**An example of current math research:** "Discriminants of some Painlevé polynomials," by David P. Roberts, to appear in Number Theory for the Millennium III.

**I. Some general things I knew beforehand** (Polynomials, their roots and discriminants.)

**II. A specific thing I knew beforehand.** (The Hermite polynomials.)

**III. How I stumbled across a situation where it seemed that I might be able to contribute something new.** (My good luck!)

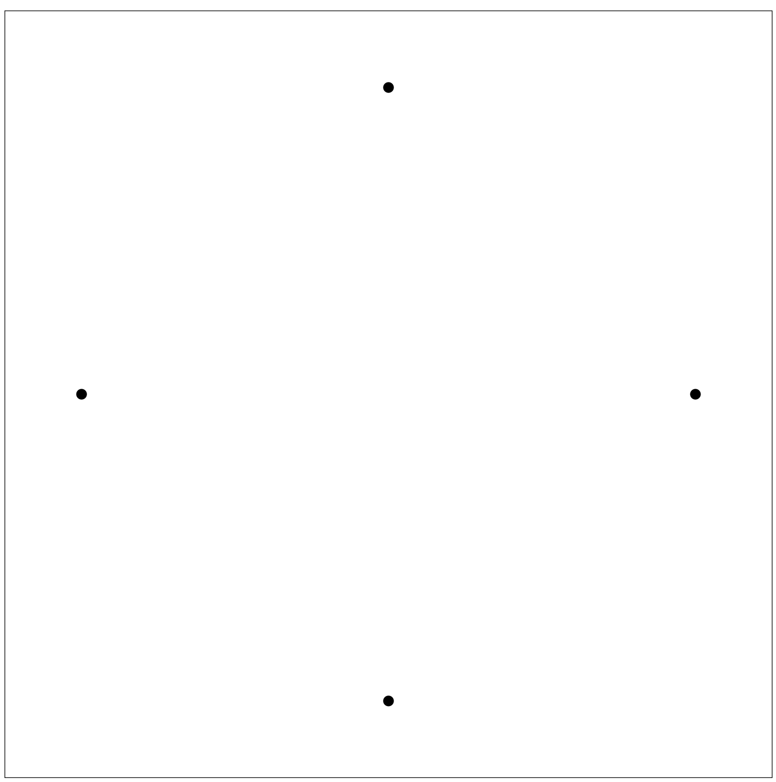
**IV. The process of actually doing the research and writing the paper.** (First conjecture, then proof.)

The paper and an accompanying *Mathematica* file are available on my homepage.

**I. Some things I knew beforehand.** The fundamental theorem of algebra says that a degree $n$ polynomial $f(x)$ factors into $n$ linear factors over the complex numbers. Each factor $(x - \alpha_j)$ corresponds to a root $\alpha_j$.

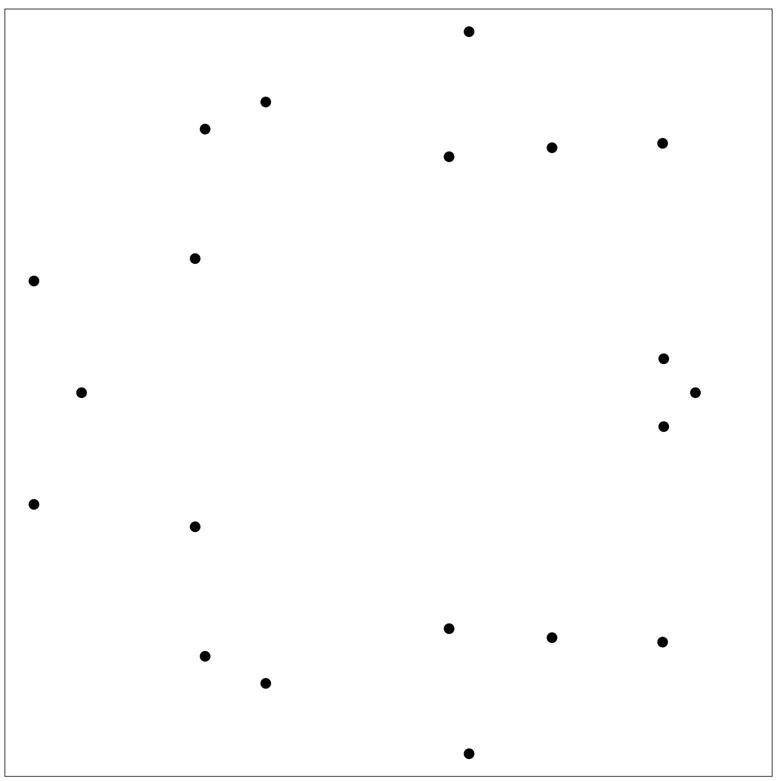
**Example 1.** A degree four polynomial.

$$\begin{aligned} f_1(x) &= x^4 - 1 \\ &= (x^2 + 1)(x^2 - 1) \\ &= (x - i)(x + i)(x - 1)(x + 1) \end{aligned}$$



**Example 2.** A degree twenty polynomial.

$$\begin{aligned} f_2(x) &= x^{20} + x^{17} - 4x^{11} + 3x^5 - 1 \\ &\approx (x - (-1.16 - 0.36i)) \cdot \\ &\quad (x - (-1.16 + 0.36i)) \cdot \\ &\quad \cdots (17 \text{ factors}) \cdots (x - 1) \end{aligned}$$



$f_2$ illustrates a principle: **Typical polynomials have rather randomly scattered roots.**

To measure how close the roots are together, mathematicians have introduced the discriminant of $f$:

$$D(f) = \left( \prod_{i<j} |\alpha_i - \alpha_j| \right)^2$$

By hand we compute

$$D(f_1) = \left(\sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot \sqrt{2} \cdot 2 \cdot 2\right)^2 = 256.$$

By machine, or by a nice linear algebra formula, we compute

$$D(f_2) = 10605575988192416385974974 54592.$$

The prime factorization of this last number is $2^{19} \cdot 7573 \cdot 979423 \cdot 272725713834 6971$.

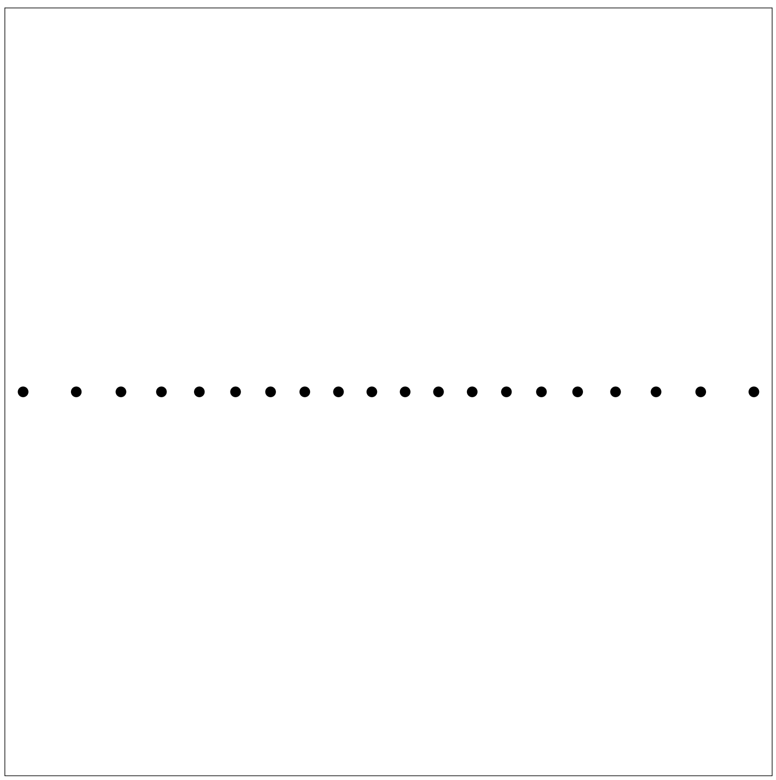
$f_2$ illustrates another principle: **Typical polynomials have discriminants which have large primes in their prime factorizations.**

**II. A specific thing I knew beforehand.** There are some interesting polynomials called **Hermite polynomials** $h_n(x)$ which arise in many places. One way they can arise is by taking successive derivatives of the function whose graph is the bell curve:

$$\begin{aligned} f(x) &= e^{-x^2/2} \\ f'(x) &= -e^{-x^2/2}x \\ f''(x) &= e^{-x^2/2}(x^2-1) \\ f^{(3)}(x) &= -e^{-x^2/2}(x^3-3x) \\ f^{(4)}(x) &= e^{-x^2/2}(x^4-6x^2+3) \\ f^{(5)}(x) &= -e^{-x^2/2}(x^5-10x^2+15) \\ &\vdots \\ f^{(n)}(x) &= (-1)^n e^{-x^2/2}h_n(x). \end{aligned}$$

The Hermite polynomial $h_n(x)$ has degree $n$.

**The Hermite polynomials are extremely atypical!** Their roots are all real, and moreover nicely spaced on the real axis. Here are the roots of $h_{20}(x)$:



Here's the discriminant of $h_{20}(x)$:

$D(h_{20}) =$
$1007637378983886143555276880035780555571894973728632372592151120203324015430217812141131135118846218463162642396377788329412456150570507893386898483318110155132911873463731724746752000000000000000000000000000000000000000000000000000$

This factors as

$$D(h_{20}) = 2^{210}3^{90}5^{50}7^{21}11^{11}13^{13}17^{17}19^{19}$$

In general, $D(h_n) = 1^1 2^2 3^3 \cdots n^n$. In "product notation" this is written

$$D(h_n) = \prod_{j=1}^{n} j^j$$

This formula was found and proven in the late 1800's. Polynomials as nice as the Hermite polynomials are extremely rare!

**III. How I stumbled across a situation where it seemed that I might be able to contribute something new.** Browsing the literature, I found the Yablonksy-Vorobiev polynomials

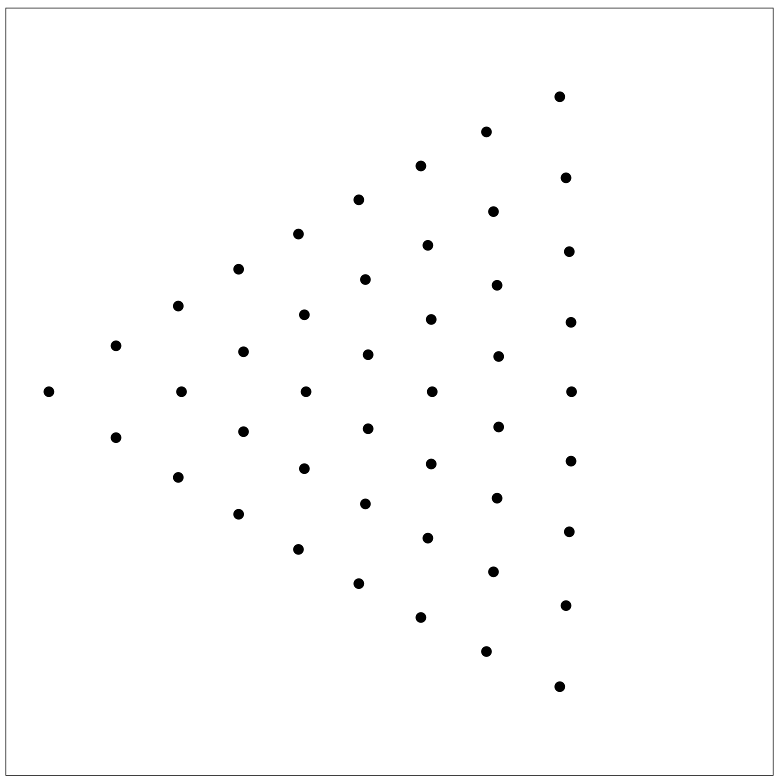
$$\begin{aligned} p_0(x) &= 1 \\ p_1(x) &= x \\ p_2(x) &= x^3 + 1 \\ p_3(x) &= x^6 + 5x^3 - 5 \\ p_4(x) &= x^{10} + 15x^7 + 175x \end{aligned}$$

I took some of their discriminants, finding for example that $D(p_9)$ is a 1096 digit number, which miraculously factors as

$$3^{702} 5^{305} 7^{252} 11^{176} 13^{117} 17^{17}.$$

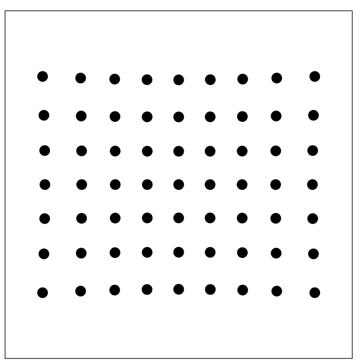
For sure, I knew the $p_n$ are quite special! While Hermite polynomials are **linear** in nature, the Yablonsky-Vorobiev polynomials are **quadratic** in nature. For example, degree($h_n$) $= n$ but degree($p_n$) $= (n^2 + n)/2$.

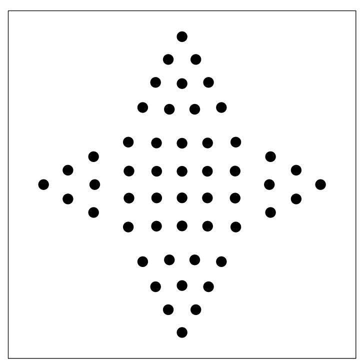
Next, I looked at the complex roots of the $p_n$. Here are the roots of $p_9$:



From this point of view too, it was totally clear that the $p_n$ are very special.

The $p_n$ are related to solutions to a differential equation called the Painlevé II equation. There are altogether six Painlevé equations and I looked through the literature on the others, finding two families of polynomials related to Painlevé IV. One family is the **biHermite polynomials** $h_{m,n}(x)$ and the other is the **Okamoto polynomials** $q_{m,n}(x)$. They too have highly factorizing discriminants and very regular roots:





On the left is $h_{9,7}$ and on the right is $q_{5,4}$ with discriminants

$$\begin{aligned} D(h_{9,7}) &= 2^{1912}3^{1152}5^{420}7^{399}11^{275}13^{117} \\ D(q_{5,4}) &= 2^{1410}5^{315}7^{252}11^{33}13^{208}19^{76} \end{aligned}$$

**IV. The process of actually doing the research and writing the paper.** To go any further, I needed conjectural discriminant formulas. I looked at the evidence:

$$\begin{aligned}
&\vdots \\
D(p_7) &= 3^{270}5^{125}7^{112}11^{44}13^{13} \\
D(p_8) &= 3^{450}5^{195}7^{175}11^{99}13^{52} \\
D(p_9) &= 3^{702}5^{305}7^{252}11^{176}13^{117}17^{17} \\
D(p_{10}) &= 3^{1026}5^{455}7^{343}11^{275}13^{208}17^{68}19^{19} \\
&\vdots
\end{aligned}$$

I conjectured that

$$D(p_n) = \prod_{j=3,5,7,9,\ldots}^{2n-1} j^{j(2m+1-j)^2/4}.$$

I did the same thing for the Hermite polynomials and the Okamoto polynomials. My conjectural formulas for $D(h_{m,n})$ and $D(q_{m,n})$ were more complicated, because there are two indices $m$ and $n$ in these cases.

To prove my formulas for $D(p_m)$ I used **induction** on $m$. For $D(h_{m,n})$ and $D(q_{m,n})$ I used a more complicated double induction. In each case, I had to also prove also that a given polynomial relates nicely to its immediate predecessors; this meant establishing **resultant formulas** as well as the desired **discriminant formulas.** Here are the roots of $p_9$, $p_8$, and $p_7$, followed by all the roots superimposed: