The exotic dodecahedron $\overline{M}_{0,5}(\mathbb{R})$

David P. Roberts
University of Minnesota, Morris

J'ai commencé a regarder $M_{0,5}$ à des moments perdus, c'est un véritable joyau, d'une géométrie très riche étroitement liée à celle de l'icosaédre.

–A. Grothendieck, Esquisse d'un Programme

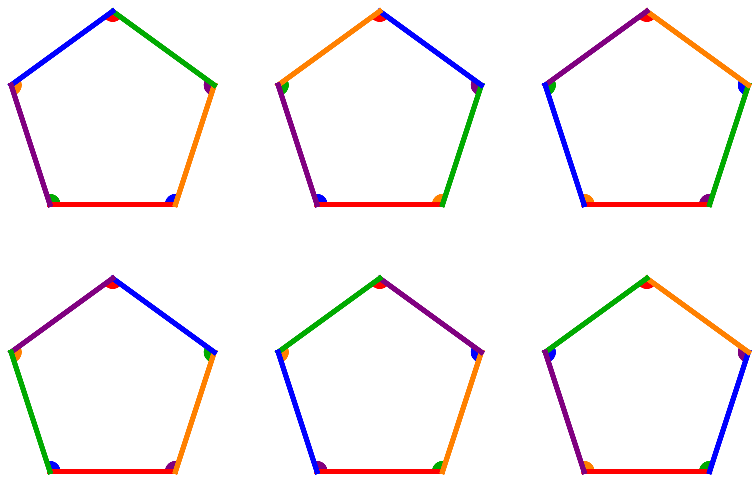
1. The exotic dodecahedron via tiling

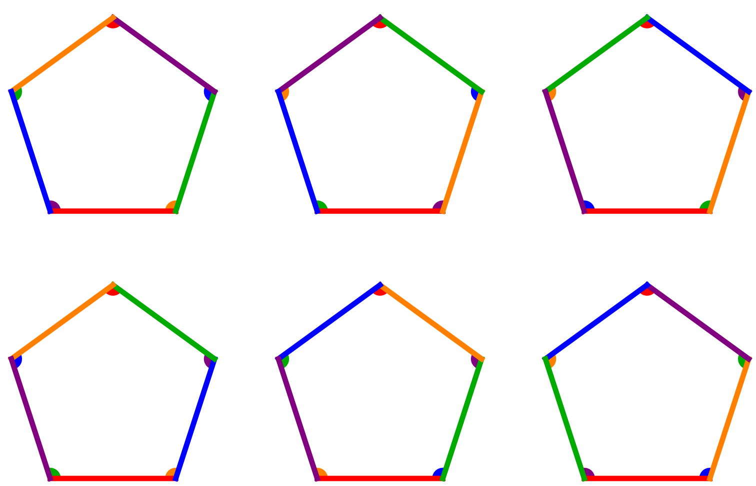
2. The exotic dodecahedron via cross-capping standard surfaces

3. The exotic dodecahedron via real algebraic geometry

4. Hints of deeper matters: $M_{0,5}(\mathbb{Z}[\frac{1}{\mathcal{P}}])$ and $\pi_1(M_{0,5}(\mathbb{C}),\star)$.

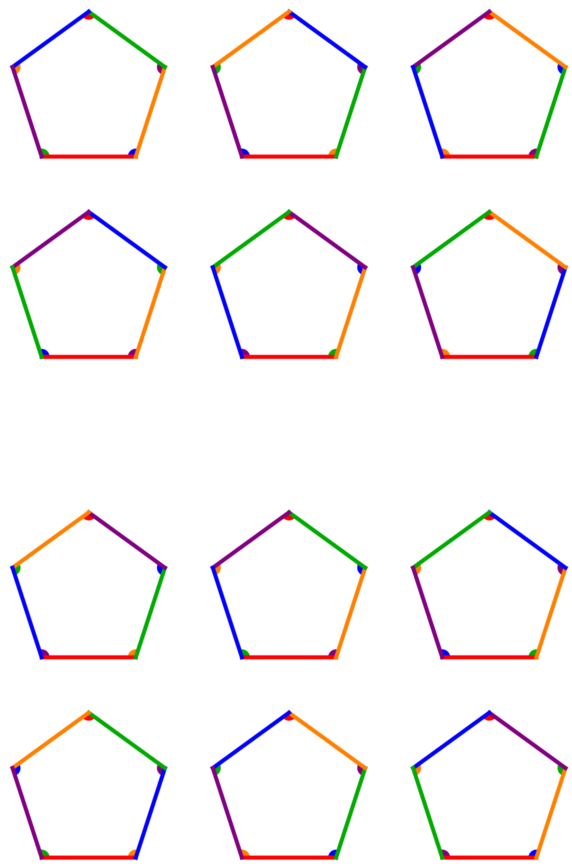
1. $\overline{M}_{0,5}(\mathbb{R})$ **via tiling.** Twelve pentagonal tiles colored with five colors satisfying obvious rules:



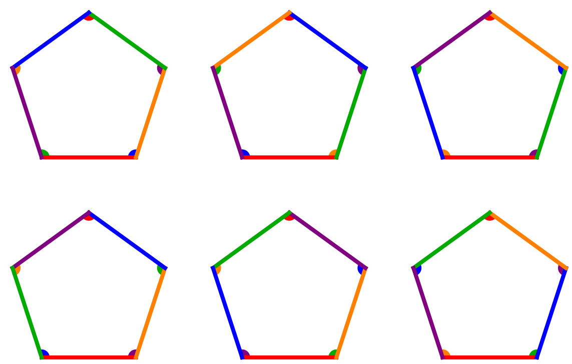


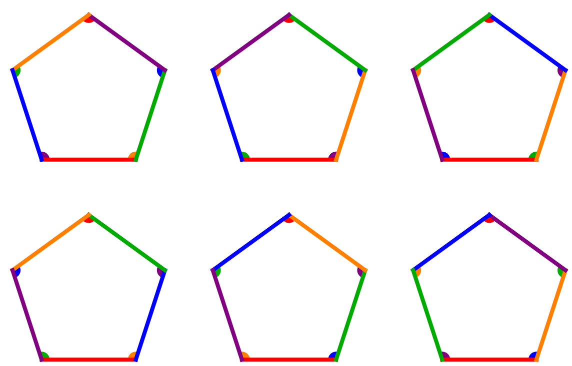
The group $S_5 = \mathrm{Sym}\{R, O, G, B, V\}$ acts on the twelve tiles, as in e.g.,

$$\begin{aligned}
(R, O)[\text{top left}] &= (R, O)[R, O, G, B, V] \\
&= [O, R, G, B, V] \\
&= [R, G, B, V, O] \\
&= [R, O, V, B, G] \\
&= [\text{bottom right}]
\end{aligned}$$



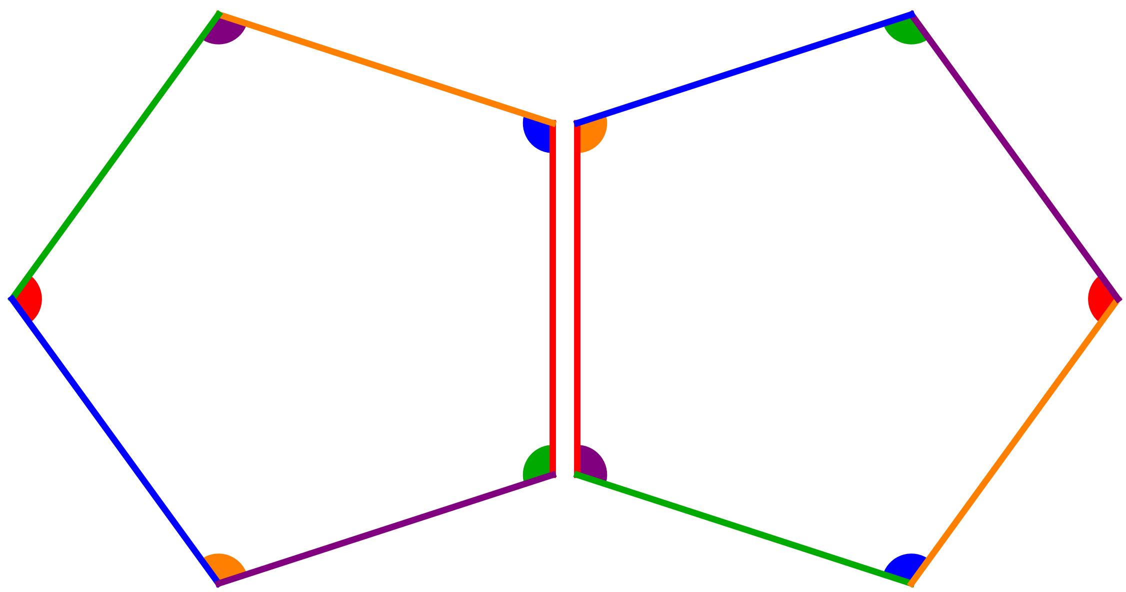
The alternating group A_5 stabilizes the two indicated subsets of six tiles.





Also each tile $[abcde]$ has an opposite tile $[acebd]$, where neighboring colors in a given tile are non-neighboring colors in its opposite and vice versa.

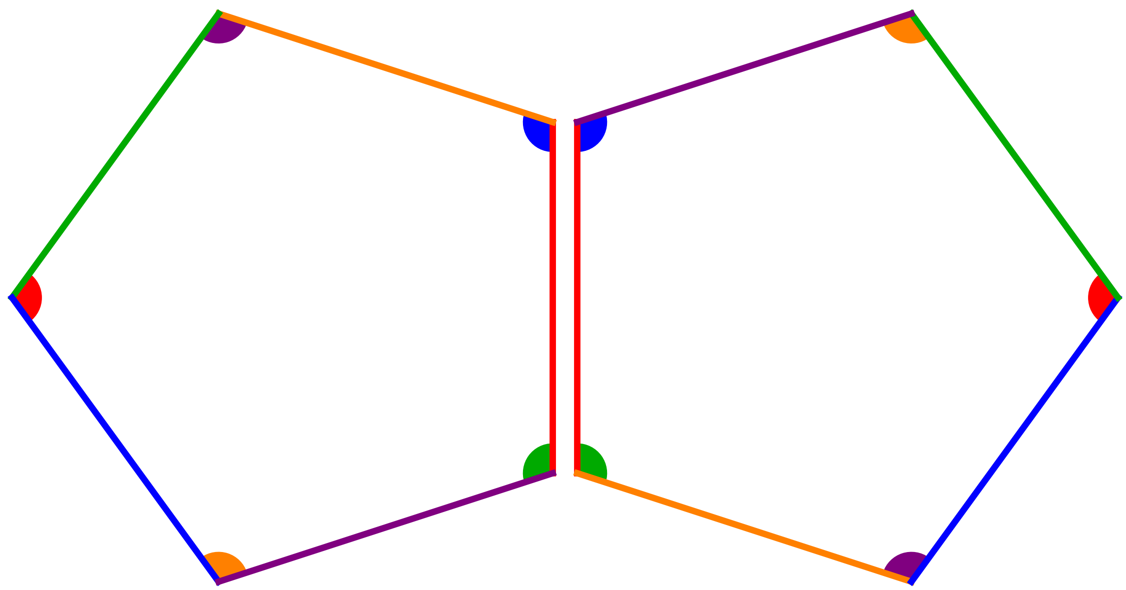
To make the standard dodecahedron follow the gluing rule:



(touching edges match colors and there are three colors at each of two touching vertices, as in picture). *Each piece determines the other.*

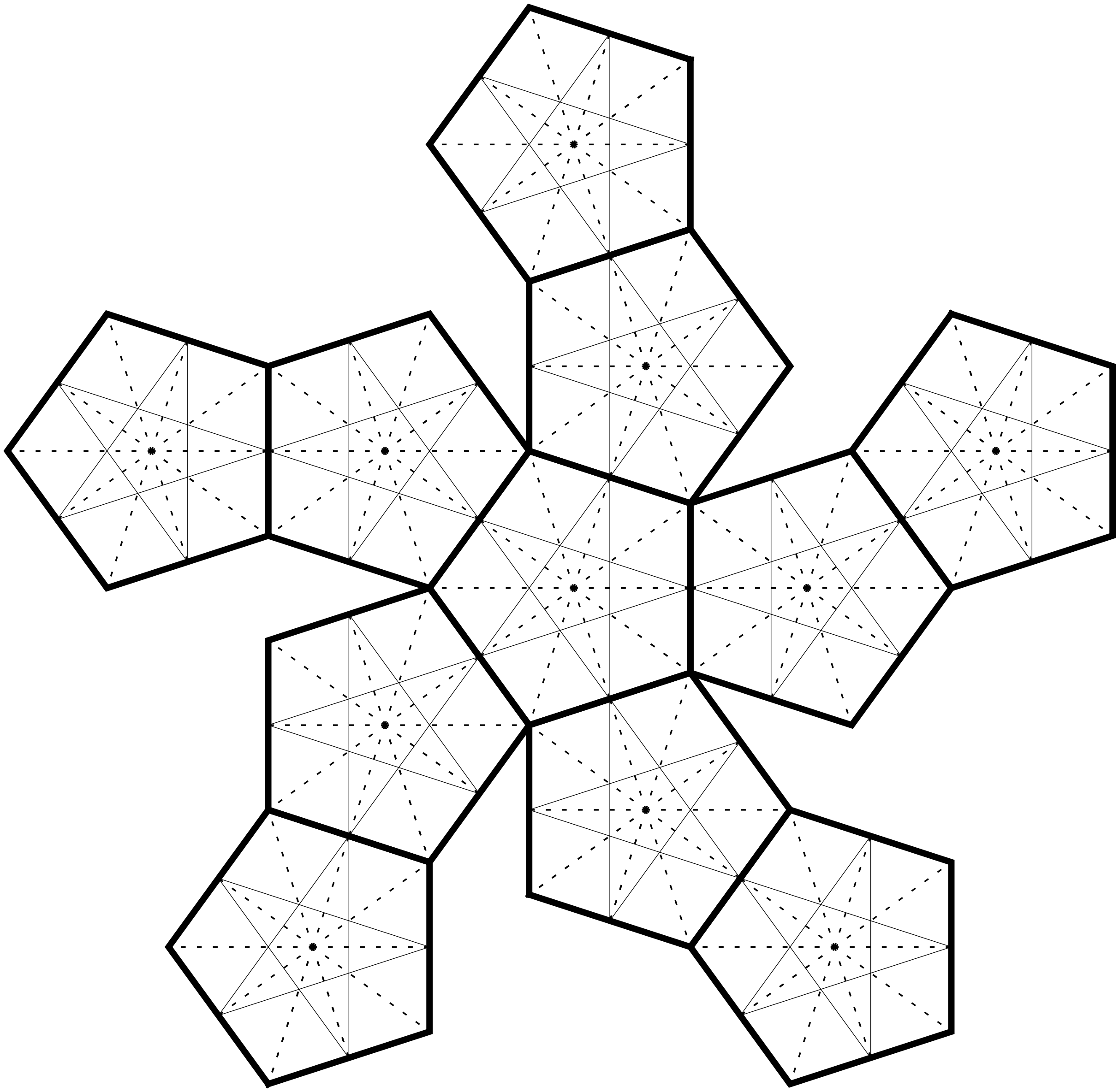
Theorem. Continuing this process yields the standard dodecahedron with three faces meeting at each vertex. Two collections of six tiles of the same parity are used.

To make the exotic dodecahedron follow the gluing rule:



(touching edges match colors, touching vertices match colors, adjacent edges don't match, as in picture). *Each piece determines the other.*

Theorem. Continuing this process yields the exotic dodecahedron with four faces meeting at each vertex. The twelve tiles are each used once.



Eleven tiles forming “pentagaman”. Gluing and including a twelfth pentagon appropriately forms either dodecahedron.

Similarities and differences:

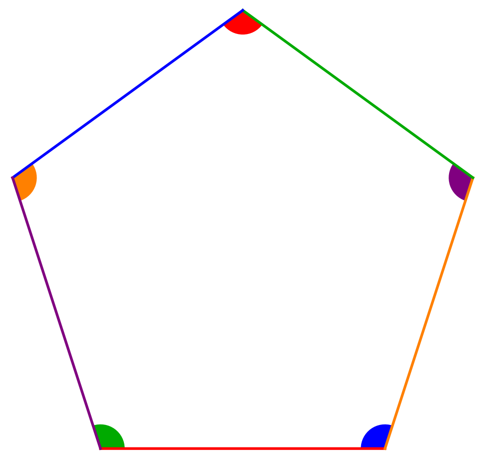
Standard dodecahedron D	Exotic dodecahedron $\overline{M}_{0,5}(\mathbb{R})$
12 pentagonal tiles	12 pentagonal tiles
30 edges	30 edges
20 vertices	15 vertices
Orientable	Nonorientable
Euler char. is 2	Euler char. is -3
Sym. group is $A_5 \times C_2$	Sym. group is S_5

Notations finally explained:

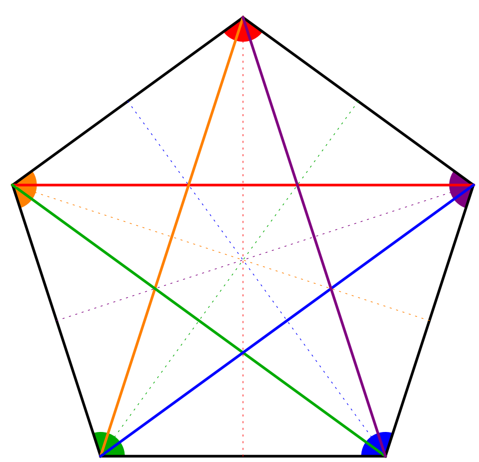
- $M_{0,5}(\mathbb{R})$ is a disconnected surface consisting of the twelve open pentagons.

- $\overline{M}_{0,5}(\mathbb{R}) - M_{0,5}(\mathbb{R})$ is a graph consisting of thirty edges and fifteen vertices. It is better thought of as ten circles meeting at fifteen points.

2. $\overline{M}_{0,5}(\mathbb{R})$ as surfaces with cross-caps. We now study only the exotic dodecahedron and switch conventions to bring out its special features. Replacing our previous tile

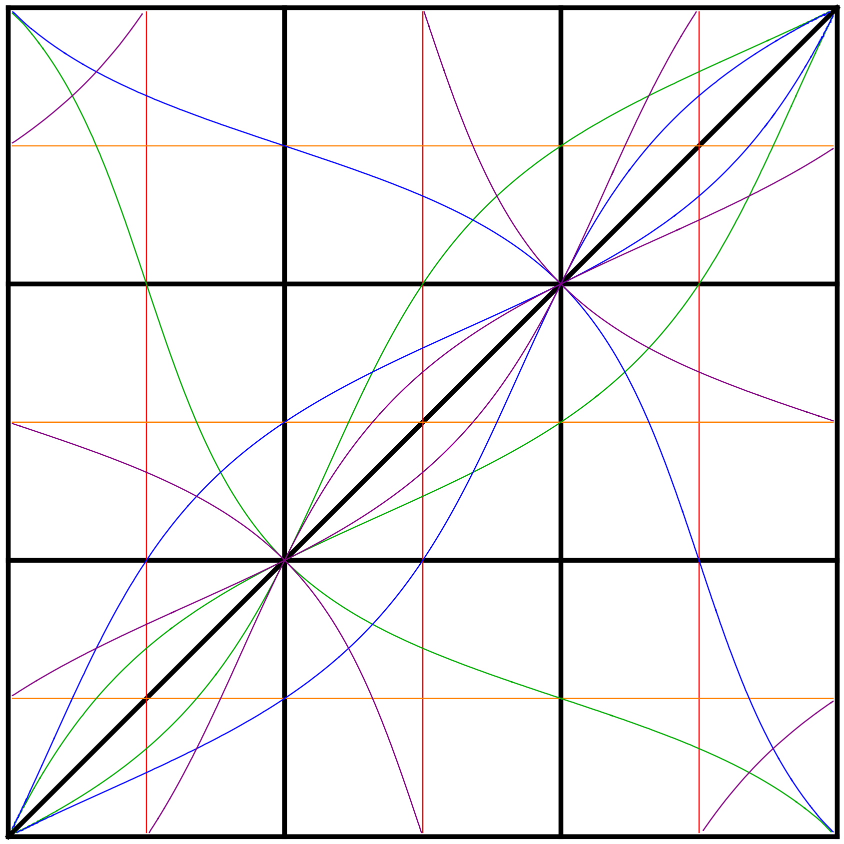


it now would be natural to use



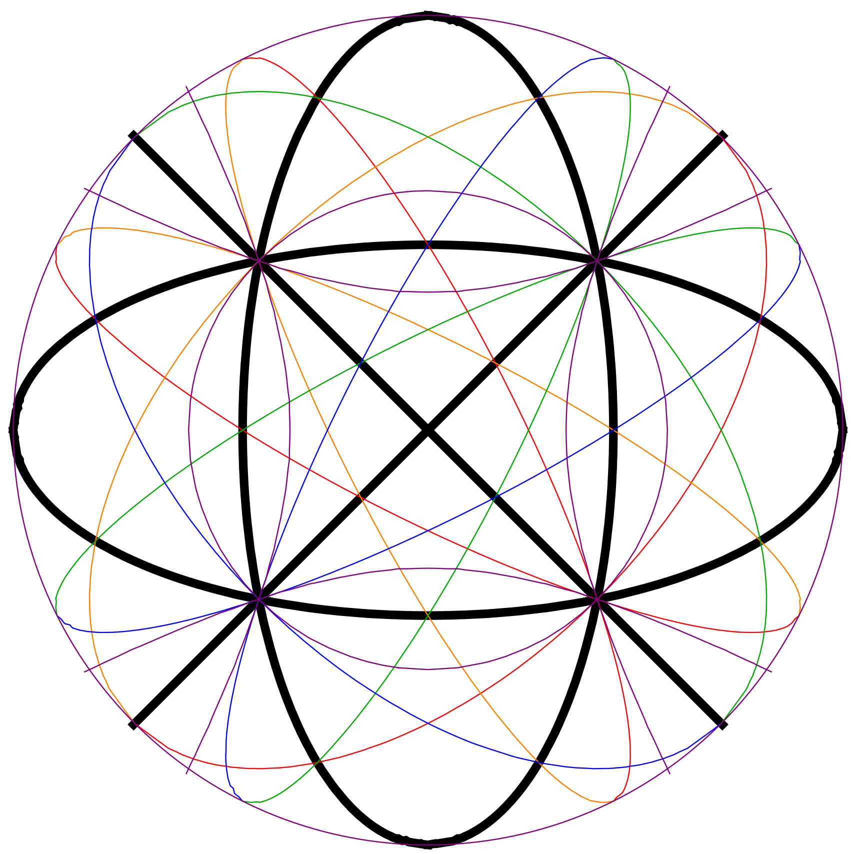
To simplify, we use only the colored star.

Toroidal view.



The drawn square has symmetry group $V = \langle (R, O), (G, B) \rangle$. Its compactification the torus has the larger symmetry group $S_2 \times S_3$ with orbits $\{R, O\}$ and $\{G, B, V\}$. The exotic dodecahedron $\overline{M}_{0,5}(\mathbb{R})$ is the torus blown up at the three black triple points.

Projective view.



The drawn disk has $D_4 = \langle (R, G, O, B), (B, G) \rangle$ symmetry. Its compactification the projective plane has S_4 symmetry belonging to the partition $\{\{R, O, G, B\}, \{V\}\}$. The exotic dodecahedron is $\overline{M}_{0,5}(\mathbb{R})$ is the torus blown up at the four black triple points.

Summary of views. For the standard dodecahedron, the group A_5 of rotational symmetries can be visually understood all at once. The rest of the symmetry group can be understood by means of reflections.

For the exotic dodecahedron, our visual understanding is more abstract:

	Euclidean Symmetry		Larger Symmetry	
View	E	$\lvert E\rvert$	H	$\lvert H\rvert$
Pentagaman	C_5	5	F_5	20
Torus with three CC	V	4	S_2S_3	12
Plane with four CC	D_4	8	S_4	24

Each of our three viewpoints breaks the symmetry so that the full group S_5 of symmetries is not visually evident. Instead we have a group $E \subset S_5$ of completely obvious Euclidean symmetries and a larger group $H \subset S_5$ of symmetries which requires a bit more imagination to see.

3. $M_{0,5}(\mathbb{R})$ via real algebraic geometry.

A broad context: In algebraic geometry, the schemes $M_{g,n}$ are very important. They have relative dimension $3g - 3 + n \geq 0$ over Spec $\mathbf{Z}$.

The schemes $M_{g,n}$ determine sets $M_{g,n}(K)$ for any field K. The key property of these sets is that if K is algebraically closed then $M_{g,n}(K)$ is the set of isomorphism classes of smooth genus g curves over K with n marked points.

The schemes $M_{g,n}$ have natural compactifications $\overline{M}_{g,n}$ in terms of minimally singular curves called stable curves.

We are concerned with the case $g = 0$ today. The first case outside of the $g = 0$ context is $M_{1,1}$ which classifies the elliptic curves. One has $M_{1,1}(K) = K$ and $\overline{M}_{1,1}(K) = K \cup \{\infty\}$.

Direct *description of* $M_{0,n}$ *and* $\overline{M}_{0,n}$. Cases with $g = 0$ can be described directly in terms of $\mathbb{P}^1(K) = K \cup \{\infty\}$. Let

$$C_{0,n}(K) \;=\; \Big\{(x_1, \ldots, x_n) \in \mathbb{P}^1(K) : \text{ the } x_i \text{ are all different}\Big\}.$$

Then the desired moduli sets are obtained by quotienting by fractional linear transformations:

$$M_{0,n}(K) \;=\; C_{0,n}(K)/PGL_2(K).$$

The slightly larger sets $\overline{M}_{0,n}(K)$ have a similar direct description. Relevant for us is that $\overline{M}_{0,5}(K) - M_{0,5}(K)$ consists of ten projective lines meeting at 15 points. While the five points are generically all different, $x_i = x_j$ over $L_{ij}(K)$. Also $x_i = x_j$ and $x_k = x_\ell$ over the single intersection point $P_{ij,kl}(K) = L_{ij}(K) \cap L_{k\ell}(K)$. More degenerate configurations, such as $x_i = x_j = x_k$ are not relevant because they are not stable.

Very direct *description of* $M_{0,n}$ *and* $\overline{M}_{0,n}$. The fact that any triple (x_1, x_2, x_3) of distinct points in $\mathbb{P}^1(K)$ can be taken to the standard triple $(0, 1, \infty)$ by a unique fractional linear transformation in $PGL_2(K)$ lets one be even more explicit.

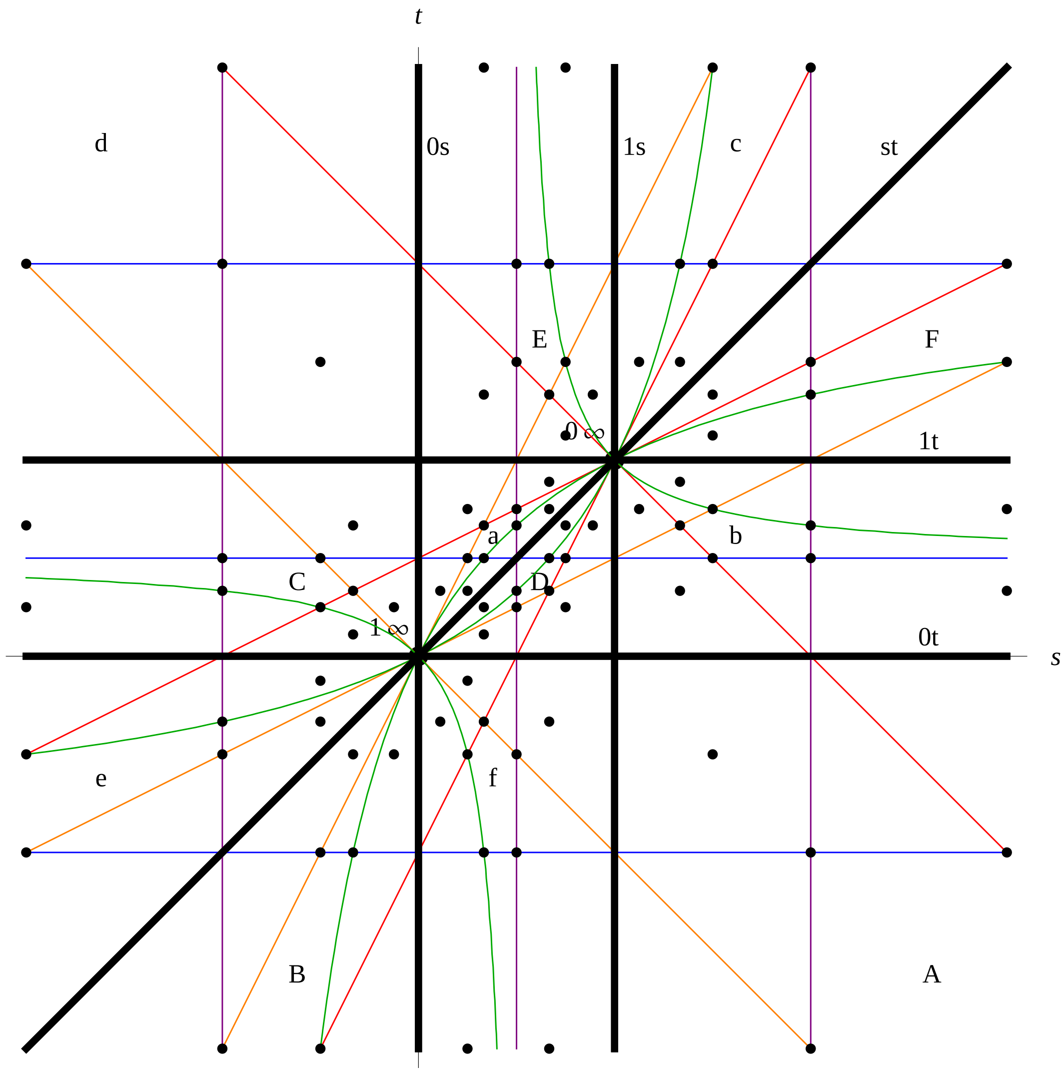
In the example relevant for us $(x_1, x_2, x_3, x_4, x_5)$ can be uniquely normalized to $(0, 1, \infty, s, t)$. Accordingly

$$M_{0,5}(K) = \{s, t \in K - \{0, 1\} : s \neq t\}.$$

We color the five points via $0, 1, \infty, s, t$. The very direct description is ideal in many respects, but it obscures the S_5 action which is still present. For example, there is still a line $L_{01}(\mathbb{R})$ at infinity, corresponding to where the points called 0 and 1 collide.

The toroidal and projective views can now be revisited with everything generalized from $\mathbb{R}$ to K and given algebro-geometric meaning (something with no analog for the standard dodecahedron!).

Toroidal viewpoint revisited: there are five projections $\overline{M}_{0,5} \to \mathbb{P}^1$:



0: red lines through $(1,1)$,

1: orange lines through $(0,0)$,

∞: green hyperbolas,

s: blue horizontal lines,

t: violet vertical lines.

4. Hints of deeper matters. *Arithmetic.* Sets $M_{g,n}(R)$ are defined for any ring. For example, for R a domain,

$$M_{0,5}(R) = \Big\{(s,t) \in R^2 : \\ s, t, (s-1), (t-1), \text{ and } (s-t) \\ \text{are all invertible.}\}$$

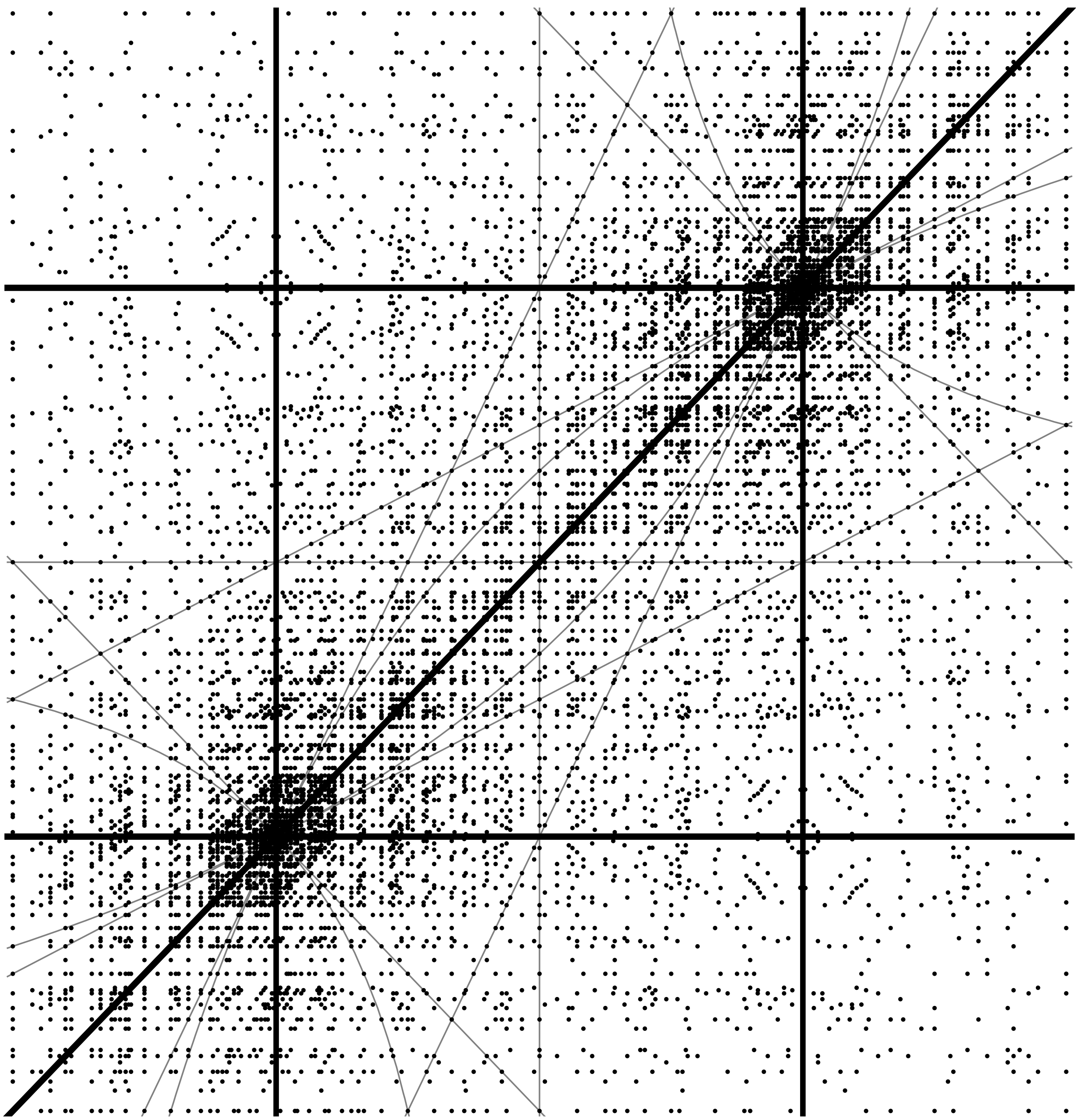
An interesting case is when $R = \mathbb{Z}[\frac{1}{\mathcal{P}}]$ is the ring obtained from $\mathbb{Z}$ by inverting all primes in a finite set $\mathcal{P}$. Then $\mathbb{Z}[\frac{1}{\mathcal{P}}]$ is known to be finite and a natural arithmetic problem is to identify it.

Example: $\mathcal{P} = \{2,3\}$. Then clearly $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{1}{4}, \frac{3}{4})$ are both in $M_{0,5}(\mathbb{Z}[\frac{1}{\{2,3\}}])$. Each gives an S_5-orbit, which has only 60 elements because each is stabilized by the involution

$$(s,t) \mapsto (1-t, 1-s)$$

of $M_{0,5}$. These two orbits form all of $M_{0,5}(\mathbb{Z}[\frac{1}{\{2,3\}}])$, as drawn above.

Larger example: The set $M_{0,5}(\mathbb{Z}[\frac{1}{\{2,3,5,7\}}])$ has 19800 points:



Topology. Suppose given a monic polynomial $f(s,t,x) \in \mathbb{C}[s,t,x]$ where s and t are viewed as a parameters and x as a variable. Suppose its discriminant has the special form

$$D(s,t) = Cs^*(s-1)^*t^*(t-1)^*(t-s)^*.$$

Then a standard and important question is how the root sets $X_{s,t} \subset \mathbb{C}$ vary as (s,t) varies over $M_{0,5}(\mathbb{C}) \subset \mathbb{C}^2$.

For visualization, define

$$\mathbb{C}^{1.5} = \{(s,t) \in \mathbb{C}^2 : \mathrm{Im}(s) = -\mathrm{Im}(t)\}$$

Let parameters vary in $M_{0,5}(\mathbb{C}) \cap \mathbb{C}^{1.5}$, with $\mathrm{Re}(s)$ and $\mathrm{Re}(t)$ as horizontal variables and $h = \mathrm{Im}(s)$ as a vertical variable. Then $M_{0,5}(\mathbb{C}) \cap \mathbb{C}^{1.5}$ consists of an upper half space $h > 0$ and a lower half space $h < 0$ connected by the twelve components of $M_{0,5}(\mathbb{R})$ at $h = 0$ viewed as windows.

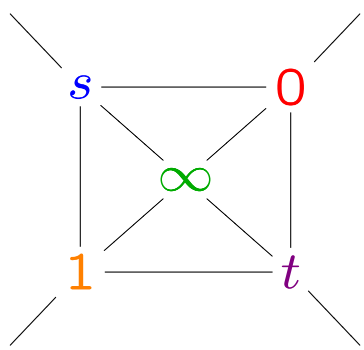
Work with base point $\star = (i, -i)$. Then the issue is how roots permute themselves as one leaves the upper half space through a window and comes back through a different window.

If $f(s,t,x) \in \mathbb{R}[s,t,x]$, one has 12 *complex conjugation operators* σ_L on $X_\star$, indexed by faces.

Carefully making choices, use them to define 10 monodromy operators, e.g.

$$\begin{aligned} m_{s0} &= \sigma_d \sigma_E & &(= \sigma_C \sigma_a, \sigma_B \sigma_f) \\ m_{1\infty} &= \sigma_C \sigma_f & &(\not\sim \text{ other choices}) \\ m_{st} &= \sigma_a \sigma_D & &(\neq \text{other choices}) \end{aligned}$$

Have these monodromy operators even when $f(s,t,x) \notin \mathbb{R}[s,t,x]$.



The diagram organizes many properties and non-properties of the m_{ij}, e.g.

$$m_{ij_1} m_{ij_2} m_{ij_3}, m_{ij_4} = 1$$

is only guaranteed when j_1, j_2, j_3, j_4 circle i in a counterclockwise order.