Some Belyi covers
unexpectedly* defined over $\mathbb{Q}$

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(* from a purely 3-point viewpoint)

1. Background on Hurwitz moduli spaces of r-point covers of P^1

2. $r = 3$: the setting for general Belyi covers

3. $r = 4$: Hurwitz curves as a broad class of very special Belyi covers

4. $r = 5$: Curves in Hurwitz surfaces as another broad class of very special Belyi covers

Sections 2-4 will center on five examples, with equations explicitly given.

1. **Background:** *Hurwitz parameters.*

Definition. *Let* $r \in \mathbb{Z}_{\geq 3}$. *A* r*-point* Hurwitz parameter *is a triple* $h = (G, C, \nu)$ *where*

- G *is a finite centerless group.*

- $C = (C_1, \ldots, C_s)$ *is a list of distinct conjugacy classes in* G.

- $\nu = (\nu_1, \ldots, \nu_s)$ *is a list of positive integers such that* $\prod [C_i]^{\nu_i} = 1$ *in* G^{ab}.

Some group-theoretic quantities.

Given a Hurwitz parameter h, let

$$\begin{aligned}\mathcal{G}_h \;=\; \{ & (g_{1,1},\ldots,g_{1,\nu_1},\ldots,g_{s,1},\ldots,g_{s,\nu_s}) : \\ & g_{i,j} \in C_i, \\ & g_{1,1}\cdots g_{1,\nu_1}\cdots g_{s,1}\cdots g_{s,\nu_s} = 1, \\ & \langle g_{i,j}\rangle = G.\}\end{aligned}$$

G acts on $\mathcal{G}_h$ by simultaneous conjugation. Define the *fiber* associated to h to be $\mathcal{F}_h = \mathcal{G}_h/G$.

Mass formulas plus inclusion-exclusion let one determine the *degree* $N := |\mathcal{F}_h|$ exactly. Let

$$\mu = \frac{\prod |C_i|^{\nu_i}}{|G'||G|}.$$

Then

$$N \approx \mu$$

is usually close and often exact.

Hurwitz covers. Let Conf_ν be the "configuration" space of $(D_1, \ldots, D_s)$, where the $D_i \subset \mathbf{P}^1$ are disjoint divisors with $|D_i| = \nu_i$.

The theory of Hurwitz schemes gives a degree N cover

$$\pi_h : \mathsf{Hur}_h \to \mathsf{Conf}_\nu$$

of complex algebraic varieties. After choosing an auxiliary embedding $G \subseteq S_n$, a point $x \in \mathsf{Hur}_h$ over $(D_1, \ldots, D_s)$ indexes a degree n cover

$$Y_x \to \mathbf{P}^1$$

with local monodromy in C_i about points in D_i and global monodromy G.

The theory says that if the C_i are rational, then the cover descends to a map of varieties over $\mathbb{Q}$. Moreover, this map has good reduction outside of the primes dividing $|G|$.

Reduced Hurwitz covers. The group PGL_2 acts on Conf_ν by fractional linear transformations. When there are no complications related to descent, the action lifts to Hur_h. Let

$$\pi_h : X_h \to U_\nu$$

be the quotient. So these spaces have dimension $r-3$.

When $r = 3$, this moduli situation is trivial over $\mathbb{C}$: U_ν is a single point and X_h consists of N points.

For $r \geq 4$, the fundamental group of U_ν with respect to a base point $\star$ is a braid group Br_ν. This braid group acts naturally on the fiber $\mathcal{F}_h = \pi_h^{-1}(\star)$. When G is close enough to being simple, then the global monodromy group is often A_N or S_N.

2. $r = 3$: the setting for general Belyi covers. Take $\nu = (1,1,1)$ always with $D_1 = \{0\}$, $D_2 = \{1\}$, and $D_3 = \{\infty\}$.

Rigid cases. A great many $(G,(C_1,C_2,C_3))$ with $N = 1$ have been studied in the literature. Simple examples include that

$$(G,(C_1,C_2,C_3)) = (S_n,(n,2\,1^{n-2},(n-1)1))$$

yields the cover $\mathbb{P}^1_y \to \mathbb{P}^1_t$ given by

$$t = \frac{y^n}{yn - n + 1}.$$

By specialization from rigid cases, many instances of the inverse Galois problem have been solved (e.g. the existence of number fields with Galois group the monster M, coming from $h = (M,(3B,2A,29A),(1,1,1))$).

Non-rigid cases illustrated by Example A. The case

$$h = (S_{12}, (642, 2^5 1^2, 532^2), (1,1,1))$$

has moduli degree $N = 24$. To find the 24 functions $\pi_i : \mathbb{P}^1_y \to \mathbb{P}^1_t$ consider rational functions

$$F(y) = -\frac{A(y)}{C(y)}$$

where

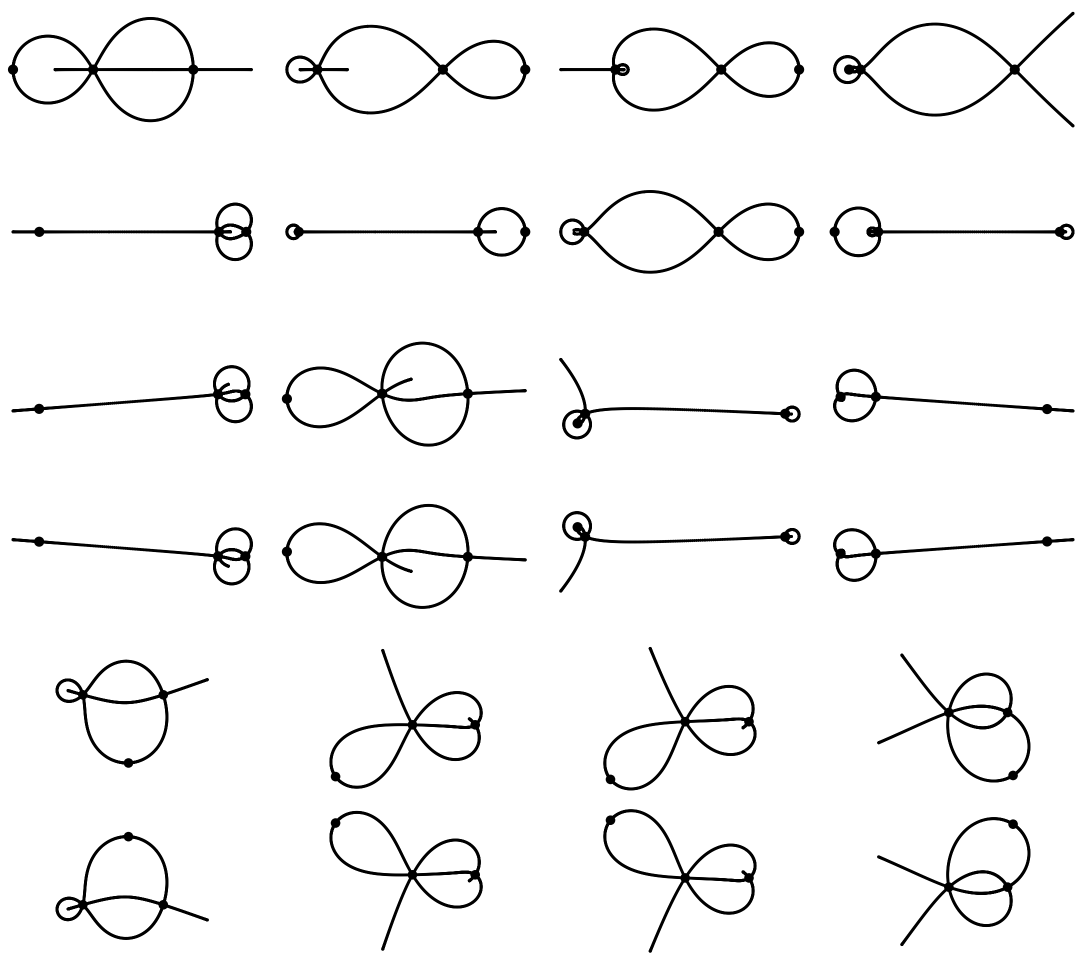
$$\begin{aligned} A(y) &= y^6(y-1)^4(y-x)^2 \\ B(y) &= (y^5 + ay^4 + by^3 + cy^2 + dy + e)^2 \\ &\quad (y^2 + fy + g) \\ C(y) &= (y+u)^3(y^2 + vy + w) \end{aligned}$$

and

$$A(y) + B(y) + C(y) = 0.$$

Gröbner basis techniques gives 24 different $x_i \in \mathbb{C}$ with each x_i then determining the complete solution $(x_i, a_i, b_i, c_i, d_i, e_i, f_i, g_i, u_i, v_i, w_i)$.

The twenty-four x_i are the roots of a polynomial $f(x) \in \mathbb{Z}[x]$. Writing $K_f = \mathbb{Q}[x]/f(x)$, one has $X_h = \text{Spec}(K_f)$. The twenty-four dessins $\pi_i^{-1}([0,1])$ are



For general three-point h, all prime factors of the discriminant of K_f divide $|G|$. Typically, $f(x)$ has Galois group S_N.

However in our example, $f(x)$ factors:

$$(5x+4)\cdot$$
$$(48828125x^{23} + 283203125x^{22} - 4345703125x^{21}$$
$$-21400390625x^{20} + 134842187500x^{19} + 461968375000x^{18}$$
$$-1670830050000x^{17} - 2095451850000x^{16} + 7249113240000x^{15}$$
$$+6576215456000x^{14} - 23053309281280x^{13} - 10284915779584x^{12}$$
$$+50191042453504x^{11} + 9449308979200x^{10} - 74715419574272x^{9}$$
$$+5031544553472x^{8} + 71884253429760x^{7} - 35243151065088x^{6}$$
$$-41613745192960x^{5} + 29347637362688x^{4} + 14541349978112x^{3}$$
$$+1765701320704x^{2} + 100126425088x + 2684354560).$$

The cover $\mathbb{P}^1_y \to \mathbb{P}^1_t$ corresponding to $x_1 = -4/5$ is

$$t = \frac{5^5 y^6 (y-1)^4 (5y+4)^2}{2^4 3^3 (2y+1)^3 \left(5y^2 - 6y + 2\right)^2}.$$

This is an example of a cover "unexpectedly" defined over $\mathbb{Q}$. Also unexpected is that it has bad reduction only at 2, 3, and 5.

Q1. Why does it split off?

Q2. Why are 7 and 11 primes of good reduction?

3. $r = 4$: Hurwitz curves as a broad class of very special Belyi covers. For $\nu = (1,1,1,1)$, $(2,1,1)$, and $(3,1)$, the reduced configuration space U_ν can be identified with $\mathbb{P}^1 - \{0,1,\infty\}$.

So a Hurwitz parameter $h = (G, C, \nu)$ determines a Belyi map $X_h \to U_\nu$. Its global monodromy and in particular the ramification partitions β_0, β_1, β_∞ can be computed by braid group techniques.

Example B. Let

$$h = (A_5, (5A, 5B, 311, 221), (1,1,1,1))$$

with $N = 12$. Because of the outer involution of A_5, the cover $X_h \to U_{1,1,1,1}$ can be descended to a degree 12 cover $X_h^* \to U_{2,1,1}$. It has $(\beta_0, \beta_1, \beta_\infty) = (642, 2^5 1^2, 532^2)$. It thus coincides with π_1 of the previous section. This answers Q1 and Q2.

Example C. Let $G = PSU_3(\mathbb{F}_3) = G_2(\mathbb{F}_2)'$. It has order $6048 = 2^5 3^3 7$ and is the twelfth smallest non-abelian simple group. For

$$h = (G, (2B, 4D), (3, 1)),$$

the degree is $N = 40$. The global monodromy group is A_{40}, the ramification triple is

$$(\beta_0, \beta_1, \beta_\infty) = (3^{12}1^4, 2^{20}, 12\ 8^2\ 7\ 3\ 2),$$

and $\mu \approx 1623$. We'll look *only* for $\pi : X_h \to U_{3,1}$ and not its many cousins.

The cover we seek has good reduction at 5. Finding it first over $\mathbb{F}_5$, then lifting 5-adically gives $\mathbb{P}^1_x \to \mathbb{P}^1_j$. Explicitly,

$$j = \frac{A(x)^3 B(x)}{2^8 3^{12} \left(2x^2 - 4x + 3\right)^8 x^7 (x-2)^3 (x+1)^2}$$

with

$$\begin{aligned} A(x) &= 64x^{12} - 576x^{11} + 2400x^{10} - 5696x^9 + 7344x^8 \\ &\quad -3168x^7 - 4080x^6 + 8640x^5 - 7380x^4 \\ &\quad -1508x^3 + 8982x^2 - 7644x + 2401 \\ B(x) &= 4x^4 - 20x^3 + 78x^2 - 92x + 49. \end{aligned}$$

Example D. Two independent and interesting phenomena are

- Sometimes one can determine $X_h \to U_\nu$ for many h at once.

- Sometimes a central extension of the monodromy group G in which all C_i are split forces a given cover X_h to be disconnected.

A five-parameter family $h(a,b,c,d,e)$ based on the combinatorics of the icosahedron illustrates both phenomena. In one instance,

$$h(8,2,1,-6,-3) = (S_{11}, (3\,1^8,\ 8\,2\,1,\ 6\,3\,2),\ (2,1,1)).$$

Here $N = 164$ and the cover splits into X_a and X_b. Each part has global monodromy group S_{82}, ramification triple

$$(\beta_0, \beta_1, \beta_\infty) = (11^2\,8\,7^4\,6\,4^2\,3^3 1,\ 2^{41}, 5^7\,4\,3\,2^{20}),$$

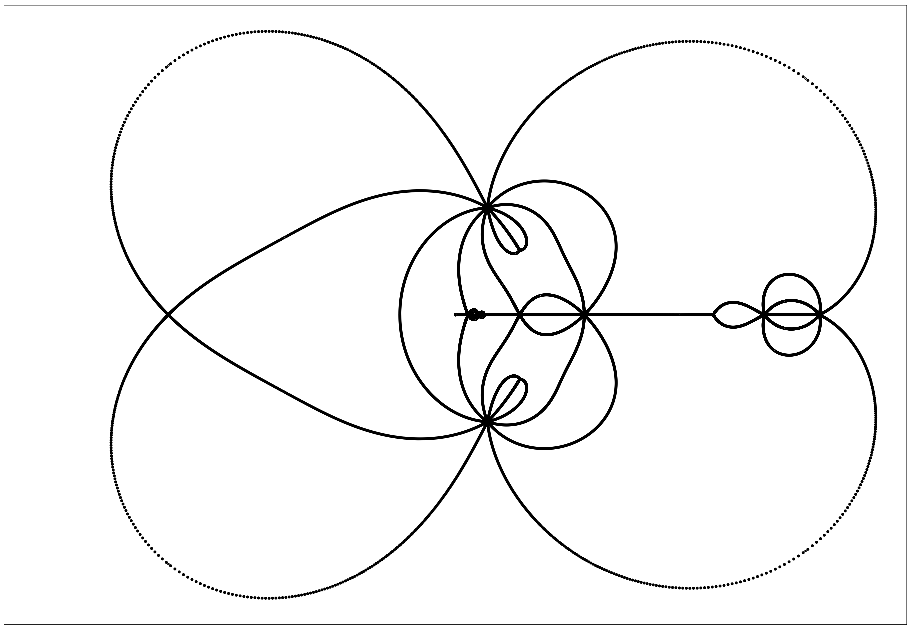
and $\mu \approx 3.09 \times 10^{14}$.

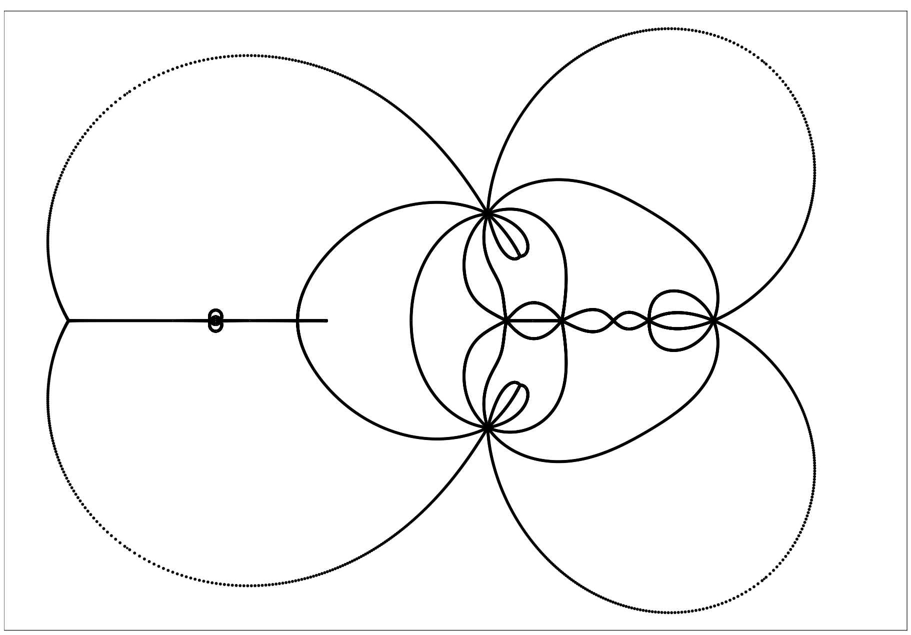
Polynomials defining covers $\mathbb{P}^1_x \to \mathbb{P}^1_w$ are:

$$
\begin{aligned}
&f_a(w,x) = \\
&\quad 2^{10}\left(4x^2-2x+3\right)^{11}(x-1)^8\left(2x^2-6x+1\right)^7\left(4x^2-10x+1\right)^7 \\
&\qquad (2x-1)^6\left(2x^2+4x-1\right)^4\left(2x^2-2x+1\right)^3(x-2)^3x \\
&\quad +w(x+1)^3 \\
&\qquad \left(192x^7-1056x^6+2148x^5-1980x^4+916x^3-180x^2+9x+1\right)^5 \\
&\qquad \Big(262144x^{20}-3997696x^{19}+25821184x^{18}-90701824x^{17} \\
&\qquad +183734272x^{16}-216097792x^{15}+240297984x^{14} \\
&\qquad -788157696x^{13}+2540448288x^{12}-5231714088x^{11} \\
&\qquad +7324268208x^{10}-7380740172x^9+5516284328x^8 \\
&\qquad -3092311406x^7+1296979268x^6-401203533x^5+89194284x^4 \\
&\qquad -13709316x^3+1371392x^2-79877x+2048\Big)^2
\end{aligned}
$$

$$
\begin{aligned}
&f_b(w,x) = \\
&\quad 2^{10}\left(4x^2-2x+3\right)^{11}(x-2)^8\left(2x^2+2x-3\right)^7\left(4x^2+6x-3\right)^7 \\
&\qquad (2x-3)^6\left(2x^2-3\right)^4\left(2x^2-2x+1\right)^3(x+3)^3(x+1) \\
&\quad +wx^3 \\
&\qquad \left(192x^7+160x^6-924x^5-336x^4+1708x^3-288x^2-783x+432\right)^5 \\
&\qquad \Big(262144x^{20}-1376256x^{19}-589824x^{18}+17629184x^{17} \\
&\qquad -29061120x^{16}-62555136x^{15}+235740160x^{14}-85084416x^{13} \\
&\qquad -614464224x^{12}+977666328x^{11}+59320728x^{10}-1697521860x^9 \\
&\qquad +1918037988x^8-313452990x^7-1429815078x^6+1887180525x^5 \\
&\qquad -1283333787x^4+548937000x^3-150220656x^2+24564384x \\
&\qquad -1889568)^2
\end{aligned}
$$

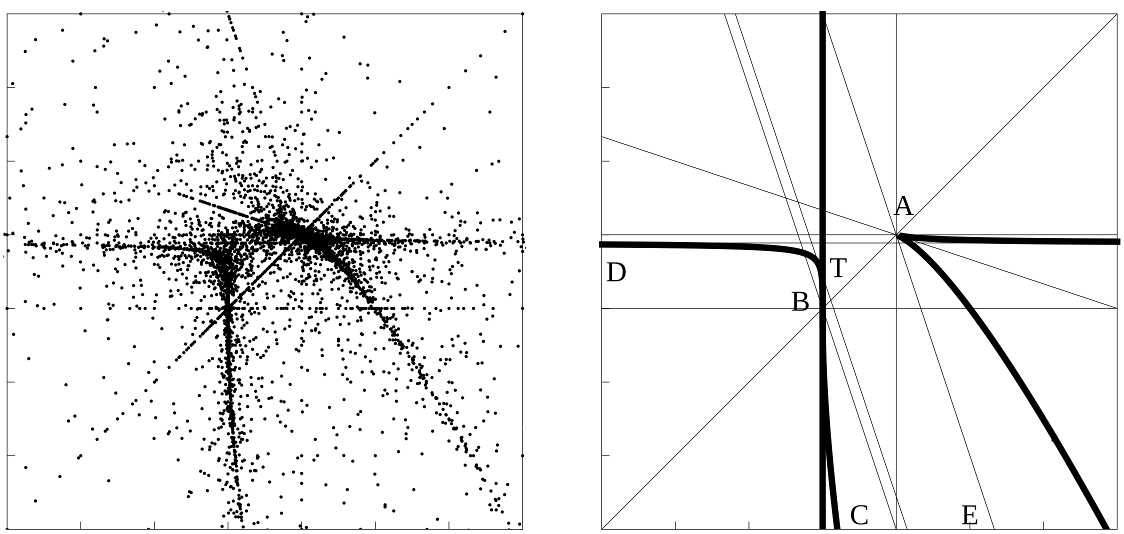
Dessins are:





4. Curves in Hurwitz surfaces as another broad class of very special Belyi covers

Reduced configuration varieties U_ν can contain rational curves with only three points at infinity. Here is a window on $U_{4,1}(\mathbb{R})$ with some points in $U_{4,1}(\mathbb{Z}[1/30])$ drawn and the rational curves on which they cluster highlighted.



Covers $X_h \to U_\nu$ restricted to one of these rational curves give Belyi maps. Global monodromy and hence $(\beta_0, \beta_1, \beta_\infty)$ can be computed by braid group methods.

Example E. Let $G = PSL_2(\mathbb{F}_7) = GL_3(\mathbb{F}_2)$ be the simple group of order $168 = 2^3\,3\,7$. The Hurwitz parameter

$$h = (G, (2A, 3A), (4, 1))$$

was studied by Malle and has degree $N = 192$. Reducing via the outer involution gives a cover $X_h^* \to U_{4,1}$ of degree 96.

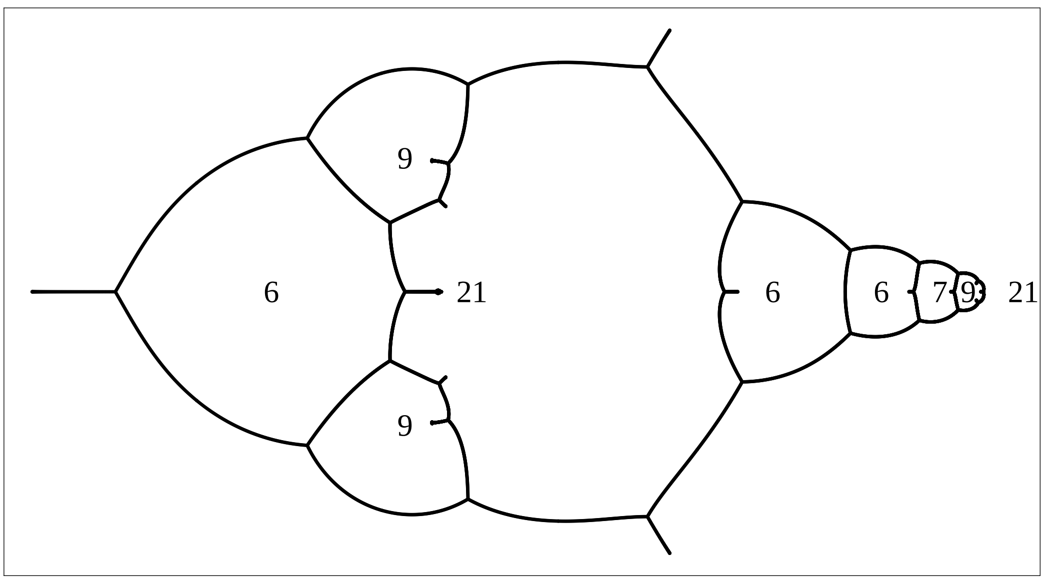
Restricted to the diagonal line AB the cover has monodromy group A_{96}. The ramification triple

$$(\beta_0, \beta_1, \beta_\infty) = (3^{32}, 2^{40}1^{16}, 21^2 9^3 7^1 6^3 2)$$

has $\mu \approx 3.10 \times 10^{15}$.

The rational map $\mathbb{P}^1_x \to \mathbb{P}^1_j$ and dessin:

$$\Big(7411887x^{32} - 316240512x^{31} + 5718682592x^{30} - 57608479936x^{29}$$
$$+345466405984x^{28} - 1143902168192x^{27} + 500924971008x^{26} + 20121596404224x^{25}$$
$$-178485128485440x^{24} + 1076315934382080x^{23} - 4902849972088320x^{22}$$
$$+16964516971136000x^{21} - 45252388465854976x^{20} + 95197078307043328x^{19}$$
$$-161987009378324480x^{18} + 229049096903122944x^{17} - 277106243726667264x^{16}$$
$$+295558502345637888x^{15} - 284898502452436992x^{14} + 250987121290100736x^{13}$$
$$-200876992270295040x^{12} + 143474999551229952x^{11} - 89556680876359680x^{10}$$
$$+47950288840949760x^{9} - 21681369027919872x^{8} + 8162827596988416x^{7}$$
$$-2520589064601600x^{6} + 626540088655872x^{5} - 122178152300544x^{4}$$
$$+17986994307072x^{3} - 1878160048128x^{2} + 123834728448x - 3869835264\Big)^3$$
$$-2^{20}jx^6(3x-2)^2\left(x^2+2x-2\right)^6\left(7x^2-14x+6\right)^{21}\left(2x^3-15x^2+18x-6\right)^9.$$



Concluding remark. The covers in this talk were all computed with old methods. It would be great to apply new methods and compute more $X_h \to U_\nu$.