NEWFORMS WITH RATIONAL COEFFICIENTS

DAVID P. ROBERTS

Abstract. We consider the set of classical newforms with rational coefficients and no complex multiplication. We study the distribution of quadratic-twist classes of these forms with respect to weight k and minimal level N. We conjecture that for each weight $k \geq 6$, there are only finitely many classes. In large weights, we make this conjecture effective: in weights $18 \leq k \leq 24$, all classes have $N \leq 30$, in weights $26 \leq k \leq 50$, all classes have $N \in \{2, 6\}$, and in weights $k \geq 52$, there are no classes at all. We study some of the newforms appearing on our conjecturally complete list in more detail, especially in the cases $N = 2$, 3, 4, 6, and 8, where formulas can be kept nearly as simple as those for the classical case $N = 1$.

1. Introduction

1.1. **A finiteness conjecture, effective in large weights.** Classical newforms, as reviewed in next section, are certain power series $g = \sum_{n=1}^{\infty} a_n q^n \in \mathbb{C}[[q]]$ which play an important role in arithmetic geometry. This paper is a contribution to cataloging newforms for which all the coefficients a_n are rational. We exclude newforms with complex multiplication, as CM newforms with rational coefficients have been comprehensively treated by Schütt [Sch09]. We quotient out by the operation of quadratic twisting, thereby replacing infinitely many newforms g^{χ} by a single twist-class of newforms $[g]$. Our focus is then on the finite sets $Q_k(N)$ of twist-classes of newforms with rational coefficients, no complex multiplication, and minimal level N. Here the weight k runs over positive even integers, while the minimal level N runs over positive integers.

The computer algebra system *Magma* [BCP97] lets one easily identify $Q_k(N)$ for kN sufficiently small. Table 1.1 presents the sizes $|Q_k(N)|$ for $k \leq 50$ and $N \leq 30$, as well as related information. We expect in particular that for $k \geq 18$, the table accounts for everything:

Conjecture 1.1. *For $6 \leq k \leq 50$ there is a largest N_k with $|Q_k(N_k)| \geq 1$. For $18 \leq k \leq 50$, this N_k is either 30, 10, 6, or 2, as reported on Table 1.1. For $k \geq 52$, there are no non-CM newforms with rational coefficients at all.*

Restricted to $N = 1$, our conjecture is a weak version of the well-known Maeda conjecture [HM97]. For $N > 1$, our conjecture is similarly related to a natural generalization of the Maeda conjecture [Tsa14, DT16]. The boldface entries on Table 1.1 reflect factorizations of Hecke polynomials that this generalization says never happens for k sufficiently large. Thus a novelty of Conjecture 1.1 is its effectivity: it says that for $k \geq 18$, there are only the indicated exceptions for $N = 3$, 15, and 22.

1.2. **Further discussion of the conjecture.** Let $\#_k = \sum_{N=1}^{\infty} |Q_k(N)|$, which *a priori* may be infinite or finite. The situation changes as k increases as follows.

	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
1						1		1	1	1	1		1												
2				1	1		2	1	1	2	2	1	1	2	2	1	1	2		1	1			1	
3			1	1	2	1	1	2	1	1	$\underline{\mathbf{2}}$	1													
4			1		1	1	1	1		1															
5		1	1	1	1	1																			
6		1	1	1	1	3	1	3	3	3	3	3	3	3	3	3	3	1	3	1	1	1	1		1
7		1	1	1																					
8		1	1	2	2	1	1	$\underline{\mathbf{2}}$																	
9																									
10		1	3	1	3	3	3	3	1	3	1		1												
11	1		1																						
12		1		2	1	2	2	1	2		1														
13		1		**1**																					
14	1	2	2	2	2	2	2	1	1																
15	1	2	2	2	2	1	1				**1**														
16																									
17	1	1	$\underline{\mathbf{2}}$	**1**																					
18			1		1	1	1	1		1															
19	1	1	$\underline{\mathbf{2}}$																						
20	1	1	1	1	1	1																			
21	1	2	$\underline{\mathbf{4}}$	1	1	**2**																			
22		3	3	3	3					**1**															
23		1																							
24	1	1	3	3	3	3	1	1																	
25		1																							
26	2	3	1	3	$\underline{\mathbf{3}}$																				
27		1																							
28		2	2																						
29																									
30	1	2	2	6	6	6	6	6	6	2	2	2													
$\#'_k$			312	142	67	44	29	28	16	15	13	7	6	5	5	4	4	3	3	2	2	1	1	1	1
N'_k			980	690	330	114	60	150	30	30	30	30	10	6	6	6	6	6	6	6	6	6	6	2	6
C_k			1000	700	450	300	200	200	150	150	150	150	100	100	100	50	50	50	50	30	30	30	30	30	30

TABLE 1.1. The number $|Q_k(N)|$ of twist-classes of non-CM modular forms with rational coefficients, weight k, and minimal level N. The number $|Q_k^u(N)|$ of such classes where rationality is unforced is 0 (regular type), 1 (boldface), or 2 (underlined boldface).

By the modularity result of Wiles et al. [Wil95, TW95, BCDT01], the set $Q_2(N)$ is naturally identified with the set of twist-and-isogeny classes of non-CM elliptic curves of minimal conductor N. Via this connection, Cremona has identified the sets $Q_2(N)$ for $N \leq 400{,}000$ [LMF16] and it easy to see that $\#_2$ is infinite. The case $k = 4$ is similarly related to rigid Calabi-Yau three-folds [GY11] and examples have been systematically pursued [Mey05]; it seems to us premature to speculate whether $\#_4$ is infinite or finite. The cases $k \geq 6$ have been studied [Yui13, PR15], but there do not seem to be any systematic non-modular sources: this lack of sources contributes to our expectation of finiteness for these $\#_k$.

The last block of Table 1.1 includes quantities N'_k, and C_k. Direct calculation is supportive of Conjecture 1.1 as follows. For $6 \leq k \leq 50$, we have computed $Q_k(N)$ for all $N \leq C_k$, finding it last non-empty at $N = N'_k$. In weights $k = 6$ and 8, non-empty $Q_k(N)$ become increasingly rare as N approaches C_k, as illustrated by

Figure 4.2. The thinning is rapid enough that we expect overall finiteness, although we also expect that the observed maximum N'_k may be considerably less than the conjectured actual maximum N_k. In each of the weights $k = 10$, 12, 14, and 16, we think it more likely than not that $N'_k = N_k$. In weights $18 \leq k \leq 50$, the ratio C_k/N'_k is always at least five, giving us considerable confidence in $N'_k = N_k$, as asserted by the conjecture. Similarly, we have carried our computations with cutoffs $C_k \geq 6$ for $52 \leq k \leq 100$, always finding $Q_k(N)$ to be empty.

The last block of the table also contains the lower bound $\#'_k = \sum_{N=1}^{C_k} Q_k(N)$ to $\#_k$. Reformulating some of the previous discussion, our expectation is that $(\#_2, \#_4, \#_6, \dots)$ takes the form $(\infty, \#_4, 312'', 142'', 67', 44', 29', 28', 16, 15, \dots)$. Here $\#_4$, whether it is infinite or finite, should be substantially large than $\#_6$. Also A'' indicates a number slightly larger than A, and B' indicates a number either equal to or very slightly larger than B. It seems likely that the sequence $\#_k$ is weakly decreasing, which would conform to general expectations in an unexpectedly sharp way.

1.3. **Content of the sections.** Section 2 gives the promised review of modular forms, using the case of $N = 1$ and the familiar ring $M(1) = \mathbb{C}[E_4, E_6]$ as an example. Included in this section is a brief summary of the classification of CM newforms, which compares interestingly with our Conjecture 1.1. Section 3 discusses a decomposition of $Q_k(N)$ into its "forced" and "unforced" parts:

$$Q_k(N) = Q_k^f(N) \coprod Q_k^u(N). \tag{1.1}$$

It explains how Conjecture 1.1 with $Q_k(N)$ replaced by $Q_k^f(N)$ would become provable, with finiteness also for $k = 2$ and $k = 4$, and all N_k identifiable. As to $Q_k^u(N)$, it gives a quantitative model, supporting our expectation of emptiness for large k. Section 4 explains the simple calculations underlying Table 1.1.

There is a large literature devoted to the explicit study of rational newforms in weight two, mostly through their connection with elliptic curves. Our viewpoint is that rational newforms in weight greater than two are worthy of at least a modest fraction of this detailed attention. Sections 5 and 6 are in this spirit, and study the cases $N = 2$, 3, 4, 6, and 8. Section 5 focuses on rings $M(N)$ of modular forms, describing them via interesting overrings $\widehat{M}(N)$, all of which are free on two generators like $\widehat{M}(1) = M(1)$. Section 6 presents many congruences between newforms for a given N and interprets these congruences in terms of explicit number fields. Finally, Section 7 returns the focus to Conjecture 1.1, and briefly discusses possible future directions.

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2. Review of modular forms

This section presents a brief synopsis of the theory of modular forms, as presented in more detail in e.g. [Kob93] and [Ste07]. Our purpose is to make this paper immediately accessible to a broad range of readers. We restrict to the case of trivial character until the very last subsection. Throughout, the classical case of level $N = 1$ is used as an example.

2.1. **Rings of modular forms.** For any pair (k, N) consisting of a non-negative even integer weight k and a positive integer level N, one has a corresponding finite-dimensional complex vector space $M_k(N)$ of modular forms. These modular forms are functions on the upper half plane $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ satisfying certain transformation laws which become less demanding as N becomes more divisible. In particular, the functions can be expressed as power series in $q = e^{2\pi i z}$ and for us it suffices to simply regard all the spaces $M_k(N)$ as subspaces of the ring $\mathbb{C}[[q]]$ of formal power series in q.

The sum of all these $M_k(N)$ together forms a graded ring $M(N)$. Each space $M_k(N)$ contains a subspace $S_k(N)$ of cusp forms, and these cusp forms together form an ideal $S(N)$ in the ring $M(N)$. The ring $M(1)$ was studied in 1916 by Ramanujan [Ram00]. To make room for later subscripting conventions, we use Ramanujan's notations for certain Eisenstein series, writing $Q = E_4$, $R = E_6$. The ring then takes the form $M(1) = \mathbb{C}[Q, R]$ with

$$Q = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n \quad = 1 + 240q + 2160q^2 + 6720q^3 + \cdots \qquad \in M_4(1),$$

$$R = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n \quad = 1 - 504q - 16632q^2 - 122976q^3 - \cdots \quad \in M_6(1).$$

Here the formulas refer to the usual sum of positive divisors, $\sigma_j(n) = \sum_{d|n} d^j$. The ideal of cusp forms has generator

$$\Delta = \frac{Q^3 - R^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + 4830q^5 - 6048q^6 - \cdots .$$

Let $\eta = q^{1/24} \prod_{n=1}^{\infty}(1 - q^n)$. Then one has the remarkable alternative expression $\Delta = \eta^{24}$.

2.2. **Operators and newforms.** There are many important operators on the spaces $M_k(N)$. Among these are the push-up operators, corresponding to positive integers t. The operator for t takes the form $g = \sum a_n q^n$ in $M_k(N)$ into the form $g_t = \sum a_n q^{tn} \in M_k(tN)$. Also playing an explicit role for us is the commuting family of Atkin-Lehner involutions w_{p^e} of the graded ring $M(N)$, one for each prime power p^e exactly dividing N. Finally, one has a commuting family of Hecke operators T_p on $S_k(N)$, indexed by primes p not dividing N. These Hecke operators commute with the Atkin-Lehner operators and are given by the explicit formula

$$T_p\left(\sum a_n q^n\right) = \sum (a_{pn} + p^{k-1} a_{n/p}) q^n.$$

Here a_x is understood to be 0 if x is non-integral.

A form $q + \cdots \in S_k(N)$ which is a basis for a one-dimensional eigenspace of the Hecke operators is called a newform. We let $P_k(N) \subset S_k(N)$ be the set of newforms. These newforms span the new subspace $S_k^{\mathrm{new}}(N)$. As (M, g, t) runs over triples, with M a divisor of N, g a newform in $P_k(M)$, and t a divisor of N/M, the push-ups g_t form a basis of $S_k(N)$.

Suppose $q = p^e$ exactly divides N and $g = \sum a_n q^n$ is a newform in $P_k(N)$ on which w_q acts with the eigenvalue ϵ_q. If $e = 1$, then one has the formula

$$a_p = -\epsilon_p p^{k/2-1}. \tag{2.2}$$

This formula is very useful because it identifies ϵ_p. If $e > 1$, then one has the simpler but not so useful formula $a_p = 0$.

The ring $M_k(1)$ is different from all the other $M_k(N)$ in that no push-up or Atkin-Lehner operators are involved in its description. However the T_p behave completely typically. For all N and $g = \sum_n a_n q^n \in P_k(N)$, one has the simple formula $T_p g = a_p g$. Also if m and n are relatively prime then $a_{mn} = a_m a_n$. For example, the $a_6 = -6048$ for $\Delta \in P_{12}(1)$ is indeed $a_2 a_3 = -24 \cdot 252$.

2.3. **Dimension formulas.** We are most directly interested in the spaces $S_k^{\text{new}}(N)$. There is an exact formula [Ste07, Prop 6.1] for the dimension of the larger space $S_k(N)$. Taking the largest term as an approximation gives

$$\dim S_k(N) \approx \frac{k-1}{12} \prod_{p^e || N} p^{e-1}(p+1). \tag{2.3}$$

The exact general formula for $S_k(N)$ passes to one for $S_k^{\text{new}}(N)$. The approximate formula becomes

$$\dim S_k^{\text{new}}(N) \approx \frac{k-1}{12} \prod_{p^e || N} m(p,e), \tag{2.4}$$

with

$$m(p,e) = \begin{cases} p-1 & \text{if } e = 1, \\ p^2 - p - 1 & \text{if } e = 2, \\ (p-1)^2(p+1)p^{e-3} & \text{if } e \geq 3. \end{cases}$$

Because the spaces $S_k(d)$ are involved for all $d|N$, secondary terms are more complicated in the exact formula for $\dim S_k^{\text{new}}(N)$.

Fix $N = \prod_{i=1}^m q_i$ with $q_i = p_i^{e_i}$ and $p_1 < \cdots < p_m$. Then the Atkin-Lehner operators give decompositions

$$M_k(N) = \sum_\epsilon M_k(N)^\epsilon, \quad S_k(N) = \sum_\epsilon S_k(N)^\epsilon, \quad S_k^{\text{new}}(N) = \sum_\epsilon S_k^{\text{new}}(N)^\epsilon.$$

Here ϵ runs over the 2^m sign strings $(\epsilon_{q_1}, \ldots, \epsilon_{q_m})$, with w_{q_i} acting by ϵ_{q_i}. As one might expect, an approximate formula for $S_k(N)^\epsilon$ is $1/2^m$ times (2.3). However for $S_k^{\text{new}}(N)^\epsilon$, one has to replace each $m(p_i, e_i)$ by the appropriate $m^{\epsilon_{q_i}}(p_i, e_i)$. Here

$$m^\pm(p,e) = \begin{cases} (p-1)/2, & \text{if } e = 1, \\ (p^2 - p - 1 \mp 1)/2, & \text{if } e = 2, \\ (p-1)^2(p+1)p^{e-3}/2, & \text{if } e \geq 3. \end{cases} \tag{2.5}$$

For fixed N and increasing $k > 2$, all approximations discussed in this section are off by a function which is periodic in k. An interesting feature of (2.5) is $m^+(2,2) = 0$, and indeed $P_k(N)^\epsilon$ is empty whenever $\text{ord}_2(N) = 2$ and $\epsilon_4 = +$.

As an example, from the description of $M(1)$ and $S(1)$ above, one has the exact formula

$$\dim(S_k(1)) = \frac{k-1-\delta_k}{12},$$

with $\delta_k = 13, 3, 5, 7, 9, -1$ for $k = 2, 4, 6, 8, 10, 12$ and satisfying $\delta_{k+12} = \delta_k$. We are interested particularly in one-dimensional spaces. These occur for $k = 12, 16, 18, 20, 22$, and 26. The corresponding unique newforms are

$$\Delta, \qquad Q\Delta, \qquad R\Delta \qquad Q^2\Delta \qquad QR\Delta, \qquad Q^2R\Delta. \tag{2.6}$$

These newforms were all studied by Ramanujan [Ram00]; they provide six explicit illustrations of the objects of our title, *Newforms with rational coefficients.*

2.4. **Quadratic twists.** Quadratic number fields are classified by their discriminants D. These integers, and also the discriminant 1 of the quadratic algebra $\mathbb{Q}\times\mathbb{Q}$, are called fundamental discriminants. Explicitly, a fundamental discriminant is an integer of the form td, where d is a square-free integer congruent to 1 modulo 4, and $t \in \{1, -4, 8, -8\}$. Fundamental discriminants form a set of representatives in $\mathbb{Q}^\times$ of the group $\mathbb{Q}^\times/\mathbb{Q}^{\times 2}$. Each fundamental discrimininant D gives a character $\chi_D : (\mathbb{Z}/D)^\times \to \{-1, 1\}$ given by the quadratic residue symbol, $\chi_D(n) = (D/n)$.

The infinite group of these quadratic Dirichlet characters acts on the set of newforms of a given weight k by twisting. Suppose $g = \sum_{n=1}^\infty a_n q^n \in P_k(N)$ and $\chi = \chi_D$. Then g^χ is a newform with level N_{g^χ} dividing $\mathrm{LCM}(N, D^2)$. It is characterized by $a_n^\chi = \chi(n)a_n$ for n not dividing $\mathrm{LCM}(N, D^2)$. Also, equality holds in $\mathrm{ord}_p(N_{g^\chi}) \le \max(\mathrm{ord}_p(D^2), \mathrm{ord}_p(N))$ if $\mathrm{ord}_p(D^2) \neq \mathrm{ord}_p(N)$.

For a form g, one says its *minimal level* is the minimum of the levels of all its twists. Say that a form is *minimal* if its level is equal to its minimal level. If g is minimal of level N then g has $t(N)$ minimal twists, where t is the multiplicative function satisfying

$$t(2^e) = \begin{cases} 1 & \text{if } e \in \{0,1,2,3\}, \\ 2 & \text{if } e \in \{4,5\}, \\ 4 & \text{if } e \ge 6. \end{cases} \quad \text{and, for } p \text{ odd, } t(p^e) = \begin{cases} 1 & \text{if } e \in \{0,1\}, \\ 2 & \text{if } e \ge 2. \end{cases}$$

The naturality of this definition is apparent from its alternative description: $t(N)$ is the number of fundamental discriminants D such that $D^2|N$. As a variant of the standard notion of squarefree, say that an integer N is *quadfree* if $t(N) = 1$. So N is quadfree if $e := \mathrm{ord}_2(N) \le 3$ and its odd part $N/2^e$ is squarefree.

2.5. **CM newforms and the set $Q_k(N)$ of interest.** We can now define some of the terms used in the introduction. Most newforms g satisfy $g = g^\chi$ only for the trivial character χ. The remaining newforms satisfy $g = g^\chi$ for exactly one non-trivial character $\chi = \chi_D$ and moreover D is negative. Such a newform is said to have CM by D. Necessarily, its level is divisible by D^2. So for most N, we do not encounter CM newforms; for a very few N we do, and then we just discard them.

In practice, the CM newforms to be discarded are immediately recognizable by their Fourier expansions: while coefficients a_p are rarely or perhaps even never zero for a non-CM newform, they are always zero whenever $\chi_D(p) = -1$ for newforms with CM by D. On a rigorous level, we can confirm that a newform with apparent CM by D really does have CM by D by a general structure theorem [Sch09, Theorem 2.4]. Namely quadratic twist-classes of CM newforms in weight $k = 1 + e$ are in bijection with imaginary quadratic fields $\mathbb{Q}(\sqrt{D})$ with class group of exponent dividing e. Also minimal levels are known [Sch09, Table 1].

The classification just mentioned lets one see that the finiteness assertions of Conjecture 1.1 are true to some extent in the parallel situation of CM newforms, but false in their full statement, as follows. For fixed k, the complete list of minimal levels N of CM newforms is sometimes known. For example for $k = 2$, the discriminants D include -3, -4, and -8, with associated minimal levels $N = 27$, 32 and 256. The remaining discriminants are $D = -p$ with $p \in \{7, 11, 19, 43, 67, 163\}$, always with minimal level $N = p^2$. The twenty-six possible D for $k = 4$ and their levels N are likewise listed in [Sch09, Table 3]. For general fixed k, finiteness of the list of N is expected, as it is implied by the Riemann hypothesis for L-functions of odd real Dirichlet characters [Sch09, Theorem 2.1]. For fixed N on the other hand, the set of weights k for which there is at least one rational CM newform with minimal level N can easily be infinite. For example, this set of weights is all positive even integers for the six $N = p^2$ above.

Returning to our main focus, all terms in the definition of $Q_k(N)$ from the introduction have now been defined: $Q_k(N)$ is the set of all twist-classes of non-CM newforms with minimal level N and rational coefficients. Such a class $[g]$ has exactly $t(N)$ representing newforms of level N. In the common case that N is quadfree, it has just one such representative.

2.6. **General character.** It is standard to work in greater generality, defining spaces $M_k(N, \chi)$ for general Dirichlet characters χ and general weights $k \in \mathbb{Z}_{\geq 0}$. These spaces can be nonzero only if the conductor of χ divides N and $\chi(-1) = (-1)^k$. We have summarized standard material in the setting of trivial character only, meaning our $M_k(N)$ agrees with $M_k(N, \chi_1)$.

We are focusing on the case of trivial character because a non-CM newform with rational coefficients necessarily has trivial character. In Section 5 below, we will encounter cases of non-trivial character. We will even allow weights to become half-integral. However our summary here is adequate for following the discussion there.

3. Forced vs. unforced rationality

This section discusses forced versus unforced rationality. We include a small *Magma* program in this section and two more in the next. These programs allow readers not familiar with *Magma* to directly see some of the calculations underlying this paper. Run on small enough arguments, the programs finish in less than the two minutes allowed by the free online *Magma* calculator.

3.1. **The Galois action.** Fix a weight $k \in 2\mathbb{Z}_{\geq 1}$ and a level $N \in \mathbb{Z}_{\geq 1}$. The group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on the corresponding set $P_k(N)$ of newforms by conjugating coefficients. Let p be a prime number not dividing N. Let $f_{k,N,p}(x)$ be the characteristic polynomial of T_p acting on the space $S_k(N)^{\mathrm{new}}$. Then, assuming $f_{k,N,p}(x)$ is separable, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set $P_k(N)$ agrees with its action on the coefficients a_n, these being the roots of $f_{k,N,p}(x)$.

A general program computing the $f_{k,N,p}(x)$ is obtained by concatenating built-in *Magma* commands. First, to obtain output in a standard form, one can introduce the variable x by `_<x>:=PolynomialRing(Integers());`. The general program is then

```
charpol := func<k,N,p|
Factorization(CharacteristicPolynomial(
HeckeOperator(NewSubspace(CuspForms(N,k)),p)
))>;
```

To compute say $f_{50,3,2}(x)$, one inputs all of the above and then `charpol(50,3,2);`. In about a second, one finds that $f_{50,3,2}(x)$ factors as an irreducible quartic times an irreducible quintic.

The action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $P_k(N)$ passes to an action on $Q_k(N)$. In this paper we are interested in the fixed points of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $Q_k(N)$. When N is quadfree, as defined at the end of §2.4, it is simplest to think of $Q_k(N)$ as being simply its single representing form in $P_k(N)$. Similarly for general N, a fixed point on $Q_k(N)$ can be thought of using §2.4 as $t(N)$ fixed points, all twists of one another, on $P_k(N)$.

There are different ways of expressing the Galois action. One convenient way is to let $E_{k,N} \subseteq \mathrm{End}(S_k^{\mathrm{new}}(N))$ be the $\mathbb{Q}$-algebra generated by all the Hecke operators T_p. One has $E_{k,N} = \mathbb{Q}[x]/f_{k,N,p}(x)$ whenever $f_{k,N,p}(x)$ is separable. In this case the factorization of $E_{k,N}$ into fields, which is the issue under study, is exactly reflected in the factorization of $f_{k,N,p}(x)$ into irreducible polynomials.

3.2. **The case $N = 1$.** The Maeda conjecture [HM97, Conj 1.2] says that the image of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ in its action on $P_k(1)$ is always the full symmetric group on the degree d_k. A slightly strengthened version [GM12, Conj 1.1] includes also the separability of all $f_{k,1,p}(x)$. In other words, it says that the Galois group of $f_{k,1,p}(x)$ is always S_{d_k}. The reference [GM12] also proves the strengthened conjecture for $p = 2$ and $k \leq 14000$, and surveys other results related to the conjecture.

In the six cases when $d_k = 1$, the set $P_k(1)$ coincides with its subset of rational forms, these forms having been listed in (2.6). For the k with $d_k \geq 3$, the Maeda conjecture is a stronger statement than Conjecture 1.1's assertion that $Q_k(1)$ is empty. The computed Hecke fields $E_{k,1} = \mathbb{Q}[x]/f_{k,1,2}(x)$ have very large discriminants, rapidly increasing with k. For example, the first case beyond rationality is the quadratic field $E_{24,1}$, and its discriminant is already the fairly large prime number 144169. The large discriminants constitute further heuristic evidence that $f_{k,1,2}(x)$ is always irreducible.

3.3. **Quadfree N.** For general N, the action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $P_k(N)$ stabilizes the Atkin-Lehner subsets $P_k(N)^\epsilon$. A very plausible analog of the Maeda conjecture for general N was formulated in [Tsa14] and strengthened to be numerically more precise and include Galois groups in [DT16]. For the case of quadfree N, it essentially says that, a finite number of exceptional spaces aside, each $P_k(N)^\epsilon$ behaves qualitatively like $P_k(1)$. More precisely, let p be the smallest prime not dividing N. Then, outside of finitely many k, the characteristic polynomial $f^\epsilon_{k,N,p}(x)$ should have Galois group the full symmetric group on its degree.

As an example of generic behavior, consider again the polynomial $f_{50,3,2}(x)$ computed in §3.1. The Galois groups of its irreducible factors are S_4 and S_5, and the field discriminants have 51 and 79 digits. The factorization is completely expected as the summands in the decomposition $P_{50}(3) = P_{50}(3)^+ \coprod P_{50}(3)^-$ have size 4 and 5 respectively.

3.4. **General N.** For N which are not quadfree, there are structures on the set $P_k(N)$ which go beyond the decomposition induced by the Atkin-Lehner operators. To give an indication of this phenomenon, consider the case $N = 25$. Then $P_k(25)$ breaks into six parts, which are conveniently described via twisting by $\chi = \chi_5$.

Write $g \in P_k(25)^{\epsilon}_{\omega}$ if $g \in P_k(25)^{\epsilon}$ and $g^{\chi} \in P_k(25)^{\omega}$. Then

$$P_k(25) = P_k(25)^{\pm} \coprod P_k(25)^{\mp} \coprod P_k(25)^{=} \coprod \text{Rest}.$$

Here we do not particularly care about Rest, as it consists of the forms $g \in P_k(25)^+$ with $f^{\chi} \in P_k(1)$, $P_k(5)^+$, or $P_k(5)^-$. The number of possibilities for $N = 25$ replaced by a general p^e is given in [DT16, Prop 3].

In general, one has a decomposition of $P_k(N)$ into $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ stable summands $P_k(N)^{\delta}$ refining the Atkin-Lehner involution. Just as ϵ is a string of signs indexed by the prime powers exactly dividing N, so too is δ a string of symbols indexed by these prime powers. For example, take $N = 150 = 2 \cdot 3 \cdot 5^2$. The set $P_k(150)$ breaks into $2 \cdot 2 \cdot 6 = 24$ parts, with $2 \cdot 2 \cdot 3 = 12$ corresponding to minimal level 150. For large k, one can expect $f_{k,150,7}(x)$ to factor into twenty-four irreducible polynomials, one for each type δ. However $f_{16,150,7}(x)$ factors into twenty-five irreducible polynomials. Here the number of irreducible polynomials arises as $25 = 24 - 1 + 2$, as follows. Not so interestingly, $P_{16}(6)^{+-}$ being empty implies that one of the δ is not represented. However there is a rational newform with type $\delta' = (+, +, \mp)$, namely

$$g = q - 128q^2 - 2187q^3 + 16384q^4 + 279936q^6 - 511994q^7 + \cdots .$$

Its twist g^{χ_5} has type $\delta'' = (-, -, \pm)$. These two exceptional δ each contribute an extra irreducible factor, as

$$\begin{aligned} f_{16,150,7}^{+,+,\mp}(x) &= (x + 511994)\cdot \\ &\quad (x^3 + 701247x^2 - 5978366987397x + 3322646963771081149) \end{aligned} \tag{3.7}$$

and $f_{16,150,7}^{-,-,\pm}(x) = f_{16,150,7}^{+,+,\mp}(-x)$.

Many $P_k(N)^{\delta}$ cannot possibly have $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ fixed points, because the structures on $P_k(N)^{\delta}$ imply that all factors of the Hecke algebra $E_{k,N}^{\delta}$ contain a specified cyclotomic field larger than $\mathbb{Q}$. This phenomenon is familiar from the case of weight 2 forms and elliptic curves. It holds without change for $k \geq 4$. In particular, rationality of a form in $P_k(N)^{\delta}$ implies that $\mathrm{ord}_2(N) \leq 8$, $\mathrm{ord}_3(N) \leq 5$, and $\mathrm{ord}_p(N) \leq 2$ for larger primes p.

3.5. **The dichotomy.** We can now explain the decomposition of $Q_k(N)$ into its two parts $Q_k^f(N)$ and $Q_k^u(N)$. Let $[g] \in Q_k(N)$ with representing newform $g \in P_k(N)$. We say that the rationality of g is *locally forced*, or simply *forced*, if g is the only non-CM newform in its refined part $P_k(N)^{\delta}$. We write $[g] \in Q_k^f(N)$ in this case and $[g] \in Q_k^u(N)$ otherwise. Exact formulas for $|P_k(N)^{\delta}|$ are not available at the moment, but they are surely within reach. Using these formulas and also formulas or upper bounds for the number of CM newforms, one could compute a cutoff c_k such that $Q_k^f(c_k)$ is nonempty but $Q_k^f(N)$ is empty for $N > c_k$. In fact, we are asserting in Conjecture 1.1 that for $k \in [18, 50]$, the cutoff c_k is the number listed as N_k' on Table 1.1. Similarly, the next paragraph gives strong evidence that $c_{16} = 42$, and (4.10) below suggests further than $(c_{14}, c_{12}, c_{10}) = (42, 90, 210)$.

To see 5=2+2+1 unforced instances of rationality, consider $k = 16$. As reported on Table 1.1, rationality is forced for all the newforms with $N \leq 30$, except for the two with $N = 8$. For just one $N \in [31, 149]$ is the set $Q_{16}(N)$ non-empty, namely $N = 42$, where it has size four. For this exceptional level, $f_{16,42,5}(x)$ factors into four linear and five quadratic irreducible factors. Closer inspection shows that

$|P_{16}(42)^\epsilon| = 1$ for $\epsilon = (-,-,-)$ and $(-,+,+)$, while otherwise $|P_{16}(42)^\epsilon| = 2$. Thus $|Q^f_{16}(42)| = |Q^u_{16}(42)| = 2$. In fact, the source of the unforced rationality is

$$f^{+,+,-}_{16,42,5}(x) = (x + 58290)(x - 296442).$$

For the example presented in the last subsection, the unexpected factorization (3.7) is saying that $|Q^f_{16}(150)| = 0$ and $|Q^u_{16}(150)| = 1$.

3.6. **A heuristic model.** This subsection describes a heuristic model for the factorization of the Hecke algebras $E^\delta_{k,N}$, obtained by considering the factorization of defining polynomials $f^\delta_{k,N,p}(x)$. While the model is very rough, we feel it complements our catalog of newforms by supporting Conjecture 1.1 in a different way.

Fix (k, N, δ) and a prime p not dividing N. The quantity $w = p^{(k-1)/2}$ plays the role of a scaling factor. Let $d = |P_k(N)^\delta|$, so that the monic polynomial $f^\delta_{k,N,p}(x) \in \mathbb{Z}[x]$ has degree d. Its d roots are all real with absolute value at most $2w$. The approximate number of such polynomials has the form

$$V_d(w) = \left[\frac{2^d}{d!} \prod_{j=1}^{d} \left(\frac{2j}{2j-1} \right)^{d+1-j} \right] w^{\Delta(d)}. \tag{3.8}$$

Here $\Delta(d) = d(d+1)/2$ is the d^{th} triangular number and the complicated coefficient is a volume computed in [DH98, Prop 2.2.1].

Now let $r < s$ be positive integers summing to d. The chance that a polynomial in the ensemble of degree d polynomials under consideration factors into a degree r polynomial times a degree s polynomial is approximately $\text{Prob}_{r,s}(w) = V_r V_s / V_d$. If $r = s$, then this formula double counts, and one needs to insert a 2 in the denominator. Applying (3.8) three times and simplifying, one gets

$$\text{Prob}_{r,s}(w) = \left[\frac{d!}{2^{\delta_{rs}} r! s!} \prod_{j=1}^{d} \left(\frac{2j-1}{2j} \right)^{j-1} \right] \frac{1}{w^{rs}}. \tag{3.9}$$

For example, the first non-trivial case is $r = s = 1$. Here the chance that a quadratic polynomial from the ensemble splits is approximately $3/(4w) = 3/(4p^{(k-1)/2})$.

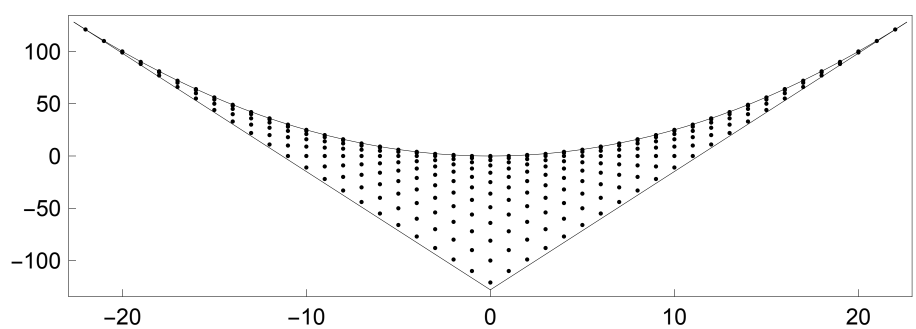

FIGURE 3.1. Illustration in the b-c plane of the heuristic that a quadratic Hecke polynomial for the prime $p = 2$ and weight $k = 6$ has approximately approximately a 1/8 chance of factoring.

The first non-trivial case in more detail goes as follows. The factorizing polynomials are simply $(x - a_1)(x - a_2)$ with $|a_i| \le 2w$ and $a_1 \le a_2$. All allowed polynomials are $x^2 + bx + c$ with $|b| \leq 4w$ and $w(|b| - w) \leq c \leq w^2/2$. The regions of the a_1-a_2 and b-c planes given by the inequalities respectively have area $V_1(w)^2/2 = 8w^2$ and $V_2(w) = 32w^3/3$. Their ratio is indeed $P_{1,1}(w) = 3/(4w)$. Figure 3.1 draws this case for $(p, k) = (2, 6)$ so that $w = 2^{5/2}$. Factoring polynomials are represented as points. There are 276 of them, which is indeed close to the approximation $V_1(w)^2/2 = 256$. The total number of polynomials $x^2 + bx + c$ is 1951 while $V_2(w) \approx 1930.9$.

One way that our model is very rough is that it does not account for the fact that roots of Hecke polynomials should be distributed according an approximation of the Sato-Tate measure. More seriously, the number $\mathrm{Prob}_{r,s}(p^{(k-1)/2})$ depends on p, whereas the factorization behavior of $E^{\delta}_{k,N}$ is independent of p. To proceed further, we take as our heuristic that the chance of $E^{\delta}_{k,N}$ factoring into a degree r algebra times a degree s algebra is $\mathrm{Pr}_{r,s}(k) := \mathrm{Prob}_{r,s}(2^{(k-1)/2})$, whether or not 2 divides N.

Some explicit numerics are as follows. Corresponding to three of the $\underline{\mathbf{2}}$'s on Table 1.1, factorization patterns for $E^{+}_{6,17}$, $E^{-}_{16,8}$, and $E^{+}_{22,3}$ are all $1 + 1$. Corresponding splitting probabilities are $\mathrm{Pr}_{1,1}(k)$ for $k = 6$, 16, and 22, namely 13.3%, 0.4%, and 0.05%. The other $\underline{\mathbf{2}}$ on Table 1.1 comes from $E^{+}_{6,19}$ and $E^{-}_{6,19}$ having splitting behavior $2 + 1$ and $4 + 1$ respectively. The probability here is $\mathrm{Pr}_{2,1}(k)\mathrm{Pr}_{4,1}(k) \approx 5.8\% \cdot 0.24\% = 0.014\%$.

Some qualitative phenomena which the heuristic seems to get right are as follows. First, the chance of splitting decreases rapidly as the weight k increases. Second, for fixed k, this chance also decreases rapidly with increasing degree d. Third, factorizations of the form $d = 1+(d-1)$ are much more common than factorizations of any other type. In fact, we have observed no other factorizations for quadfree N in weights $k \geq 6$ except for $|P_6(3 \cdot 23)^{+-}| = 2 + 4$, $|P_6(3 \cdot 5 \cdot 17)^{++-}| = 3 + 4$, $|P_6(455)^{+--}| = 2+10$, and $|P_8(3\cdot 17)^{--}| = 2+4$. This third phenomenon is one of the reasons that this paper concentrates not on general factorizations, but rather on just those which produce newforms with rational coefficients.

On a more quantitative level, there are definitely more factorizations than the heuristic predicts. In weight two, the heuristic correctly predicts that there are infinitely many elliptic curves, but considerably underestimates the number of elliptic curves per level. As examples in higher weight, one of the rational newforms discussed in §4.4 has associated probability $\mathrm{Pr}_{1,12}(10) \approx 2.2\times 10^{-16}$ while one from §7.2 corresponds to the even smaller number $\mathrm{Pr}_{1,83}(4) \approx 3.4 \times 10^{-37}$. One is thus led to ask for a conceptual source of rational newforms such as these, something we will briefly pursue in §7.2.

4. Assembling the main table

In this section we discuss how we drew up Table 1.1, as well as its unprinted extension to a larger region in the N-k plane. We present a number of examples similar to those of the last subsection, but now with more reference to Table 1.1 itself.

4.1. **Computing the cardinality $|Q_k(N)|$.** Combining several built-in *Magma* functions, we define a new one:

```
RatNewforms := func<k,n|
[fs : fs in Newforms(CuspForms(Gamma0(n),k))|#fs eq 1]>;
```

Following this definition by `RatNewforms(22,3)` then returns approximations to the two elements in $Q_{22}(3)$:

$$\begin{aligned} f_1 &= q + 1728q^2 - 59049q^3 + 888832q^4 - 41512770q^5 - \cdots, \\ f_2 &= q - 2844q^2 - 59049q^3 + 5991184q^4 + 3109950q^5 + \cdots. \end{aligned}$$

This particular computation takes less than a second and accounts for the $\underline{\mathbf{2}}$ in the row 3 column 22 of Table 1.1. On the other hand, for N a highly factorizing level close to our cutoff C_k, `RatNewforms(`k`,`N`)` takes around an hour.

To convert the output of `RatNewforms(`k`,`N`)` into cardinalities $Q_k(N)$ one needs to take quadratic twisting into account. If no CM newform is present, for example if a prime exactly divides N, then this cardinality is the number of newforms returned, divided by the number $t(N)$ of twists allowed from §2.4. When CM newforms are present, they are in practice easily recognized by their a_p being zero for many p. We used the theory of CM newforms as summarized in §2.5 to confirm that CM is really present, discarded the forms, and then divided the number of remaining forms by $t(N)$.

4.2. **Identifying the decomposition** $Q_k(N) = Q_k^f(N) \coprod Q_k^u(N)$**.** In the example of the previous subsection, both coefficients of q^3 are -3^{10}, so by (2.2) they both belong to $S_{22}^{\text{new}}(N)^+$. They thus contribute to the unforced part $Q_{22}^u(3)$ of $Q_{22}(3)$. To systematically separate the forced part from the unforced part in the case of quadfree N, we used the following refinement of our previous `charpol`:

```
charpol2 := function(k,N,signs,p)
fN := Factorization(N); fN2 := [g[1]^g[2]:g in fN];
New := NewSubspace(CuspForms(N,k));
return Factorization(CharacteristicPolynomial(
HeckeOperator(New,p)*(&*[(AtkinLehnerOperator(New,fN2[i])^2+
signs[i]*AtkinLehnerOperator(New,fN2[i]))/2: i in [1..#fN2]])));
end function;
```

Running `charpol2(22,3,[1],2)` and then `charpol2(22,3,[-1],2)` lets one conclude $\det(x - T_2|S_{22}(3)^+) = (x-1728)(x+2844)$ and $\det(x - T_2|S_{22}(3)^-) = x^2 - 666x - 2464992$. This computation shows again that rationality of these two newforms is unforced. Some non-quadfree cases can also be done via this program, as illustrated by the example of $(k,N) = (16,150)$ in §3.4.

4.3. **Sample calculations at small level.** Table 4.2 gives the result of running `charpol2` for three hundred different (k, N, ϵ). All polynomials obtained were irreducible, except for the cases $(22,3,+)$ and $(16,8,-)$ where the result was a factorizing quadratic. On the table, q is either 2, 4, or 8.

The column m gives the mass belonging to (N,ϵ), calculated as a product of local masses (2.5). The absence of a line for $(N,\epsilon_4) = (4,+)$ reflects the vanishing $m^+(2,2) = 0$ mentioned after (2.5). The many **1**'s on the lines for $N = 2$ arise from the small value $m^{\pm}(2,1) = 1/2$. The similarly many **1**'s for each line belonging to $N = 6$ arise because $m^{\pm}(2,1) = 1/2$ is not increased by the second factor $m^{\pm}(3,1) = 1$.

Table 4.2 clarifies the lines corresponding to $N = 2$, 3, 4, 6, and 8 of Table 1.1. More importantly, it serves as an overview of the newforms studied in the next two sections.

N ϵ_q ϵ_3	m	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50
1	1						**1**		**1**	**1**	**1**	**1**	2	**1**	2	2	2	2	3	2	3	3	3	3	4	3
2 +	0.5				**1**			**1**	**1**		**1**	**1**	**1**	**1**	**1**	**1**	2	**1**	**1**	2	2	**1**	2	2	2	2
2 −	0.5					**1**		**1**		**1**	**1**	**1**		2	**1**	**1**	**1**	2	**1**	2	**1**	2	2	2	**1**	3
3 +	1				**1**	**1**	**1**	**1**	**1**	**1**	2	**2**	2	2	2	2	3	3	3	3	3	3	4	4	4	4
3 −	1			**1**		**1**		2	**1**	2	**1**	2	**1**	3	2	3	2	3	2	4	3	4	3	4	3	5
4 −	1			**1**		**1**	**1**	**1**	**1**	2	**1**	2	2	2	2	3	2	3	3	3	3	4	3	4	4	4
6 + +	0.5		**1**				**1**		**1**	**1**	**1**		**1**	**1**	2	**1**	**1**	**1**	2	**1**	2	2	2	**1**	2	2
6 + −	0.5					**1**	**1**			**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	2	2	**1**	**1**	2	2	2	2	2
6 − +	0.5			**1**				**1**	**1**	**1**		**1**	**1**	**1**	**1**	2	**1**	**1**	**1**	2	2	2	**1**	2	2	2
6 − −	0.5				**1**		**1**		**1**		**1**	**1**	2		**1**	**1**	2	**1**	2	**1**	2	**1**	2	2	3	**1**
8 +	1.5		**1**		**1**	**1**	2	**1**	2	2	3	2	3	3	4	3	4	4	5	4	5	5	6	5	6	6
8 −	1.5			**1**	**1**	**1**	**1**	2	**2**	2	2	3	3	3	3	4	4	4	4	5	5	5	5	6	6	6

TABLE 4.2. Dimensions of Atkin-Lehner subspaces of $S_k(N)^\epsilon$, with sources of rational newforms highlighted in bold.

4.4. **Sample computations at larger level.** To represent our computations for larger level, consider the weights 10, 12, 14, and 16 just below the weights $k \geq 18$ where Conjecture 1.1 becomes effective. For these weights, the sequences of observed minimal levels, with multiplicities, end as follows:

$$\begin{aligned} k = 10: &\quad \ldots, 210, \mathbf{210}, \mathbf{210}, \mathbf{285}, \mathbf{294}, \mathbf{330}; \\ k = 12: &\quad \ldots, 90, \mathbf{96}, \mathbf{114}; \\ k = 14: &\quad \ldots, 42, \mathbf{60}; \\ k = 16: &\quad \ldots, 42, 42, \mathbf{42}, \mathbf{42}, \mathbf{150}. \end{aligned} \tag{4.10}$$

We discussed the case $k = 16$ in §3.4 and §3.5. The other cases are similar, with ordinary type indicating forced rationality and boldface unforced rationality. For example, the forced 210 comes from the one-element set $P_{10}(210)^{----}$, while the unforced **210**, **210** comes from the two-element set $P_{10}(210)^{+--+}$. The largest set $P_k(N)^\delta$ where we have observed splitting in $k \geq 10$ is $P_{10}(285)^{-+-}$, which splits as $1 + 12$.

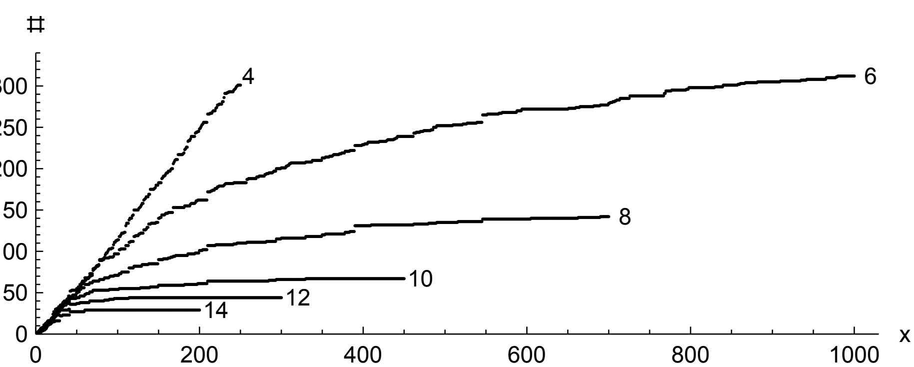

FIGURE 4.2. Graphs of the summatory functions $\#_k(x) = \sum_{N \leq x} |Q_k(N)|$, for $k = 4$, 6, 8, 10, 12, and 14.

Figure 4.2 graphs the summatory functions indicated by its caption. One has $\#_2(1000) = 1612$ and $\#_4(1000) = 802$, the cases $k = 2$ and $k = 4$ being not drawn and partially drawn respectively. For $k = 6$ and $k = 8$, the graphs are still slightly rising as x reaches the cutoff C_k. For $k = 10$, 12, and 14, the complete flattening of the graph makes it plausible that the last minimal level seen N'_k is indeed the last minimal level N_k. The undrawn graph for $k = 16$ almost coincides with the drawn graph for $k = 14$, although for $k = 16$ the last minimal level seen $N'_{16} = 150$ is close to the cutoff $C_{16} = 200$.

5. Rings $M(N)$ for small N

According to Conjecture 1.1, non-CM newforms in large weight k rarely have rational coefficients. Our viewpoint is that those newforms which do have rational coefficients are of particular interest and deserve to be exhibited explicitly, in the style of Ramanujan's formulas (2.6). We present some such explicit formulas here, systematically working in analogs of the classical ring $M(1) = \mathbb{C}[Q, R]$. As a general convention, if $P_k(N)^\epsilon$ has just one element, then we call it $\Delta^\epsilon_{k,N}$.

We treat the cases $N = 2$, 3, 4, 6, and 8. The new rings $M(N)$ contain $M(1)$ with indices 3, 4, 6, 12, and 12. We deal with the greater complexity by first embedding these rings into yet larger rings $\widehat{M}(N)$, all of which, like $M(1)$, are free on two generators. We then keep formulas concise by systematically exploiting Atkin-Lehner operators in the cases $N = 2$, 3, 6, and 8 and by discarding old forms cleanly in the cases $N = 4$ and 8.

An overall theme is that the five cases we treat are remarkably similar to the classical case $M(1)$. For example, the cuspidal ideal $S(N)$ is always generated by an η-product analogous to the generator $\Delta_{12,1} = \eta^{24}$ of $S(1)$. As analogs of Ramanujan's six newforms, Table 4.2 for $N = 2$, 3, 4, 6, and 8 lists 23, 13, 6, 47, and 10 newforms. We keep things relatively brief by giving explicit formulas only for a small subset of these newforms. Proofs of all statements are straightforward and omitted. To proceed similarly for other N, *Magma*'s `Relations` command would be very useful.

5.1. **Theta series.** We build all our modular forms from two types of theta series,

$$\Theta = \sum_{(x,y)\in\mathbb{Z}^2} q^{x^2+xy+y^2} \qquad = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \cdots ,$$

$$\theta = \sum_{x\in\mathbb{Z}} q^{x^2} \qquad = 1 + 2q + 2q^4 + 2q^9 + 2x^{16} + \cdots .$$

These theta series are modular forms in their own right, with weights 1 and 1/2 respectively, and certain characters. Using standard notation, $\Theta \in M_1(3, \chi_{-3})$ and $\theta \in M_{1/2}(4, \chi_1)$. As mentioned in §2.6, these two forms are outside of the context of the main review in Section 2. However, we are using them only to build forms which are in that context.

Using the second push-up operator $q \mapsto q^2$ of §2.2, which does not change the weight, we obtain the graded rings

$$\widehat{M}(6) = \mathbb{C}[\Theta_1, \Theta_2], \qquad \widehat{M}(8) = \mathbb{C}[\theta_1, \theta_2].$$

The rings of §2.1 are then the subrings

$$M(6) = \mathbb{C}[\Theta_1^2, \Theta_1\Theta_2, \Theta_2^2], \qquad M(8) = \mathbb{C}[\theta_1^4, \theta_1^2\theta_2^2, \theta_2^4].$$

These rings relate to their common subring $M(1)$ via

$$\begin{aligned} Q &= \Theta_1\left(5\Theta_1^3+12\Theta_1^2\Theta_2-16\Theta_2^3\right) \\ &= \theta_1^8+56\theta_1^6\theta_2^2-40\theta_1^4\theta_2^4-32\theta_1^2\theta_2^6+16\theta_2^8, \end{aligned}$$

$$\begin{aligned} R &= \left(\Theta_1^2+2\Theta_1\Theta_2-2\Theta_2^2\right)\left(-11\Theta_1^4-20\Theta_1^3\Theta_2+16\Theta_1\Theta_2^3+16\Theta_2^4\right) \\ &= \left(\theta_1^4+4\theta_1^2\theta_2^2-4\theta_2^4\right)\left(\theta_1^8-136\theta_1^6\theta_2^2+152\theta_1^4\theta_2^4-32\theta_1^2\theta_2^6+16\theta_2^8\right). \end{aligned}$$

The four formulas just displayed are already a little bit long. In our treatment of the various $M(N)$, we are exploiting structure to keep analogous formulas as short as possible.

5.2. **Level 2.** The ring $M(2)$ is freely generated by

$$\begin{aligned} A &= \theta_1^4+4\theta_1^2\theta_2^2-4\theta_2^4 && \in M_2(2)^-, \\ B &= \theta_1^8-24\theta_1^6\theta_2^2+40\theta_1^4\theta_2^4-32\theta_1^2\theta_2^6+16\theta_2^8 && \in M_4(2)^-. \end{aligned}$$

The cuspidal ideal $S(2)$ is generated by

$$\Delta_{8,2}^+ = \eta_1^8\eta_2^8 = 2^{-8}(A^4-B^2) \in M_8(2)^+.$$

Table 4.2 shows 23 rational newforms in $S(2)$, all of which have explicit expressions as polynomials in A and B. The one of largest weight is

$$\begin{aligned} \Delta_{48,2}^- \quad = \quad & 2^{-16}A^2B\left(49A^4-81B^2\right)\left(25A^4-9B^2\right)\cdot \\ & \left(375531625A^8-755257890A^4B^2+379726137B^4\right)\Delta_{8,2}^+. \end{aligned}$$

Here the fact that the total degree in A and B has to be odd cuts the number of terms roughly by half. Throughout this section, there are many striking factorizations such as the one just displayed; we are not pursuing their meaning.

5.3. **Level 3.** Here we work in the graded ring $\widehat{M}(3) = \mathbb{C}[\Theta,\Phi]$ with $\Phi = 4\Theta_2^3 - 3\Theta_1^2\Theta_2 = 1-36q-54q^2-252q^3+\cdots \in M_3(3,\chi_{-3})$. The even part of this ring is exactly $M(3)$. The order four automorphism given by $\Theta \mapsto i\Theta$ and $\Phi \mapsto i\Phi$ restricts to the Atkin-Lehner involution w_3 on $M(3)$. The cuspidal ideal $S(3)$ is generated by

$$\Delta_{6,3}^- = \eta_1^6\eta_3^6 = 2^{-2}3^{-3}(\Theta^6-\Phi^2) = q-6q^2+9q^3+4q^4+\cdots.$$

The newform $\Delta_{6,3}^-$ is the first of the thirteen rational newforms with level 3 on Table 4.2. The newforms from §4.1 exhibiting unforced rationality are

$$\begin{aligned} \Delta_{22,3}^{+a} &= 6^{-1}\Theta\Phi \quad \left(75\Theta^{12}-44\Theta^6\Phi^2-25\Phi^4\right) \quad \Delta_{6,3}^- &&= q+1728q^2-\cdots, \\ \Delta_{22,3}^{+b} &= 3^{-3}\Theta\Phi\left(-869\Theta^{12}+1072\Theta^6\Phi^2-176\Phi^4\right)\Delta_{6,3}^- &&= q-2844q^2-\cdots. \end{aligned}$$

The difference of the coefficients of q^2 is $4572 = 2^2 3^2 127$. Computation quickly suggests that in fact $\Delta_{22,3}^{+a} \equiv \Delta_{22,3}^{+b}$. In fact, seeing this congruence on the coefficient of q^n for $n \le 11$ suffices to confirm the general congruence, by a version of Sturm's theorem [CKR10, Prop. 1].

5.4. **Level 4.** Here again we work in a larger graded ring $\widehat{M}(4) = \mathbb{C}[\theta, D]$, with θ in weight 1/2 and $D = \theta_1^4 - 8\theta_1^2\theta_2^2 + 8\theta_2^4 = 1 - 24q + 24q^2 - \cdots$ in weight 2. The graded ring $M(4)$ is then just the sum of the graded pieces indexed by even integers. Let ρ be an eighth root of unity. Then the automorphism $\theta \mapsto \rho\theta, D \mapsto D$ restricts to the Atkin-Lehner involution w_4 on $M(4)$. One thus has $M(4) = \mathbb{C}[C, D]$ with $C := \theta^4 \in M_2(4)^-$ and $D \in M_2(4)^+$. Note that our presentation is a variant of [Kob93, IV.1 Prop. 4], which uses the generators θ and $F = 2^{-5}(\theta^4 - D)$.

For k an odd integer, one has $\widehat{M}_k(4) = M_k(4, \chi_{-4})$. For example, the element $\Delta_{5,4} = 2^{-6}\theta^2(\theta^8 + D^2) = \eta_1^4\eta_2^2\eta_4^4 = q - 4q^2 + 16q^4 - \cdots$ of $\widehat{M}_5(4)$ is a cuspidal newform with character χ_{-4} and CM. Multiplying this element by θ^2 we get a non-CM newform which fits into our framework,

$$\Delta_{6,4}^- = \eta_2^{12} = 2^{-6}C(C^2 - D^2) = q - 12q^3 + 54q^5 - 88q^7 - \cdots . \tag{5.11}$$

Thus one has the unusual situation of two quite different eta-products with quotient just θ^2.

The $N = 4$ block of Table 4.2 has two interesting features. First, as commented already after (2.5), the $w_4 = +$ part is zero and thus missing from the block. But second, the entries on the $w_4 = -$ line are exactly those of the familiar $N = 1$ line, shifted to the left by a weight difference of six. In fact, newforms on $N = 4$ nicely separate from old forms via the formula

$$S_k^{\text{new}}(4)^- = M_{k-6}(1)_2\Delta_{6,4}^-. \tag{5.12}$$

The source of this equation is that the group S_3 acts on the graded ring $M(4)$ with quotient $M(1)_2$, with the isotypical component corresponding to the sign character of S_3 being exactly $S^{\text{new}}(4)^-$.

To be more explicit about (5.12), one has $M(1)_2 = \mathbb{C}[Q_2, R_2]$ with $Q_2 = 2^{-2}(3\theta^8 + D^2)$ and $R_2 = 2^{-3}D(9\theta^8 - D^2)$. Ramanujan's six rational forms (2.6) in $S(1)$ becomes six rational forms in $S^{\text{new}}(4)^-$ via the simultaneous replacements $\Delta_{12,1} \mapsto \Delta_{6,4}^-$, $Q \mapsto Q_2$, and $R \mapsto R_2$. For example, the largest weight rational form on Table 4.2 is $\Delta_{20,4}^- = Q_2^2R_2\Delta_{6,4}^-$.

5.5. **Level 6.** The ring $\widehat{M}(6) = \mathbb{C}[\Theta_1, \Theta_2]$ and its even weight subring $M(6)$ have already been introduced in §5.1. The space $M_2(6)$ has dimension three, with a basis consisting of a sum, product, and difference:

$$s = \Theta_1^2 + 2\Theta_2^2 \in M_2(6)^{+-}, \qquad p = \Theta_1\Theta_2 \in M_2(6)^{-+}, \qquad d = \Theta_1^2 - 2\Theta_2^2 \in M_2(6)^{--}.$$

Via the equation $d^2 - s^2 = 8p^2$, every element in $M(6)$ can be written in the canonical form $f_1(s,p) + df_2(s,p)$.

The cuspidal ideal $S(6)$ is generated by

$$\Delta_{4,6}^{++} = \eta_1^2\eta_2^2\eta_3^2\eta_6^2 = 2^{-2}3^{-2}(9p^2 - s^2).$$

Illustrations of how $\Delta_{4,6}^{++}$ is indeed a generator include

$$\Delta_{6,6}^{-+} = p\Delta_{4,6}^{++}, \qquad \Delta_{8,6}^{--} = sp\Delta_{4,6}^{++}, \qquad \Delta_{10,6}^{+-} = 2^{-1}3^{-2}(5s^2 - 39p^2)s\Delta_{4,6}^{++}.$$

None of these expressions involve the generator d. However the canonical expression for the rational newform of largest weight does:

$$\Delta_{50,6}^{--} = 2^{-2}3^{-9}d\cdot$$
$$\big(140349306081007255050000p^{22} - 1116591205016604926700000p^{20}s^2$$

$$
\begin{aligned}
&+27589681151783316300150p^{18}s^4 + 2577120214736187574830p^{16}s^6 \\
&-3234565067472714047760p^{14}s^8 + 921682623552505460496p^{12}s^{10} \\
&-149165289449290130931p^{10}s^{12} + 15554206382841117045p^8s^{14} \\
&-1070217851875219680p^6s^{16} + 47245789680492400p^4s^{18} \\
&-1218365734678125p^2s^{20} + 14004203846875s^{22}\big)\,\Delta^{++}_{4,6}.
\end{aligned}
$$

In comparison with the newform of the next largest weight, $\Delta^-_{48,2}$ from §5.2, the expression for $\Delta^{--}_{50,6}$ is much longer. Like for the case $N=2$, our treatment of $N=6$ fully exploits the Atkin-Lehner operators, but does not cleanly discard old forms. The difference in complexity can be attributed to the fact that asymptotically 1/3 of $\dim(S_k(2)^-)$ comes from newforms but only 1/6 of $\dim(S_k(6)^{--})$ does. The complexity of the displayed expression underscores the usefulness of discarding old forms cleanly, as in (5.12) for $N=4$ and (5.13) for $N=8$.

5.6. **Level** 8. The ring $\widehat{M}(8) = \mathbb{C}[\theta_1, \theta_2]$ has already been described in §5.1. Its integral weight part is generated by three forms in weight one, $\theta_1^2, \theta_2^2 \in M_1(8, \chi_{-4})$ and $\theta_1\theta_2 \in M_1(8, \chi_{-8})$. The ring $M(8) = \sum_k M_k(8, \chi_1)$ constitutes half of the even weight part of $\widehat{M}(8)$. A monomial $\theta_1^i\theta_2^j$ with total weight $(i+j)/2$ is in $M(8)$ if both i and j are even, and in the other half $\sum_k M_k(8, \chi_8)$ if both i and j are odd.

The cuspidal ideal $S(8)$ of $M(8)$ is generated by

$$\Delta^+_{8,4} = \eta_2^4\eta_4^4 = 2^{-2}\theta_1^2\theta_2^2(-\theta_1^2+\theta_2^2)(\theta_1^2-2\theta_2^2) = q - 4q^3 - 2q^5 + 24q^7 - \cdots .$$

Asymptotically, $S_k^{\mathrm{new}}(8)$ has one-quarter the dimension of $S_k(8)$. Analogously to (5.12), each Atkin-Lehner eigenspace on newforms can be conveniently isolated via

$$S_k^{\mathrm{new}}(8)^\epsilon = M(2)_2^\epsilon \Delta^+_{8,4}. \tag{5.13}$$

So all the **1**'s in the $N=8$ block of Table 4.2 have monomial formulas of the form $E\Delta^+_{8,4}$, with E as follows:

	4	6	8	10	12	14
$Q_k(8)^+$	1		A_2^2	A_2B_2		$A_2^3B_2$.
$Q_k(8)^-$		A_2	B_2	A_2^3	$A_2^2B_2$	

The unforced splitting in weight $k=16$ yielding a $\underline{\mathbf{2}}$ on Table 1.1 and a **2** on Table 4.2 is given by

$$
\begin{aligned}
\Delta^{-a}_{16,8} &= 2^{-1}B_2(23A_2^4 - 21B_2^2)\Delta^+_{8,4} &&= q + 2700q^3 - 251890q^5 + \cdots, \\
\Delta^{-b}_{16,8} &= 2^{-1}B_2(-25A_2^4 + 27B_2^2)\Delta^+_{8,4} &&= q - 3444q^3 + 313358q^5 - \cdots .
\end{aligned}
$$

Applying [CKR10, Prop. 1] as we did at the end of §5.3, one has $\Delta^{-a}_{16,8} \equiv \Delta^{-b}_{16,8}$ modulo $6144 = 2^{11}3$.

6. Reductions modulo ℓ and associated number fields

In general, the arithmetic of coefficients of newforms is governed by Galois representations which in turn can be described in terms of number fields. In particular, for a rational newform $\sum a_n q^n$, the reductions $a_n \in \mathbb{Z}/\ell^e$ are determined by a number field with Galois group inside $GL_2(\mathbb{Z}/\ell^e)$. In this section, we briefly discuss our ninety-nine newforms from this point of view. Our goal is to make clear that the newforms here are a rich source of examples, in a way which complements the much studied $N=1$ case [SD73, Bos11].

6.1. **Overview.** In the interest of brevity, we restrict to the cases $\ell^e \in \{2, 3, 5, 7\}$. For further brevity, we focus on number fields with Galois group surjecting onto

	number of fields						number of newforms governed					
$\ell \setminus N$	1	2	3	4	6	8	1	2	3	4	6	8
2												
3						1						10
5			1	1	1	2			7	6	24	7, 3
7		1	2	1	2	4		12	8, 3	6	24, 12	2, 4, 2, 2
Total number of forms:							6	23	13	6	47	10

TABLE 6.3. Summary of examples of number fields governing newforms

$PGL_2(\mathbb{F}_\ell)$. We call the other cases degenerate. The numbers of $PGL_2(\mathbb{F}_\ell)$ number fields involved in various parts of this section, blanks indicating 0, are indicated in the left half of Table 6.3. Thus the classical case $N = 1$ gives no examples in our restricted context. This degeneracy continues somewhat into our setting $N \in \{2, 3, 4, 6, 8\}$. However as ℓN increases, number fields with Galois group all of $PGL_2(\mathbb{F}_\ell)$ begin to appear. In §6.3 we give defining polynomials in all cases.

A remarkable feature of our collection of examples, indicated in the right half of Table 6.3, is that each of the sixteen number fields governs more than one rational newform. The numbers listed in the right half follow the alphabetic order of the labeling in §6.3. For example, of the thirteen newforms for $N = 3$, eight and three are governed modulo 7 by F_{3a} and F_{3b} respectively; the remaining two newforms are thus degenerate modulo 7.

6.2. **The projective correspondence for $\ell \leq 7$.** Knowledge of the a_p for a newform $\sum a_n q^n \in S_k^{\mathrm{new}}(N)$, for p running over prime numbers, determines the entire newform via the explicit formula

$$\sum_{n=1} \frac{a_n}{n^s} = \prod_{p|N} \frac{1}{1 - a_p p^{-s}} \cdot \prod_{p \nmid N} \frac{1}{1 - a_p p^{-s} + p^{k-1-s}}. \tag{6.14}$$

Accordingly, we focus attention on the a_p. When studying the reductions of the a_p to $\mathbb{F}_\ell$, we exclude the p dividing $N\ell$, which behave differently.

The a_p are governed by a Galois representation into $GL_2(\mathbb{F}_\ell)$ which we take to be semisimple, making it well-defined. To simplify, we will be working mainly not with the a_p themselves, rather with their normalized squares $s_p = a_p^2/p^{k-1} \in \mathbb{Q}$. Let $f(x) \in \mathbb{Z}[x]$ be a degree $\ell + 1$ polynomial capturing the associated Galois representation into $PGL_2(\mathbb{F}_\ell)$. Then the s_p, considered in $\mathbb{F}_\ell$, are correlated with the partitions λ_p giving the degrees of the irreducible factors of $f(x)$ over $\mathbb{Q}_p$. For $\ell \leq 7$, these correlations are as follows:

	$\ell = 2$		$\ell = 3$			$\ell = 5$					$\ell = 7$						
s_p	0	1	0	1	2	0	1	2	3	4	0	1	2	3	4	5	6
λ_p	21	3	22	31	4	222	33	411	6	51	2222	3311	44	611	71	8	8
	1^3		211	1^4	4	2211			6	1^6	22211				1^8		

The Galois group of $f(x)$ is all of $PGL_2(\mathbb{F}_\ell)$ if and only if all s_p arise.

In our restricted setting of ninety-nine newforms and four residual primes, a mod ℓ projective representation is either surjective or has cyclic image. In the latter case, the newform moreover satisfies a general congruence of the form

$$a_p \equiv p^i + p^j \ (\ell). \tag{6.15}$$

Thus our restriction to surjective representations corresponds to concentrating on the more mysterious cases.

6.3. **Explicit polynomials.** For any of the ninety-nine newforms, all of the a_p with $p \nmid 2N$ are even. Thus s_p is always 0 in $\mathbb{F}_2$ and the projective Galois representations into $PGL_2(\mathbb{F}_2) = S_3$ are all non-surjective. For $\ell = 3$, all newforms with $N < 8$ are likewise degenerate, but the ten newforms at $N = 8$ are all nondegenerate and governed by

$$\phi_8(x) = x^4 - 2x^3 - 6x + 3, \qquad \delta_8 = -2^4 3^5.$$

Here and later, next to each displayed polynomial $f(x)$ we show also the discriminant of the number field $\mathbb{Q}[x]/f(x)$. In fact, for a given k, N, and ℓ, theory, as partially summarized in §6.4 below, gives only a small list of possibilities for these field discriminants. The polynomials we display were found on the database [JR14]. Matching as in §6.2 for all small p makes it very likely that the $f(x)$ are correct. For $\ell \leq 5$, the completeness of the database confirms correctness. For all ℓ, correctness is confirmed using the Serre conjecture [Ser87, KW09a, KW09b] which implies that any $PGL_2(\mathbb{F}_\ell)$ field which is not totally real will appear already in weight $k \leq \ell+1$.

For $\ell = 5$, Table 6.3 says there are five fields. Indexing by the relevant level as we did before, defining polynomials with Galois group $PGL_2(\mathbb{F}_5)$ and the indicated discriminant are

$$\begin{aligned}
f_3(x) &= x^6 - x^5 + 5x^4 - 5x^2 + 16x - 1, & d_3 &= 3^4 5^9,\\
f_4(x) &= x^6 - x^5 + 5x^3 + 10x^2 - 27x - 23, & d_4 &= 2^4 5^9,\\
f_6(x) &= x^6 - x^5 + 30x^3 - 15x^2 + 3x + 222, & d_6 &= 2^4 3^4 5^9,\\
f_{8a}(x) &= x^6 - 2x^5 - 8x - 4, & d_{8a} &= 2^6 5^7,\\
f_{8b}(x) &= x^6 - 2x^5 + 10x + 5, & d_{8b} &= 2^6 5^9.
\end{aligned}$$

One can sometimes describe the situation much more completely while still remaining brief. For example at $N = 6$, the polynomial f_6 governs all twenty-four forms with $\epsilon_2\epsilon_3 = -1$, while all twenty-three forms with $\epsilon_2\epsilon_3 = 1$ are degenerate.

For $\ell = 7$, Table 6.3 says there are ten fields. Defining polynomials with Galois group $PGL_2(\mathbb{F}_7)$ and the indicated discriminant are

$$\begin{aligned}
F_2(x) &= x^8 - x^7 - 196x^2 + 28x - 28, & D_2 &= -2^6 7^{13},\\
F_{3a}(x) &= x^8 - 4x^7 + 21x^4 - 21x^2 - 15x - 3, & D_{3a} &= -3^6 7^{11},\\
F_{3b}(x) &= x^8 - 3x^7 - 7x^6 + 49x^5 + 42x^4, & D_{3b} &= -3^6 7^{13},\\
F_4(x) &= x^8 - x^7 - 7x^6 + 7x^2 - 27x - 1, & D_4 &= -2^4 7^{11},\\
F_{6a}(x) &= x^8 - 2x^7 + 42x^4 - 126x^3 + 84x^2 + 66x - 48, & D_{6a} &= -2^6 3^6 7^9,\\
F_{6b}(x) &= x^8 - x^7 + 21x^6 + 21x^5 - 21x^4 + 945x^3 & D_{7b} &= -2^6 3^6 7^{13},\\
&\quad - 441x^2 + 45x + 3168, & &\\
F_{8a}(x) &= x^8 - 2x^7 + 7x^4 - 14x^2 + 8x + 5, & D_{8a} &= -2^8 7^9,
\end{aligned}$$

$$\begin{aligned} F_{8b}(x) &= x^8 - 2x^7 - 14x^4 + 28x^2 - 60x + 92, & D_{8b} &= -2^8 7^{11}, \\ F_{8c}(x) &= x^8 - 2x^7 + 14x^6 + 42x^5 + 140x^4 & D_{8c} &= -2^8 7^{13}, \\ &\quad + 266x^3 + 322x^2 + 222x - 157, & & \\ F_{8d}(x) &= x^8 - 2x^7 + 49x^4 - 196x^2 - 140x - 63, & D_{8d} &= -2^8 7^{13}. \end{aligned}$$

All twelve newforms in $M(2)^+$ are governed by F_2, while the eleven newforms in $M(2)^-$ are all degenerate. The twenty-four newforms in $M(6)$ with $\epsilon_2 = +$ are governed by F_{6a}; of those in $M(6)$ with $\epsilon_2 = -$, twelve are governed by F_{6b} and eleven are degenerate.

6.4. **Field discriminants and ramification.** General facts, some used to find the polynomials of the previous subsection, are illustrated by the displayed discriminants D. Certainly, all primes dividing the discriminant must divide $N\ell$ and for odd ℓ the discriminant must be a square times $\chi_{-4}(\ell)\ell$.

Ramification at ℓ is directly related to weight. As mentioned earlier, all polynomials necessarily arise already from newforms in weight $k \leq \ell + 1$. In fact if $\mathrm{ord}_\ell(D) \geq \ell + 2$, then in this range the polynomial arises only in weight $\mathrm{ord}_\ell(D) + 2 - \ell$. Thus in the reduction bijection from $\{\Delta^+_{4,8}, \Delta^-_{6,8}, \Delta^+_{8,8}, \Delta^-_{8,8}\}$ to $\{f_{8a}, f_{8b}, f_{8c}, f_{8d}\}$, discriminants force the correspondences $\Delta^+_{4,8} \leftrightarrow f_{8a}$ and $\Delta^-_{6,8} \leftrightarrow f_{8b}$. In fact, the only ambiguity as to the canonical lowest weight source of each of our fifteen polynomials is the rest of this bijection. For $\Delta^+_{8,8}$ and $\Delta^-_{8,8}$, one has $s_3 = 6$ and 0 respectively. For f_{8c} and f_{8d}, one has $\lambda_3 = 8$ and 22211 respectively. By §6.2, the bijection is completed by $\Delta^+_{8,8} \leftrightarrow f_{8c}$ and $\Delta^-_{8,8} \leftrightarrow f_{8d}$.

In general, ramification at primes p different from ℓ is directly related to the refinement of the Atkin-Lehner decomposition discussed in §3.4. With our tiny levels, we see only a small part of this complicated theory. If p exactly divides N, then $\mathrm{ord}_p(D)$ is usually $\ell - 1$ but exceptionally can be zero. For the ten instances in §6.3, it is always $\ell - 1$. If $\mathrm{ord}_2(N)$ is 2 or 3, then the 2-decomposition group has to be S_3 and S_4 respectively; these nonabelian subgroups exclude the simple abelian behavior (6.15); they partially explain why all newforms at these levels 4 and 8 have surjective mod 5 and mod 7 projective representations. For $\mathrm{ord}_2(N) = 3$, the slope content as in [JR06] has to be $[4/3, 4/3]_3^2$, as opposed to the other possibility for S_4 2-adic fields, $[8/3, 8/3]_3^2$. This restriction accounts for the small exponents on 2 in the seven polynomials in §6.3 associated to $N = 8$.

6.5. **Congruences.** A necessary condition for two rational newforms with the same level N to reduce to the same power series in $\mathbb{F}_\ell[[q]]$ is that their projective mod ℓ representations coincide and their weights are congruent modulo $\ell - 1$. This condition is sufficient for $\ell = 2$, but not for $\ell > 2$, as there is still a sign ambiguity in each of the a_p. For example, the two newforms $\Delta^\pm_{8,8}$ satisfy the necessary condition, but differ via twisting by χ_{-3}, as illustrated by $(a_5, a_7, a_{11}, a_{13}, a_{17}, a_{19}, a_{23}, a_{29}) = (\pm1, 0, \pm1, 1, \pm2, 2, \pm1, \pm0)$.

Remarkably, the sign ambiguity can be resolved in a simple way in all our cases. Most strikingly, for $N \in \{2, 4, 6\}$ the above necessary condition is also sufficient, and in the degenerate case only the congruence condition on weights needs to be verified. As examples with $\ell = 7$,

$$\begin{aligned} \Delta^+_{8,2} \equiv \Delta^+_{14,2} \equiv \Delta^+_{20,2} \equiv \Delta^+_{26,2} &\equiv q + 6q^2 + 5q^3 + q^4 + 2q^6 + q^7 + 6q^8 + \cdots, \\ \Delta^-_{10,2} \equiv \Delta^-_{22,2} \equiv \Delta^-_{28,2} \equiv \Delta^-_{40,2} &\equiv q + 2q^2 + 5q^3 + 4q^4 + 2q^5 + 3q^6 + q^8 + \cdots. \end{aligned}$$

Other similar examples can be read off from Table 4.2.

7. Concluding discussion

Sections 5 and 6 examined particular non-CM newforms with rational coefficients. Here we return to discussing Conjecture 1.1, which concerns the landscape of all non-CM newforms with rational coefficients. Given that the Maeda conjecture has been open for twenty years, we do not expect that our similar Conjecture 1.1 will be proved soon. In contrast, our assessment is that there is still insight to be gained by continuing exploratory computations in this spirit of this paper. Our concluding discussion proposes three directions for such computations.

7.1. **Larger cutoffs.** We have simply used *Magma*'s general modular form package to compute the sets $Q_k(N)$. Programs optimized for this exact problem could likely go further, meaning larger cutoffs C_k for each given k. Steps have been taken in this direction [KM], with one of the main ideas being to first work with modular forms modulo two.

7.2. **Connections with extra vanishing.** Let $P_k(N)^\delta$ be a locally defined collection of newforms, as in §3.4. Every newform g in this set has a completed L-function $\Lambda(g,s)$, with functional equation $\Lambda(g,s) = w\Lambda(g,k-s)$. The sign w of the functional equation is the same for all g. As a consequence, the order of vanishing $r(g)$ at the central point $s = k/2$ satisfies $(-1)^{r(g)} = w$ and so has constant parity. While the rank $r(g)$ itself can certainly vary with g, the Beilinson-Bloch arithmetic interpretation of central vanishing implies that conjugate g should have identical $r(g)$.

Vanishing beyond order one therefore has the potential to "explain" some locally unforced splittings. For example, we have looked at all rational non-CM newforms g with $k = 6$ and $N \leq 400$. Eight of them have $r(g) = 2$ and none have $r(g) \geq 3$. Seven of these eight have quadfree level and are members of sets $P_6(N)^\epsilon$ as follows:

N	95	116	122	260	308	359	371
	$5 \cdot 19$	$4 \cdot 29$	$2 \cdot 61$	$4 \cdot 5 \cdot 13$	$4 \cdot 7 \cdot 11$	359	$7 \cdot 53$
ϵ	$+-$	$-+$	$+-$	$-++$	$-++$	$-$	$+-$
d	$9+1$	$6+1$	$6+1$	$4+1$	$6+1$	$83+1$	$34+1$

In each case, we work with the smallest p not dividing N. The degree d characteristic polynomial $f^\epsilon_{6,N,p}(x)$ always factors as a linear factor times a polynomial with Galois group S_{d-1}. For example $f^-_{6,359,2}(x) = (x-5)(x^{83} - 7x^{82} - \cdots)$ with the degree 83 polynomial having a 4128-digit field discriminant.

The order of vanishing $r(g^\chi)$ of quadratic twists g^χ can also be taken into consideration. As an example, let $g = q + 4q^2 + 11q^3 + \cdots$ be the unique form in the space denoted $P_6(50)^{-\pm}$ in §3.4. Both g and its twist $g^{\chi_5} = q - 4q^2 - 11q^3 + \cdots \in P_6(50)^{+\mp}$ have rank zero. However the twist $g^{\chi_{-4}}$ with level 400 has rank two. To deal with arbitrary quadratic twists of a fixed form g, it would be natural to bring in half-integral weight modular forms via the Shimura-Waldspurger correspondence as described with examples in [Kob93, IV.4].

In weight six, we computed also in levels larger than 400 and have seen several more examples of extra vanishing, all again with rank two. We have not seen any extra vanishing at all in weights ≥ 8, and so in particular we have no explanation of the various unforced splittings seen there. However, we have not computed systematically and there may be extra vanishing for twists at larger levels. In weight eight, order two vanishing has been seen in CM newforms [Wat08, §6.6.2].

7.3. **A broader notion of rationality.** This paper has addressed the problem of tabulating all non-CM newforms which satisfy the rationality condition that all their Fourier coefficients a_n are rational. There is second much weaker rationality condition that is equally natural from a motivic point of view, namely that all the $|a_n|^2$ are rational. In this larger setting, one has to work with general characters χ and allow odd weights k as well. The rings described in Section 5 include some interesting examples.

The two problems are both instances of a common general problem concerning objects in the category $\mathcal{M}(\mathbb{Q},\mathbb{Q})$ of motives over $\mathbb{Q}$ with coefficients in $\mathbb{Q}$. Roughly speaking, the problem of this paper is equivalent to classifying rank two motives in this category with Sato-Tate group the symplectic group Sp_2. The larger problem with the weaker rationality condition is equivalent to classifying rank three motives with Sato-Tate group the special orthogonal group SO_3. With this weaker notion of rationality, there would of course be more newforms to collect; however we would still expect finiteness in all weights $k \geq 5$.

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Division of Science and Mathematics, University of Minnesota-Morris, Morris, MN 56267

E-mail address: roberts@morris.umn.edu