 + 3248x^4 + 384x^2 + 16. \tag{4}$$

Its Galois group $\mathrm{Gal}(K_{16}) = 16T1506$ has order $2^4|\,\mathrm{PSL}_2(7)|$. The resulting field, with discriminant 28^{16}, however is not minimal.

Corollary 3. *There are exactly five number fields with Galois group* $16T1506$ *and discriminant* $\leq 30^{16}$*. The smallest discriminant* $2^{16}3^8 53^8 \approx 25.22^{16}$ *arises from two fields. These fields have defining polynomials* $f(x^2)$ *and* $f(-3x^2)$ *where*

$$f(x) = x^8 + 8x^7 + 32x^6 + 44x^5 + 382x^4 + 496x^3 + 656x^2 - 20x + 1.$$

The next smallest discriminant, $2^{12}3^{12}53^8 \approx 27.91^{16}$, also arises twice, and so the field defined by (4) is in fact last on the list.

5. Results for $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$

5.1. **Complete lists of** $\mathrm{PSL}_2(5)$ **and** $\mathrm{PGL}_2(5)$ **sextics.** Our second illustration of the method of mixed degree searches comes from the sextic groups $\mathrm{PSL}_2(5) = 6T12$ and $\mathrm{PGL}_2(5) = 6T14$. The fields with smallest root discriminant were obtained by a direct sextic search in [FP92] and [FPDH98]. These root discriminants are $2^1 67^{1/3} \approx 8.12$ and $2^1 3^{1/3} 7^{1/2} \approx 11.01$.

In this second illustration, the smaller degree is 5, via the isomorphisms $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$. Table 4 gives an analysis of tame ramification, with the bottom three lines being relevant to $S_5 \cong \mathrm{PGL}_2(5)$ only. As before, each partition λ_n determines the corresponding discriminant exponent c_n via formula (2). The

Table 4. Tame ramification data for $A_5 \cong \mathrm{PSL}_2(5)$ and $S_5 \cong \mathrm{PGL}_2(5)$ in degrees 5 and 6.

λ_5	λ_6	c_5	c_6
5	51	4	4
311	33	2	4
221	2211	2	2
41	411	3	3
32	6	3	5
2111	222	1	3

behavior of wild ramification in this 5-to-6 context is analogous to the case of $\mathrm{GL}_3(2)$ and $\mathrm{PSL}_2(7)$ discussed earlier and so we omit the detailed analysis. From [JR14b], one knows that the bounds suggested by the tame table hold in general: $D_5 \leq D_6 \leq D_5^2$ for $A_5 \cong \mathrm{PSL}_2(5)$ and $|D_5| \leq |D_6| \leq |D_5|^3$ for $S_5 \cong \mathrm{PGL}_2(5)$.

As pointed out in [JR14b], the first bound $|D_5| \leq |D_6|$ implies that a complete table of quintic fields up through discriminant bound B determines the corresponding complete table of sextic fields up through B. In contrast to the situation for $\mathrm{GL}_3(2) \cong \mathrm{PSL}_2(7)$, this observation gives non-empty lists in the larger degree. Taking $B = 12{,}000{,}000$ from our extension [JR14a] of [SPDyD94], one gets 78 $\mathrm{PSL}_2(5)$ sextics and 34 $\mathrm{PGL}_2(5)$ sextics with root discriminant at most $B^{1/6} \approx 15.13$.

On Table 4 there are three instances when $c_5 < c_6$. Accordingly, we can substantially reduce the quintic search space by targeting. The result for root discriminant $\delta_6 \leq 35$ is as follows.

Theorem 2. *Among sextic fields with absolute discriminant* $\leq 35^6$*, exactly* 2361 *have Galois group* $\mathrm{PSL}_2(5)$ *and* 3454 *have Galois group* $\mathrm{PGL}_2(5)$.

Lists of fields can be retrieved by searching the website for [JR14a].

In parallel with the figure for our first case, Figure 2 illustrates our second case. The regularity near the bottom boundary $\delta_6 = \delta_5^{5/6}$ is easily explained, as follows. For any pair (K_5, K_6), the ratio D_6/D_5 is always a perfect square r^2. The pair gives rise to a point on the curve $\delta_6 = r^{1/3}\delta_5^{5/6}$. In the $S_5 \cong \mathrm{PGL}_2(5)$ case, the curves corresponding to $r = 1, 2, \ldots, 14, 15$ are all clearly visible. The first "missing curve," clearly visible as a gap, corresponds to $r = 16 = 2^4$. This curve is missing because none of the 2-adic possibilities for (c_5, c_6) satisfy $c_6 - c_5 = 8$. In the $A_5 \cong \mathrm{PSL}_2(5)$ case, there are fewer 2-adic possibilities and the first four visible gaps correspond to $r = 4$, 8, 12, and 16.

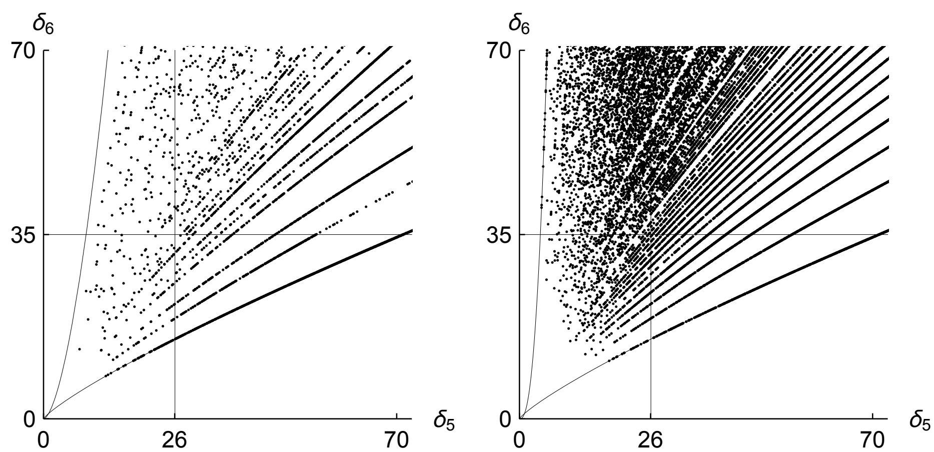

FIGURE 2. Root discriminant pairs (δ_5, δ_6) associated to $A_5 \cong \mathrm{PSL}_2(5)$ (left) and $S_5 \cong \mathrm{PGL}_2(5)$ (right), including all pairs in the window with $\delta_5 \leq 26$ from [JR14a], and all pairs with $\delta_6 \leq 35$ from Theorem 2.

5.2. **Connections with expected mass formulas.** Let $NF(G, x)$ denote the set of isomorphism classes of number fields K with $\mathrm{Gal}(K) = G$ and absolute discriminant at most x. From the quintic search in [JR14a], one knows

$$|NF(A_5, 26^5)| = 539, \qquad |NF(S_5, 26^5)| = 726862. \tag{5}$$

The ratio $539/726862 \approx 0.00074$ is an instance of the familiar informal principle "S_n fields are common but A_n fields are rare." In this light, the much larger ratio $2361/3454 \approx 0.68356$ from Theorem 2 is surprising.

However, the fact that $NF(\mathrm{PSL}_2(5), 35^6)$ and $NF(\mathrm{PGL}_2(5), 35^6)$ have such similar sizes can be explained as follows. For $g \in S_n$, let o_g be its number of cycles and let $\epsilon_g = n - o_g$. For a transitive permutation group $G \subseteq S_n$ and a conjugacy class $C \subseteq G$, define ϵ_C to be ϵ_g for any $g \in C$. Define a_G to be the minimum of the ϵ_C and b_G to be the number of classes obtaining this minimum. Then Malle conjectures an asymptotic growth rate

$$|NF(G,x)| \sim c_G x^{n/a_G} \log(x)^{b_G-1},$$

for some constant c_G [Mal02]. For A_n, the two minimizing classes have cycle types $2^2 1^{n-4}$ and 31^{n-3}, while for S_n the unique minimizing class is 21^{n-2}. Thus, consistent with numerical data like (5), one expects very different growth rates:

$$|NF(A_n,x)| \sim c_{A_n} x^{n/2} \log x, \qquad |NF(S_n,x)| \sim c_{S_n} x^n. \tag{6}$$

For $n \leq 5$, this growth rate is proved for S_n with identified constants, and it is known that the growth for A_n is indeed slower; see [Bha10] for S_5 and [BCT15] for A_5.

But now, for $G \in \{\mathrm{PSL}_2(5), \mathrm{PGL}_2(5)\}$, the unique minimizing class is $2^2 1^2$. Thus, consistent with Theorem 2, these two groups should have the same asymptotics up to a constant, both having the form $|NF(G,x)| \sim c_G x^{1/3}$. There are similar comparisons associated to our two other theorems. For the group of Theorem 1, one should have $|NF(\mathrm{PSL}_2(7),x)| \sim c_{\mathrm{PSL}_2(7)} x^2 \log x$ from 2^4 and $3^2 1^2$. This growth rate is substantially less than the expected $|NF(\mathrm{PGL}_2(7),x)| \sim c_{\mathrm{PGL}_2(7)} x^{8/3}$ from $2^3 1^2$. Indeed, there are eighteen known $\mathrm{PGL}_2(7)$ octics with root discriminant ≤ 21 [JR14a]. For the groups of Theorem 3, $|NF(\mathrm{GL}_3(2),x)| \sim c_{\mathrm{GL}_3(2)} x^{1/2}$ should grow more slowly than $|NF(A_7,x)| \sim c_{A_7} x^{1/2} \log x$. The search underlying Theorem 3 is complete through discriminant 12^7. It can be expected to be near-complete for x substantially past 12^7, and these extra fields do indicate a general increase in $|NF(A_7,x)|/|NF(\mathrm{GL}_3(2),x)|$.

5.3. **Complete lists of Artin representations.** Theorems 1 and 2 can each be viewed from a different perspective. Computing octic $\mathrm{PSL}_2(7)$ fields is essentially the same as computing Artin representations for the irreducible degree 7 character of $\mathrm{GL}_3(2)$; computing sextic $\mathrm{PSL}_2(5)$ and $\mathrm{PGL}_2(5)$ fields is equivalent to computing Artin representations for certain irreducible degree 5 characters of A_5 and S_5. In each case, discriminants match conductors.

By targeting differently, one could get lists of other Artin representations belonging to these and other small groups, which are complete up through some conductor cutoff. For example, $\mathrm{PSL}_2(7)$ and $\mathrm{PSL}_2(5)$ each have two three-dimensional representations. These representations are particularly interesting because of their low degree, which means that the associated L-functions are relatively accessible to analytic computations. We plan to draw up these lists elsewhere, as they fit into the program of [LMF15] of systematically cataloging L-functions.

6. Results for $\mathrm{GL}_3(2)$, A_7, and related groups

6.1. **Extending the known lists for $\mathrm{GL}_3(2)$ and A_7.** The first pair of fields for $\mathrm{GL}_3(2)$ and first field for A_7 were determined by Klüners and Malle in [KM01]. We extend the complete list of such fields by employing the same technique as [KM01]: a standard Hunter search modified to select polynomials with Galois group contained in A_7. The latter condition comes into play as the first step when inspecting a polynomial in the search to see if it is suitable. Testing if the polynomial discriminant is a square can be done very quickly and filters out all the polynomials with odd Galois group.

Theorem 3. *Among septic fields with discriminant* $\leq 12^7$*, exactly* 46 *have Galois group* $\mathrm{GL}_3(2)$ *and* 17 *have Galois group* A_7*.*

Carrying out the computation up to discriminant 12^7 took six and a half months of CPU time and inspected roughly 10^{12} polynomials. As in other cases, defining polynomials and other information for these fields can be obtained from the website for [JR14a].

The computation establishing Theorem 3 sheds light on the list of $\mathrm{PSL}_2(7)$ fields established by Theorem 1. The runtime of a Hunter search in degree n with discriminant bound B is proportional to $B^{(n+2)/4}$ [JR98]. Using this, our estimate for the runtime for confirming the first $\mathrm{PSL}_2(7)$ octic field by computing septic fields without targeting is approximately $(21^8/12^7)^{9/4}6.5/12 \approx 3$ million CPU-years. To get Theorem 1's complete list through discriminant 30^8 would then take $(30^8/12^7)^{9/4}6.5/12 \approx 2$ billion CPU-years.

6.2. **The first** $2^3{:}\,\mathrm{GL}_3(2)$ **field.** As observed in [KM01], a sufficiently long complete list of septic $\mathrm{GL}_3(2)$ fields can be used to determine the first octic $2^3{:}\,\mathrm{GL}_3(2)$ field. The splitting field of a $2^3{:}\,\mathrm{GL}_3(2)$ polynomial contains (up to isomorphism), two subfields K_{7a} and K_{7b} of degree 7, two K_{14a} and K_{14b} of degree 14 and Galois group $14T34$, and two subfields K_{8a} and K_{8b} of degree 8 and Galois group $8T48$. One of the septic fields is contained in both K_{14a} and K_{14b}, and the other is contained in neither.

Since the septic fields are arithmetically equivalent, they have the same discriminants, i.e., $D_{7a} = D_{7b}$. The other indices can be adjusted so that

$$D_{7x}D_{8x} = D_{14x} \quad \text{for } x \in \{a, b\}. \tag{7}$$

This comes from a character relation on the relevant permutation characters. Because of the asymmetry in the field inclusions described above, the fact that $\mathrm{GL}_3(2)$ fields come in arithmetically equivalent pairs does not play a role in our computations, and so we have forty-six septic ground fields to consider separately. Accordingly, we drop x from notation, always taking the correct octic resolvent so that (7) holds.

Of the forty-six septic $\mathrm{GL}_3(2)$ fields with discriminant $\leq 12^7$, only one has nontrivial narrow class group. This field has narrow class number two, and is $K_7 = \mathbb{Q}[x]/f(x)$ with

$$f(x) = x^7 - x^6 - x^5 - 2x^4 - 7x^3 - x^2 + 3x + 1. \tag{8}$$

The unramified quadratic extension turns out to be simply $K_{14} = \mathbb{Q}[x]/f(-x^2)$, which has Galois group $14T34$. So K_{14} and K_7 both have root discriminant $5717^{2/7} \approx 11.84$. By (7), the sibling K_8 of K_{14} has the even smaller root discriminant $5717^{1/4} \approx 8.70$.

In general, given all septic $\mathrm{GL}_3(2)$ fields up to some discriminant bound B, one can get all $14T34$ fields up to the discriminant bound B^2 via quadratic extensions. If $\mathfrak{d}$ is the relative discriminant of K_{14}/K_7, then $D_{14} = D_7^2 N_{K_7/\mathbb{Q}}(\mathfrak{d})$, and so $N_{K_7/\mathbb{Q}}(\mathfrak{d})$ must be at most B^2/D_7^2. In the same way, the stronger bound $N_{K_7/\mathbb{Q}}(\mathfrak{d}) \leq B/D_7$ is necessary and sufficient for the resolvent $8T48$ field to have discriminant $\leq B$.

Taking $B = 12^7$ now, the quotient B/D_7 decreases from $12^7/(13^2 109^2) \approx 17.85$ for the first two ground fields K_7 to $12^7/(2^6 743^2) \approx 1.01$ for the last two ground fields. For the first twenty-six ground fields, computation shows that there are no $14T34$ overfields satisfying $N_{K_7/\mathbb{Q}}(\mathfrak{d}) \leq B^2/D_7^2$. For the last twenty fields, already $B/D_7 < \sqrt{2}$ and so the lack of overfields, except for the unramified one above, follows from the narrow class numbers being 1. Hence we have the following corollary of Theorem 3.

Corollary 4. *The field* $\mathbb{Q}[x]/f(-x^2)$ *from* (8) *is the only degree fourteen field with Galois group* $14T34$ *and discriminant* $\leq 12^{14}$*. Its sibling, with defining polynomial*

$$x^8 - 4x^7 + 8x^6 - 9x^5 + 7x^4 - 4x^3 + 2x^2 + 1,$$

is the only the octic field with Galois group $2^3{:}\,\mathrm{GL}_3(2) = 8T48$ *and discriminant* $\leq 12^7$*. These two fields have discriminants* $5717^4 \approx 11.84^{14}$ *and* $5717^2 \approx 11.84^7 \approx 8.70^8$ *respectively.*

References

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust, *The Magma algebra system. I. The user language*, J. Symbolic Comput. **24** (1997), no. 3-4, 235–265, Computational algebra and number theory (London, 1993). MR MR1484478

[BCT15] Manjul Bhargava, Alina Carmen Cojocaru, and Frank Thorne, *The number of non-S_5 quintic extensions of bounded discriminant*, preprint, March 2015.

[Bha10] Manjul Bhargava, *The density of discriminants of quintic rings and fields.*, Ann. of Math. (2) **172** (2010), no. 3, 1559–1591.

[CH08] John J. Cannon and Derek F. Holt, *The transitive permutation groups of degree 32*, Experiment. Math. **17** (2008), no. 3, 307–314. MR 2455702

[CHM98] John H. Conway, Alexander Hulpke, and John McKay, *On transitive permutation groups*, LMS J. Comput. Math. **1** (1998), 1–8 (electronic).

[Coh00] Henri Cohen, *Advanced topics in computational number theory*, Graduate Texts in Mathematics, vol. 193, Springer-Verlag, New York, 2000. MR 2000k:11144

[FP92] David Ford and Michael Pohst, *The totally real A_5 extension of degree 6 with minimum discriminant*, Experiment. Math. **1** (1992), no. 3, 231–235.

[FPDH98] David Ford, Michael Pohst, Mario Daberkow, and Nasser Haddad, *The S_5 extensions of degree 6 with minimum discriminant*, Experiment. Math. **7** (1998), no. 2, 121–124.

[Iwa55] Kenkichi Iwasawa, *On Galois groups of local fields*, Trans. Amer. Math. Soc. **80** (1955), 448–469.

[JR98] John W. Jones and David P. Roberts, *Timing analysis of targeted Hunter searches*, Algorithmic number theory (Portland, OR, 1998), 1998, pp. 412–413.

[JR99] ———, *Sextic number fields with discriminant $(-1)^j 2^a 3^b$*, Number theory (Ottawa, ON, 1996), CRM Proc. Lecture Notes, vol. 19, Amer. Math. Soc., Providence, RI, 1999, pp. 141–172. MR 2000b:11142

[JR03] ———, *Septic fields with discriminant $\pm 2^a 3^b$*, Math. Comp. **72** (2003), no. 244, 1975–1985 (electronic). MR MR1986816 (2004e:11119)

[JR06] ———, *A database of local fields*, J. Symbolic Comput. **41** (2006), no. 1, 80–97, website: `http://math.asu.edu/~jj/localfields`.

[JR14a] ———, *A database of number fields*, LMS J. Comput. Math. **17** (2014), no. 1, 595–618, website: `http://hobbes.la.asu.edu/NFDB`.

[JR14b] ———, *The tame-wild principle for discriminant relations for number fields*, Algebra Number Theory **8** (2014), no. 3, 609–645.

[KM01] Jürgen Klüners and Gunter Malle, *A database for field extensions of the rationals*, LMS J. Comput. Math. **4** (2001), 182–196 (electronic). MR 2003i:11184

[LMF15] The LMFDB Collaboration, *The L-functions and modular forms database*, `http://www.lmfdb.org`, 2015, [Online; accessed November 2015].

[Mal02] Gunter Malle, *On the distribution of Galois groups*, J. Number Theory **92** (2002), no. 2, 315–329.

[PAR15] The PARI Group, Bordeaux, *PARI/GP version* `2.7.5`, 2015, available from `http://pari.math.u-bordeaux.fr/`.

[SPDyD94] A. Schwarz, M. Pohst, and F. Diaz y Diaz, *A table of quintic number fields*, Math. Comp. 63 **63** (1994), no. 207, 361–376.

[Tri68] W. Trinks, *Ein Beispiel eines Zahlkörpers mit der Galoisgruppe* $PSL(3,2)$ *über* $\mathbf{Q}$, Univ. Karlsruhe, Karlsruhe, 1968.

School of Mathematical and Statistical Sciences, Arizona State University, PO Box 871804, Tempe, AZ 85287

E-mail address: `jj@asu.edu`

Division of Science and Mathematics, University of Minnesota-Morris, Morris, MN 56267

E-mail address: `roberts@morris.umn.edu`