Shimura curves analogous to $X_0(N)$

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Abstract

Let N be a positive integer. The modular curve $X_0(N)$ plays a prominent role in arithmetic geometry. For example it is conjectured that every elliptic curve over $\mathbf{Q}$ with conductor N is uniformized by $X_0(N)$. We define for each "odd" relatively prime factorization $N = N^+N^-$ a Shimura curve X_{N^+,N^-}. $X_{N,1}$ is just another name for $X_0(N)$. In general, the curve X_{N^+,N^-} is analogous to $X_0(N)$ in several ways. First, if $X_0(N)$ uniformizes an elliptic curve E with conductor N then so does X_{N^+,N^-}. Second the geometry of X_{N^+,N^-}, considered as an arithmetic surface over Spec $\mathbf{Z}$, is similar to that of $X_0(N)$. An important difference is that at primes dividing N^- the reduction of X_{N^+,N^-} is considerably different from that of $X_0(N)$.

We begin this paper by specializing a number of general facts about Shimura curves to the particular Shimura curves X_{N^+,N^-}. Then we consider four topics particular to these Shimura curves: 1. We give a conjectural description of the bad reduction of X_{N^+,N^-} at primes p with $p^2|N^-$, the other cases being known. We assemble evidence for these conjectures. 2. We give a formula for the intersection of certain complex multiplication divisors on the arithmetic surface X_{N^+,N^-}. This formula generalizes that of Gross, Kohnen, and Zagier for the special case $X_0(N)$. 3. We identify the Jacobians of certain genus one quotients of X_{N^+,N^-} with elliptic curves given in Swinnerton-Dyer's tables. Here the identification is made by comparing bad reductions. 4. We consider a parametrization $X_{1,57} \to E$ in considerable detail. This example illustrates the intersection formula in a concrete fashion.

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Introduction

Let N be a positive integer. Let t be the number of prime factors of N. If $N = N^+N^-$ is one of the 2^t factorizations of N into two relatively prime positive integers we put

$$\Sigma(N^+, N^-) = \{p : \mathrm{ord}_p(N^-) \text{ is odd}\} \cup \{\infty\} \subset \mathrm{Places}(\mathbf{Q}).$$

We say that a pair (N^+, N^-) is even or odd according to whether the integer $\#(\Sigma(N^+, N^-))$ is even or odd. Thus if N is not a perfect square it has 2^{t-1} factorizations of each parity; in the exceptional case where N is a perfect square all 2^t factorizations are odd.

In this paper we define for each such pair (N^+, N^-) an algebraic curve X_{N^+,N^-} over $\mathbf{Q}$. $X_{N,1}$ is just another name for the usual modular curve $X_0(N)$ figuring in our title. In general the odd curves are analogous to $X_0(N)$ and equal in complexity. The even curves are also analogous to $X_0(N)$ but much simpler objects. Only the odd curves are Shimura curves in the usual sense (e.g. [Sh 71]); however the words "Shimura curves" in our title are meant to refer to both types.

In Section 1 we introduce the notion of an oriented Eichler order of type (N^+, N^-). Sections 2–4 consist mostly in specializing results about general Shimura curves to our particular Shimura curves. Section 5 contains two new conjectures concerning bad reduction of our particular Shimura curves at primes p with $p^2|N^-$. Section 6 contains an intersection formula generalizing that of [GKZ 87] from $X_0(N)$ to all the odd X_{N^+,N^-}. Sections 7 and 8 contain some new examples. Here is a section-by-section summary:

Section 1. We define what it means for a quaternionic order R to be an Eichler order of type (N^+, N^-). For example

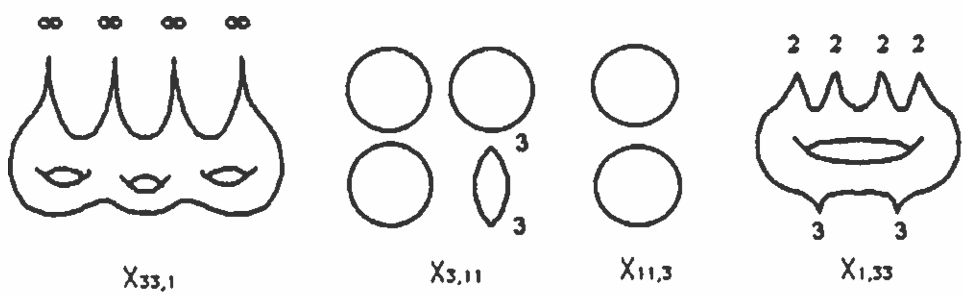
$$R_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) : N|c \right\}$$

is an Eichler order of type $(N, 1)$. If R is an Eichler order of type (N^+, N^-) then $B := R \otimes \mathbf{Q}$ has ramification locus $\Sigma(N^+, N^-)$ if (N^+, N^-) is even and $\Sigma(N^+, N^-) - \{\infty\}$ if (N^+, N^-) is odd. We define a notion of orientation on an Eichler order R of type (N^+, N^-) which will prove technically useful; such an order R has exactly 2^t orientations.

Section 2. Here we define complex analytic curves $\mathcal{X}_{N^+,N^-}$ by direct uniformization in a way analogous to the usual description

$$\mathcal{X}_0(N) := R_0(N)^\times \backslash (\mathbf{C} - \mathbf{R}) \coprod \{\text{cusps}\}.$$

For $N = 33$ the four possibilities appear as follows:



In general if (N^+, N^-) is even then $\mathcal{X}_{N^+,N^-}$ is disconnected for all but a few small N; all components have genus zero. In fact the components of $\mathcal{X}_{N^+,N^-}$ are indexed by oriented Eichler orders of type (N^+, N^-). If (N^+, N^-) is odd then $\mathcal{X}_{N^+,N^-}$ is connected; it has positive genus for all but a few small N.

Despite the rather striking difference in appearance we treat the two cases simultaneously. For example there is a notion of CM point which is defined in the same way for the two cases; the points labeled 2 and 3 above are examples of CM points. As a second example let $W(N)$ denote the group of symbols $\{w_m\}_{m||N}$, multiplication being defined by $w_{m_1,m_2} = w_{m_1 m_2/(m_1,m_2)}$. $W(N)$ is called the Atkin-Lehner group and is clearly an elementary 2-group with order 2^t. There is an action of $W(N)$ on $\mathcal{X}_{N^+,N^-}$ defined in the same way for both cases.

We explain how the Jacquet-Langlands theorem specializes to give a precise relation among the curves $\mathcal{X}_{N^+,N^-}$. For those readers already familiar with the terminology the most important part of this relation is easy to state: the Hecke modules $H^*(\mathcal{X}_{N^+,N}, \mathbf{C})^{\text{new}}$ and $H^*(\mathcal{X}_0(N), \mathbf{C})^{\text{new}}$

are isomorphic. It is also useful to sum up another part of this relation in an imprecise and informal way: as one moves primes from N^+ to N^- the complementary subspace $H^*(\mathcal{X}_{N^+,N^-}, \mathbf{C})^{\mathrm{old}}$ becomes smaller, in fact strictly smaller except for certain small values of N such as 33. Indeed, in our example $\dim(H^*(\mathcal{X}_{N^+,N^-}, \mathbf{C})^{\mathrm{new}}) = 2$ for all four factorizations $33 = N^+N^-$. Hence $\dim(H^*(\mathcal{X}_{N^+,N^-}, \mathbf{C})^{\mathrm{old}})$ is four for the cases $(N^+, N^-) = (33,1), (11,3)$ and zero for the cases $(N^+, N^-) = (3,11), (1,33)$.

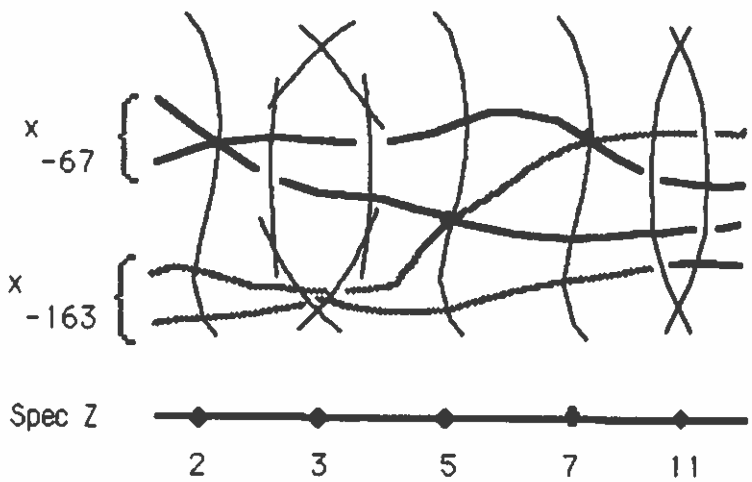
Section 3. We define X_{N^+,N^-} as algebraic curves over $\mathbf{Q}$. The even case is trivial; the odd case is highly non-trivial and involves a moduli problem. The Jacquet-Langlands theorem reviewed in Section 2 together with the Eichler-Shimura congruence and Faltings' isogeny theorem relates the Jacobians J_{N^+,N^-} of the odd curves X_{N^+,N^-} in a precise way. Again the most important point is that $J^{\mathrm{new}}_{N^+,N^-}$ is isogenous to $J_0(N)^{\mathrm{new}}$.

Let E be an elliptic curve over $\mathbf{Q}$ with conductor N. E is called modular if there exists a surjective map $X_0(N) \to E$. Whether or not E is modular can be verified by a finite computation. The relation just mentioned implies that if E is a modular elliptic curve of conductor N then there are also surjective maps $X_{N^+,N^-} \to E$ for all odd (N^+, N^-). The Taniyama-Weil conjecture adds interest to this already interesting situation: it says that one need not verify modularity as all elliptic curves are modular.

Section 4. We define the natural model $\underline{X}_{N^+,N^-}$ over $\mathbf{Z}$ for (N^+, N^-) an odd pair. Let $M = \prod_{p^2|N^-} p$. We describe this model $\underline{X}_{N^+,N^-}$ over $\mathbf{Z}[1/M]$. It is smooth over $\mathbf{Z}[1/N]$. It is — ignoring elliptic points — regular as a scheme over $\mathbf{Z}[1/M]$. The most basic phenomenon when $p \nmid N$ is that there is a finite subscheme $z_{ss} \subset (\underline{X}_{N^+,N^-})_{\mathbf{F}_p}$ which is, up to a quadratic twist, a copy of the even curve $\mathcal{X}_{N^+,N^-p}$, components having been replaced by copies of Spec $\mathbf{F}_p$. z_{ss} is called the supersingular locus. Similarly, when $p|N$ many of the phenomena can be described concretely in terms of oriented Eichler orders.

Section 5. If $p^2|N^-$ then the scheme $\underline{X}_{N^+,N^-}$ is badly singular over $\mathbf{Z}_p$. The conjectures mentioned above concern the minimal desingularization $\underline{X}'_{N^+,N^-}$ of $\underline{X}_{N^+,N^-}$. According to our conjectures, the scheme $\underline{X}_{N^+,N^-}$ can be rather completely described in terms of definite oriented Eichler orders. We give quite a lot of evidence for our conjectures.

Section 6. Let x_0 and x_1 be two points on an even curve $\mathcal{X}_{N+,N-}$. We define — ignoring elliptic points here in the introduction — their coincidence number (x_0, x_1) to be one if they lie on the same component and zero otherwise. By linearity this pairing is defined on all divisors. Now let x_0 and x_1 be two distinct scheme-theoretic points on an odd curve $X_{N+,N-}$. Let $\underline{x}_0$ and $\underline{x}_1$ be their closures on $\underline{X}'_{N+,N-}$. We define the intersection number $(\underline{x}_0, \underline{x}_1) \in \mathbf{R}$ to be $\log \#(A)$ where A is the ring of functions on the scheme-theoretic intersection $\underline{x}_0 \cap \underline{x}_1$. Again by linearity $(\cdot, \cdot)$ is defined on arbitrary divisors with disjoint support. The main result of this section is a formula for the coincidence and intersection numbers of certain divisors constructed from CM points. Here is a picture which illustrates our intersection formula.



We have drawn the arithmetic surface $\underline{X}_{1,33}$. The horizontal divisors are closures of two scheme-theoretic points associated to the discriminants -67 and -163, the first drawn solid, the second dotted. We find that these divisors intersect exactly twice, each intersection being transverse. One of these intersections occurs at a supersingular point in characteristic 5, the other at a supersingular point in characteristic 7.

Section 7. We systematically consider all involutory parametrizations $X_{N+,N-} \to E$ where E is an elliptic curve with conductor $N \leq 60$. Here a parametrization $X_{N+,N-} \to E$ is called involutory iff it factors through some genus one quotient $X_{N+,N-}/W$, W being a subgroup of the Atkin-

Lehner group. As an example let E run over the isogeny class $33A - 33D$ of modular elliptic curves of conductor 33. Here, as in Section 7, we systematically use the notation of the tables in [SD 75]. The possibilities for genus($X_0(33)/W$) are 0,2, and 3 so none of the usual parametrizations $X_0(33) \to E$ are involutory. However $X_{1,33}$ already has genus one and so all parametrizations $X_{1,33} \to E$ are involutory.

We compare the bad reduction of the genus one quotients $X_{N^+,N^-}/W$ with that given in the Table 1 in [SD 75] for modular elliptic curves E. As an example $X_{1,33}$ has bad reduction of type $I6$ at 3 and type $I2$ at 11, as drawn above. The tables in [SD 75] give

	3	11
$33A$	$I3$	$I1$
$33B$	$I6$	$I2$
$33C$	$I12$	$I1$
$33D$	$I3$	$I4$

Hence (over any field F with $X_{1,33}(F) \neq \emptyset$) we have $X_{1,33} = 33B$. These considerations serve to illustrate the generalities of the previous sections, especially the description of bad reduction given in Sections 4 and 5. Also we explain a connection with the Birch-Swinnerton-Dyer conjecture.

Section 8 We consider one of the involutory parametrizations of Section 7, namely $X_{1,57} \to 57E$ in more detail. This section serves as an illustration of Section 6.

Finally I would like to thank my thesis advisor, Benedict H. Gross, for introducing me to the circle of ideas which led to this thesis. I would also like to express my appreciation to the National Science Foundation and the Sloan Foundation for financial support during my years of graduate study.

1 Oriented Eichler orders

In the section we dispense with a number of preliminaries necessary for defining the curves X_{N^+,N^-}. The focus is on the notion of oriented Eichler order. Eichler orders in various levels of generality have been introduced by many authors; see [Gr 88] for a notion slightly more general than ours. The rather elementary idea of orienting Eichler orders, and the recognition that the notion of orientation is technically useful, seems to have appeared first in [M-O —].

We assume that the reader is familiar with quaternions; a standard reference is [Vi 80].

1.1 Rigidifying choices

We want to define our curves X_{N^+,N^-} up to unique isomorphism, not just up to isomorphism. To do this in a down-to-earth fashion we will simply make some arbitrary choices.

Definition 1.1.1 *Let p be a prime. If $p = 2$ then $\mathbf{Q}_{p^2} := \mathbf{Q}_p[i_p]/i_p^2 + 3$. If $p \neq 2$ then $\mathbf{Q}_{p^2} := \mathbf{Q}_p[i_p]/i_p^2 + d_p$ where d_p is the smallest positive integer prime to p such that $x^2 + d_p = 0$ has no solution in $\mathbf{F}_p$.*

Thus while all unramified quadratic field extensions of $\mathbf{Q}_p$ are isomorphic they are not canonically isomorphic. We have chosen one, namely $\mathbf{Q}_{p^2}$, and will view it as the standard unramified quadratic field extension of $\mathbf{Q}_p$. This is completely analogous to the usual practice of considering the field $\mathbf{C} := \mathbf{R}[i]/i^2 + 1$ as the standard algebraic closure of $\mathbf{R}$. We put $\mathbf{Z}_{p^2}$ equal to the ring of integers in $\mathbf{Q}_{p^2}$ and $\mathbf{F}_{p^2}$ equal to its residue field.

Our rigidifying choices will most typically appear in the guise of certain finite rings. For p a prime, $e \in \mathbf{Z}_{\geq 0}$, let

$$\begin{aligned} A_{p^e,1} &:= \mathbf{Z}_p/p^e \oplus \mathbf{Z}_p/p^e \\ A_{1,p^e} &:= \mathbf{Z}_{p^2}/p^e. \end{aligned}$$

So $A_{p^e,1}$ and A_{1,p^e} each have exactly p^{2e} elements. If $e \neq 0$ then they each have exactly one non-trivial automorphism, namely that induced by σ_p.

Let $N = \prod p^{e_p}$ be a positive integer with t prime factors. Let $N = N^+N^-$ be a relatively prime factorization. Then we put

$$A_{N^+,N^-} = \prod_{p|N^+} A_{p^{e_p},1} \prod_{p|N^-} A_{1,p^{e_p}}$$

so that A_{N^+,N^-} has N^2 elements and 2^t automorphisms.

1.2 Local Eichler orders

Let B_p be a quaternion algebra over $\mathbf{Q}_p$ and let R_p be a $\mathbf{Z}_p$-order in $\mathbf{Q}_p$. If R_p has reduced discriminant p^e then we call e the level of R_p. We are interested in a particular kind of quaternionic order which we call Eichler orders. They come in three types:

Definition 1.2.1

1. *If R_p has level zero then we say R_p is an Eichler order of type* $(1, 1)$.
2. *If R_p has level $e \geq 1$ and contains a quadratic order isomorphic to $\mathbf{Z}_p \oplus \mathbf{Z}_p$ then we say R_p is an Eichler order of type* $(p^e, 1)$.
3. *If R_p has level $e \geq 1$ and contains a quadratic order isomorphic to $\mathbf{Z}_{p^2}$, we say R_p is an Eichler order of type* $(1, p^e)$.

We refer to these three types as "level zero", "split", and "twisted". We warn the reader that the notion of Eichler order varies somewhat in the literature, perhaps the most common encompassing only our level zero and split orders. However, from our point of view it is natural to consider split and twisted Eichler orders as being on the same footing.

Before proceeding we construct one Eichler order of each type. We will consider these examples to be our standard Eichler orders. Naturally we take $R_{1,1} := M_2(\mathbf{Z}_p)$ as our standard level zero Eichler order. More generally, for $e \in \mathbf{Z}_{\geq 0}$ we put

$$R_{p^e,1} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_p) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \quad (p^e) \right\}.$$

Turning now to twisted Eichler orders we put

$$R_{1,p} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}_{p^2}) : a = d^\sigma,\ c = pb^\sigma \right\}.$$

$R_{1,p}$ is the unique maximal order in its quotient skew-field B_p^{ram}. More generally, for $e \in \mathbf{Z}_{\geq 1}$ put

$$R_{1,p^e} = \begin{cases} \mathbf{Z}_{p^2} + p^{e/2} M_2(\mathbf{Z}_p) \subset M_2(\mathbf{Q}_p) & e \text{ even} \\ \mathbf{Z}_{p^2} + p^{(e-1)/2} R_{1,p} \subset B_p^{\mathrm{ram}} & e \text{ odd.} \end{cases}$$

Eichler orders of positive level are not quite rigid enough for us. We will rigidify them by orienting them:

Definition 1.2.2 *Let* $(e^+, e^-) \in (\mathbb{Z}_{\geq 0}, 0) \cup (0, \mathbb{Z}_{\geq 0})$. *Let* R_p *be an Eichler order of type* (p^{e^+}, p^{e^-}). *Then an orientation on* R_p *is a homomorphism* $f_p : R_p \to A_{p^{e+}, p^{e-}}$.

If R_p has type $(1, 1)$ then an orientation on R_p is clearly no extra data as $A_{1,1}$ is the zero ring. We will be considering oriented Eichler orders all the time and will typically let a single symbol like $\vec{R}_p$ denote an oriented Eichler order (R_p, f_p).

Before proceeding we give our standard Eichler orders their standard orientations. We orient $R_{p^e,1}$ by

$$\begin{array}{rcl} f_p : R_{p^e,1} & \longrightarrow & A_{p^e,1} \\ \begin{pmatrix} a\, b \\ c\, d \end{pmatrix} & \longmapsto & (a, d). \end{array}$$

We orient R_{1,p^e} by

$$\begin{array}{rcl} f_p : R_{1,p^e} & \longrightarrow & A_{1,p^e} \\ \begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix} & \longmapsto & a. \end{array}$$

By an isomorphism from one oriented Eichler order (R_p, f_p) to another (R'_p, f'_p) we mean an isomorphism of rings $g : R_p \to R'_p$ such that

$$\begin{array}{ccc} R_p & \xrightarrow{f_p} & A_{p^{e+}, p^{e-}} \\ \downarrow g & & \| \\ R'_p & \xrightarrow{f'_p} & A_{p^{e+}, p^{e-}} \end{array}$$

commutes.

Proposition 1.2.3

1. *An Eichler order of positive level has exactly two orientations.*
2. *Two oriented Eichler orders are isomorphic iff they have the same type. An oriented Eichler order of type* (p^{e^+}, p^{e^-}) *lies in a ramified quaternion algebra iff* e^- *is odd.*

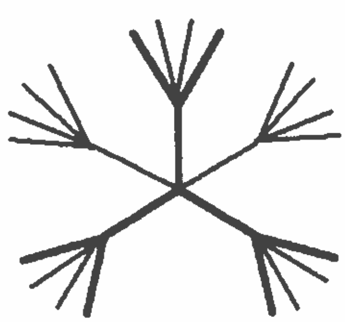
3. All automorphisms of an oriented Eichler order are inner.

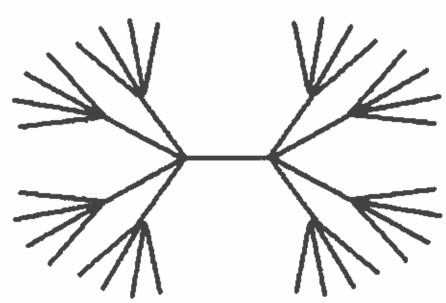
Proof. In the next subsection we will translate our entire situation into geometric terms. From this new point of view all three statements will be obvious. □

Remark 1.2.4 Let $\vec{R}_p = (R_p, f_p)$ be an oriented Eichler order of positive level. Let L_p be a right R_p-module which is free of rank one. Let R'_p be the left order of L_p i.e. $\mathrm{End}(L_p)_{R_p}$. Then R'_p is isomorphic with R_p canonically up to inner automorphisms. Namely let $\ell_p \in L_p$ be a basis for L_p as an R_p-module. Then ℓ_p determines an isomorphism $i_{\ell_p} : R'_p \to R_p$ by $r'_p\ell_p = \ell_p i_{\ell_p}(\bar{r}'_p)$. We then define $f'_p := f_p$ via any of these identifications $R'_p = R_p$. This remark will be used repeatedly and often without comment in the sequel, mostly in Section 4.

1.3 Trees and symmetric spaces

We will translate our entire situation into geometric terms, leaving the details of the translation to the reader. Let B_p be a quaternion algebra over $\mathbb{Q}_p$. Let B_{p^2} be the base-change of B_p to $\mathbb{Q}_{p^2}$. B_{p^2} is split. Consider the set $\mathcal{V}$ of orders of level zero in B_{p^2}. If we identify B_{p^2} with $M_2(\mathbb{Q}_{p^2})$ then it is elementary that $\mathcal{V}$ consists of $M_2(\mathbb{Z}_{p^2})$ and its conjugates. $\mathcal{V}$ is naturally the set of vertices of a tree $\mathcal{T}(B_{p^2})$, two vertices $v_{R_{p^2}}$ and $v_{R'_{p^2}}$ being connected by an edge iff $\mathrm{level}(R_{p^2} \cap R'_{p^2}) = 1$. $\mathcal{T}(B_{p^2})$ has p^2+1 edges incident upon each vertex and the distance function on vertices has the property $d(v_{R_{p^2}}, v_{R'_{p^2}}) = \mathrm{level}(R_{p^2} \cap R'_{p^2})$. The reader can check that the action of σ_p on $\mathcal{T}(B_{p^2})$ is as indicated by the following two pictures, drawn for $p = 2$:





Here the left diagram represents the split case. σ_p fixes precisely the solid lines. The right diagram represents the ramified case. σ_p acts by reflection in the vertical line of symmetry.

We have bijections

1.	σ_p-fixed vertices	$\leftrightarrow$	Level zero Eichler orders
2a.	σ_p-invariant geodesics	$\leftrightarrow$	Split subalgebras $K_p \subset B_p$
2b.	Oriented σ_p-invariant geodesics	$\leftrightarrow$	Embeddings $\mathbf{Q}_p \oplus \mathbf{Q}_p \hookrightarrow B_p$
3a.	σ_p-antiinvariant geodesics	$\leftrightarrow$	Inert subalgebras $K_p \subset B_p$
3b.	Oriented σ_p-antiinvariant geodesics	$\leftrightarrow$	Embeddings $\mathbf{Q}_{p^2} \hookrightarrow B_p$
4a.	σ_p-invariant segments of length e	$\leftrightarrow$	Eichler orders with type $(p^e, 1)$
4b.	Oriented such segments	$\leftrightarrow$	Oriented such orders
5a.	σ_p-antiinvariant segments of length e	$\leftrightarrow$	Eichler orders with type $(1, p^e)$
5b.	Oriented such segments	$\leftrightarrow$	Oriented such orders

Here a vertex $v_{R_{p^2}}$ is on a segment s_{S_p} iff $S_p \subset R_{p^2}$. Similarly a vertex $v_{R_{p^2}}$ is on a geodesic γ_{K_p} iff $K_p \cap R_{p^2}$ is the full ring of integers in K_p. There are actually two natural bijections in each of the four cases 2b-5b. To choose one of them, say in cases 4b and 5b, one has to make a convention identifying our algebraic notion of orientation with the usual geometric notion of orientation. This is quite easy to do explicitly but not necessary for us.

Symmetric spaces. We now give an archimedean analog of the above considerations. Let B_∞ be a quaternion algebra over $\mathbf{R}$.

Definition 1.3.1 $\mathcal{Y} := \{b \in B_\infty : t(b) = 0,\ n(b) = 1\}$.

Here t denotes the reduced trace and n denotes the reduced norm. Consider the quadratic form on the three-dimensional space $B_\infty^0 := \{b \in B_\infty : t(b) = 0\}$. n has signature $+++$ if B_∞ is definite and $+--$ if B_∞ is indefinite. Thus $\mathcal{Y}$ is a sphere in the definite case and a two-sheeted hyperboloid in the indefinite case. $B_\infty^\times$ acts transitively through its quotient $B_\infty^\times/\mathbf{R}^\times$: $b \cdot y := byb^{-1}$. If B_∞ is definite then $\mathcal{Y}$ has a unique B_∞-invariant Riemannian metric with constant curvature 1. Similarly if B_∞ is indefinite then $\mathcal{Y}$ has a unique B_∞-invariant Riemannian metric with constant curvature -1.

The points of $\mathcal{Y}$ can be interpreted as parametrizing embeddings $h : \mathbf{C} \to B_\infty$, via $h_y(i) := y$. We turn $\mathcal{Y}$ into a Riemann surface by declaring that $h_y(e^{\pi i/4})$ acts by i on the tangent space to y. If $B_\infty = M_2(\mathbf{R})$ then we have just constructed the upper and lower half-plane (from a rather non-standard point of view!). In general $\mathcal{Y}$ is a symmetric space canonically associated to B_∞.

p-adic upper half plane. Returning to the ultrametric case, suppose B_p is split. Then one has a formal scheme $\underline{\mathcal{T}}$ over $\mathbf{Z}_p$, often called the p-adic upper half plane. We refer to [Mu 72], [Dr 76], or [Te —] for full definitions from several points of view. The special fiber $\underline{\mathcal{T}}^0$ is easy to describe and we describe it here. $\underline{\mathcal{T}}^0$ is a reduced scheme. All components have genus zero. The components are in bijection with level zero orders in B_p. Here C_{R_p} is characterized by the fact that it is stabilized exactly by $\mathbf{Q}_p^\times R_p^\times$. Alternatively C_{R_p} is characterized by the fact that it is pointwise fixed exactly by $\mathbf{Q}_p^\times(\mathbf{Z}_p + pR_p)^\times$. It is thus canonically isomorphic to the conic in the projective plane R_p^0/pR_p^0 defined by the norm. The $\mathbf{F}_p$-rational points on C_p are thus in bijection to Eichler suborders S_p of R_p of type $(p, 1)$. The components are glued together pairwise along their $\mathbf{F}_p$-rational points in the obvious way (so that the dual graph of $\underline{\mathcal{T}}^0$ is exactly the tree $\mathcal{T}(B_p)$).

1.4 Ideal theory

Let $\vec{R}_p = (R_p, f_p)$ be an oriented Eichler order of type (p^{e^+}, p^{e^-}) and level e (so that $e = \max\{e^+, e^-\}$). Let $B_p = R_p \otimes \mathbf{Q}_p$ as usual. We now turn to consideration of the set

$$\mathrm{Lat}(B_p)_{R_p} = \{\text{Lattices in } B_p \text{ with right order } R_p\}.$$

$\mathrm{Lat}(B_p)_{R_p}$ is identified with the coset space $B_p^\times/R_p^\times$ via $bR_p \mapsto bR_p^\times$. Thus $\mathrm{Lat}(B_p)_{R_p}$ is naturally a left $B_p^\times$-space.

$\mathrm{Lat}(B_p)_{R_p}$ supports several natural $B_p^\times$-equivariant correspondences. Namely we certainly have $S_pL_p := pL_p$. More interestingly in the case $e = 0$ we have the Hecke correspondences

$$T_{p^k}L_p := \sum_{L'_p \subset L_p} L'_p.$$

Here the sum is over $L'_p \subset L_p$ such that $\mathrm{length}_{\mathbf{Z}_p} L_p/L'_p = 2k$. In the complementary case $e \neq 0$ we have the equally important Atkin-Lehner automorphism:

$$w_{p^e} := J_p L_p.$$

Here J_p is the kernel of the orientation map f_p. We have the relations

$$T_{p^k} = T_{p^{k-1}} T_p - p T_{p^{k-2}} S_p \qquad (1.4.1)$$
$$w_{p^e}^2 = S_{p^e}. \qquad (1.4.2)$$

If one mods out by scalars it is easy to treat interpret these correspondences geometrically. Namely if $e = 0$ then homothety classes in $\mathrm{Lat}(B_p)_{R_p}$ correspond to level zero Eichler orders in B_p (via "take left order"). If $e > 0$ then homothety classes correspond to oriented Eichler orders of type (p^{e^+}, p^{e^-}) (again via "take left order", now using Remark 1.2.4 to fix the orientation). We then have the geometric descriptions

$$\bar{T}_{p^k}(v) = \sum_{\substack{d(w,v) \le k \\ d(w,v) \equiv k \, (2)}} w$$

if $e = 0$ and

$$\bar{w}_{p^e}(\vec{s}) = \vec{t}$$

if $e \geq 1$. Here $\vec{t}$ is the same segment as $\vec{s}$ but with the opposite orientation. Note that from this geometric point of vew relations 1.4.1-2 become obvious.

1.5 Embedding invariants

Let K_p be a semisimple quadratic algebra over $\mathbf{Q}_p$ with maximal order $\mathcal{O}_p$. Let $\mathcal{O}_p^c$ be the order in K_p with conductor $c \in \mathbf{Z}_{\geq 0}$, i.e. let $\mathcal{O}_p^c := \mathbf{Z}_p + p^c \mathcal{O}_p$. Let $(e^+, e^-) \in (\mathbf{Z}_{\geq 0}, 0) \coprod (0, \mathbf{Z}_{\geq 0})$. We consider pairs $(h, \vec{R}_p)$ where $\vec{R}_p = (R_p, f_p)$ is an oriented Eichler order of type (p^{e^+}, p^{e^-}) and $h : \mathcal{O}_p^c \to R_p$ is an embedding with $R_p / h(\mathcal{O}_p^c)$ torsion-free. We call such a pair an optimal embedding.

Definition 1.5.1 *$\mathcal{E}_{p^{e^+}, p^{e^-}}(\mathcal{O}_p^c)$ denotes the set of isomorphism classes of optimal embeddings. An element of $\mathcal{E}_{p^{e^+}, p^{e^-}}(\mathcal{O}_p^c)$ is called an embedding invariant.*

Note that two optimal embeddings $(h, \vec{R}_p)$ and $(h', \vec{R}_p)$ are isomorphic iff there exists $r \in R_p$ with $h(\cdot) = r h'(\cdot) r^{-1}$.

Theorem 1.5.2 *The number $\#(\mathcal{E}_{p^{e^+},p^{e^-}}(\mathcal{O}_p^c))$ is as given in the following table.*

	e^+	e^-	$\mathcal{O}_p$ split	$\mathcal{O}_p$ ramified	$\mathcal{O}_p$ inert
$0 = c < m$	$2m$	0	2	0	0
	0	$2m$	0	0	2
	$2m+1$	0	2	0	0
	0	$2m+1$	0	0	2
$0 \neq c < m$	$2m$	0	$2(p+1)p^{c-1}$	0	0
	0	$2m$	0	0	$2(p-1)p^{c-1}$
	$2m+1$	0	$2(p+1)p^{c-1}$	0	0
	0	$2m+1$	0	0	$2(p-1)p^{c-1}$
$0 = c = m$	0	0	1	1	1
	1	0	2	1	0
	0	1	0	1	2
$0 < c = m$	$2m$	0	$(p+2)p^{c-1}$	$(p+1)p^{c-1}$	p^c
	0	$2m$	p^c	$(p-1)p^{c-1}$	$(p-2)p^{c-1}$
	$2m+1$	0	$(p+1)p^{c-1}$	p^c	0
	0	$2m+1$	0	p^c	$(p-1)p^{c-1}$
$c > m = 0$	0	0	1	1	1
	$2m+1$	0	2	2	2
	0	$2m+1$	0	0	0
$c > m > 0$	$2m$	0	$(p+1)p^{m-1}$	$(p+1)p^{m-1}$	$(p+1)p^{m-1}$
	0	$2m$	$(p-1)p^{m-1}$	$(p-1)p^{m-1}$	$(p-1)p^{m-1}$
	$2m+1$	0	$2p^m$	$2p^m$	$2p^m$
	0	$2m+1$	0	0	0

Proof. Computations of this sort have been done in many places, often under the name of "orbital integrals". We will translate the problem into the geometric setting of §1.3 and then leave it for the reader. For a more algebraic approach see e.g. [H-P-S 89].

Fix a quaternion algebra B_p over $\mathbb{Q}_p$, split if e^- is even and ramified if e^- is odd. If K_p is split and B_p is ramified then certainly $\mathcal{E}_{p^{e^+},p^{e^-}}(\mathcal{O}_p^c)$ is empty exactly as asserted on the table. In the remaining cases K_p embeds in B_p. Fix such an embedding $h : K_p \to B_p$. Consider the set of oriented Eichler orders $\vec{R}_p$ in B_p such that $h^{-1}(R_p) = \mathcal{O}_p^c$. We need to count the orbits of this set under the action by conjugation of $h(K_p)^\times$.

We therefore have a fixed tree $\mathcal{T}(B_{p^2})$ and a fixed subtree t_K consisting of either a σ-invariant geodesic ($\mathcal{O}_p$ split), a σ-stable edge ($\mathcal{O}_p$ ramified), or a σ-antiinvariant ($\mathcal{O}_p$ inert) geodesic. We are considering varying oriented segments $\vec{s}$ of length e, σ-invariant if $e = e^+$ and σ-antiinvariant if $e = e^-$, such that

$$\max_{v \in s}(\text{distance}(t_K, v)) = c.$$

The problem is to count such segments up to the action of $h(K_p)^\times$. □

Here are some important special cases:

1. $\#(\mathcal{E}_{1,1}(\mathcal{O}_p^c)) = 1$ irrespective of the type of $\mathcal{O}_p$. This can be viewed as a Skolem-Noether theorem on an integral level.

2. Define the Eichler symbol by

$$\left\{\frac{\mathcal{O}_p^c}{p}\right\} = \begin{cases} \left(\frac{K}{p}\right) & \text{if } \mathcal{O}_p^c \text{ is maximal } (c = 0) \\ 1 & \text{if } \mathcal{O}_p^c \text{ is nonmaximal } (c \geq 1). \end{cases}$$

   Then

$$\#(\mathcal{E}_{p,1}(\mathcal{O}_p^c)) = 1 + \left\{\frac{\mathcal{O}_p^c}{p}\right\} \quad (1.5.3)$$

$$\#(\mathcal{E}_{1,p}(\mathcal{O}_p^c)) = 1 - \left\{\frac{\mathcal{O}_p^c}{p}\right\} \quad (1.5.4)$$

   The general trace formula alluded to in §2.2 makes use of the full theorem. However when N is squarefree — the case treated in [Ei 56] — one only has to use the simple statement just given.

3. Suppose $\mathcal{O}_p^c$ is maximal, i.e. $\mathcal{O}_p^c = \mathcal{O}_p$. If $\mathcal{O}_p$ is split or inert then

$$\#(\mathcal{E}_{p^e,1}(\mathcal{O}_p)) = 1 + \left\{\frac{\mathcal{O}_p}{p}\right\} \quad (1.5.5)$$

$$\#(\mathcal{E}_{1,p^e}(\mathcal{O}_p)) = 1 - \left\{\frac{\mathcal{O}_p}{p}\right\} \quad (1.5.6)$$

   for all $e \geq 1$. Moreover the natural maps

$$\mathcal{E}_{p^e,1}(\mathcal{O}_p) \longrightarrow \mathrm{Hom}(\mathcal{O}_p, \mathbf{Q}_p \oplus \mathbf{Q}_p) \quad (\mathcal{O}_p \text{ split})$$
$$\mathcal{E}_{1,p^e}(\mathcal{O}_p) \longrightarrow \mathrm{Hom}(\mathcal{O}_p, \mathbf{Q}_{p^2}) \quad (\mathcal{O}_p \text{ inert})$$

   are bijections (of two element sets). This fact will appear critically in our intersection formula of Section 6, and hence in the detailed example worked out in Section 8. Less importantly, if $\mathcal{O}_p$ is ramified with discriminant p^f (= 8, 4 or an odd prime p) then

$$\#(\mathcal{E}_{p^e,1}(\mathcal{O}_p)) = \#(\mathcal{E}_{1,p^e}(\mathcal{O}_p)) =: \eta_{p^f,p^e} =: \begin{cases} 1 & \text{if } e \leq 1 \\ 0 & \text{if } e \geq 2. \end{cases}$$

1.6 Global Eichler orders

Let R be a quaternionic order over $\mathbb{Z}$. Let $N^+ = \prod p^{e_p^+}$ and $N^- = \prod p^{e_p^-}$ be a pair of relatively prime integers.

Definition 1.6.1 *R is an Eichler order of type (N^+, N^-) iff R_p is an Eichler order of type (p^{e^+}, p^{e^-}) for all primes p.*

Suppose that R is an Eichler order of type (N^+, N^-). An orientation on R is an orientation f_p on each of its localizations R_p. Equivalently an orientation on R is is a homomorphism $f : R \longrightarrow A_{N^+,N^-}$. Thus if $N = N^+N^-$ has t prime factors then R has 2^t orientations, permuted simply transitively by the automorphisms of A_{N^+,N^-}.

Definition 1.6.2 *$\overrightarrow{\mathrm{Ord}}_{N^+,N^-}$ denotes the set of oriented Eichler orders of type (N^+, N^-) up to isomorphism.*

Let $\vec{R}$ be an oriented Eichler order of type (N^+, N^-). Then we have a bijection

$$\begin{array}{rcl} B^\times \backslash \hat{B}^\times / \hat{R}^\times & \longrightarrow & \overrightarrow{\mathrm{Ord}}_{N^+,N^-} \\ L & \longmapsto & \text{Left order}(L). \end{array}$$

Theorem 1.6.3

1. *$B^\times \backslash \hat{B}^\times / \hat{R}^\times$ is finite.*
2. *If B is indefinite then $B^\times \backslash \hat{B}^\times / \hat{R}^\times$ has one element. Moreover $R^\times$ contains an element of negative norm.*

Proof. See [Vi 80]. We will give a numerical refinement of these assertions, also citing [Vi 80] for a proof, in Theorem 2.1.3. Statement 2 is referred to as "strong approximation". More precisely let U be any open compact subgroup of $\hat{B}^\times$. Then the norm map

$$B^\times \backslash \hat{B}^\times / U \longrightarrow \mathbb{Q}^\times \backslash \hat{\mathbb{Q}}^\times / n(U)$$

is a bijection; it is this fact which is referred to as strong approximation. In our case $U = \hat{R}^\times$ so $n(U) = \hat{\mathbb{Z}}^\times$; Statement 2 follows because $\#(\mathbb{Q}^\times \backslash \hat{\mathbb{Q}}^\times / \hat{\mathbb{Z}}^\times) = 1$ precisely because $\mathbb{Q}$ has class number one. □

2 Analytic curves $\mathcal{X}_{N^+,N^-}$

Fix for this and the next four sections a pair (N^+, N^-). In this section we will allow (N^+, N^-) to have either parity. In §1 we define a complex analytic curve $\mathcal{X}_{N^+,N^-}$. In §2 we discuss the cohomology groups of this curve. In §3 we interpret $\mathcal{X}_{N^+,N^-}$ as the coarse solution to a certain moduli problem. Since (N^+, N^-) is fixed we will often drop it from the notation.

2.1 Definition via uniformization

Choose an oriented Eichler order $\vec{\underline{R}} = (\underline{R}, \underline{f})$ of type (N^+, N^-). Here the underline is to distinguish our fixed Eichler order from other Eichler orders which will arise. Choose further identifications

$$\vec{\underline{R}}_p = \vec{R}_{p^{e+},p^{e-}} \tag{2.1.1}$$

the right side being the standard local oriented orders of §1.2; these identifications are for convenience only and will be used only occasionally. Let $B := \underline{R} \otimes \mathbf{Q}$ and let $\mathcal{Y} \subset B_\infty$ be the associated symmetric space. Put $N^-_{\max}$ equal to the squarefree part of N^-. Our identifications 2.1.1 give us a preferred oriented Eichler order in B of type $(1, N^-_{\max})$ and we denote it $\underline{R}_{\max}$.

First we define a possibly non-compact space $\mathcal{X}^f := \mathcal{X}^f_{N^+,N^-}$.

Definition 2.1.2 $\mathcal{X}^f := B^\times \setminus \mathcal{Y} \times \hat{B}^\times / \hat{\underline{R}}^\times$.

Up to isomorphism $\mathcal{X}^f$ depends only on the datum (N^+, N^-). Up to unique isomorphism it depends on the rigidifying choices made in §1.1 (which were needed to define the notion of orientation in the twisted case). We emphasize that $\mathcal{X}^f$ does not depend on the extra choice of $\vec{\underline{R}}$ just made. In fact one can give an alternative description of $\mathcal{X}^f$ which makes no reference to $\vec{\underline{R}}$. Namely points of $\mathcal{X}^f$ are in bijection with isomorphism classes of pairs $(h, \vec{R})$; here $\vec{R}$ is an oriented Eichler order of type (N^+, N^-) and h is an embedding $\mathbf{C} \to R_\infty$.

This alternative description is useful for describing the set $\pi_0(\mathcal{X}^f)$ of components as well. Namely $\pi_0(\mathcal{X}^f)$ is in bijection with the set $\overline{\mathrm{Ord}}_{N^+,N^-}$ of isomorphism classes of oriented orders of type (N^+, N^-). If (N^+, N^-)

is odd then $\mathcal{X}^f$ is connected by Theorem 1.6.3. If (N^+, N^-) is even then $\mathcal{X} = \mathcal{X}^f$ is in general disconnected and we write

$$\mathcal{X} = \coprod_{\bar{R} \in \overline{\mathrm{Ord}}_{N^+,N^-}} \mathcal{X}_{\bar{R}}.$$

The topological sphere $\mathcal{X}_{\bar{R}}$ has volume $4\pi/u(R)$ where $u(R) := [R^\times, \mathbf{Z}^\times] = \#(R^\times)/2$.

Theorem 2.1.3 Area$(\mathcal{X}^f)$ *is finite and given by the formula*

$$\frac{\pi N}{3} \prod_{p|N^+} (1+p^{-1}) \prod_{p|N^-} (1-p^{-1}).$$

Proof. The key point is that $\mathcal{X}^f_{1,N^-_{\max}}$ has area

$$\frac{\pi N^-_{\max}}{3} \prod_{p|N^-_{\max}} (1-p^{-1}).$$

This statement is proved in [Vi 80]. The general case follows because

$$\#(R^\times_{1,N^-_{\max}}/R^\times_{N^+,N^-}) = \frac{N}{N^-_{\max}} \prod_{p|N^+} (1+p^{-1}). \ \square$$

Now we discuss the crucial notion of a CM point. We recall that imaginary quadratic orders in $\mathbf{C}$ are indexed by negative integers congruent to $0, 1 \bmod 4$. Namely given $\mathcal{O}$, put D equal to -4 times the area of a fundamental parallelogram. Conversely given D put

$$\mathcal{O} = \mathbf{Z}[(\bar{D} + \sqrt{D})/2]$$

where

$$\bar{D} = \begin{cases} 0 & \text{if } D \equiv 0 \ (4) \\ 1 & \text{if } D \equiv 1 \ (4). \end{cases} \tag{2.1.4}$$

Definition 2.1.5 *Let $\mathcal{O}$ be a imaginary quadratic order in $\mathbf{C}$. Let $x = B^\times(y, L) \in \mathcal{X}^f$ and let $h_y : \mathbf{C} \to B_\infty$ be the associated embedding. x is called a CM point with order $\mathcal{O}$ iff $h_y^{-1}($ Left order$(L)) = \mathcal{O}$. The set of such points is denoted $x_{\mathcal{O}}$.*

Equally important is the action of $\hat{K}^\times/\hat{\mathcal{O}}^\times$ on $x_{\mathcal{O}}$.

Definition 2.1.6 *$\hat{K}^\times/\hat{\mathcal{O}}^\times$ acts on $x_{\mathcal{O}}$ by*

$$\hat{k}(y,L) := (y, h_y(\hat{k})L).$$

It is clear that this action of $\hat{K}^\times/\hat{\mathcal{O}}^\times$ on $x_{\mathcal{O}}$ factors through the quotient group $\mathrm{Cl}(\mathcal{O}) := K^\times \setminus \hat{K}^\times/\hat{\mathcal{O}}^\times$ and that this quotient group acts freely.

To a CM point $x \in x_{\mathcal{O}}$ we associate an embedding invariant $i(x) \in \mathcal{E}_{N^+,N^-}(\mathcal{O}) := \prod \mathcal{E}_{p^{e+},p^{e-}}(\mathcal{O}_p)$. Namely let $x = B^\times(y,L)$ have order $\mathcal{O}$ and define $i(x)$ to be the class of $h_y : \mathcal{O} \to$ Left order(L) in $\mathcal{E}_{N^+,N^-}(\mathcal{O})$. Chasing through the definitions we find that $i(\cdot)$ is well-defined on $x_{\mathcal{O}}$ and that $i(x_1) = i(x_2)$ iff x_1 and x_2 are in the same orbit under $\mathrm{Cl}(\mathcal{O})$. This proves the following formula:

Proposition 2.1.7 $\#(x_{\mathcal{O}}) = h(\mathcal{O})\#(\mathcal{E}_{N^+,N^-}(\mathcal{O}))$. □

We remark that there are simple algorithms for determining $h(\mathcal{O})$. See e.g. [B-S 66] for one such algorithm due to Lagrange and Gauss. [B-S 66] also contains tables of $h(\mathcal{O})$ for D small.

We can now treat elliptic points quite explicitly. Let $x = B^\times(y,L) \in \mathcal{X}^f$. Then the stabilizer of (y,L) in $B^\times$ is finite and contains $\{\pm 1\}$. We put $2e(x)$ equal to its cardinality. Clearly if x is not a CM point then $e(x) = 1$. On the other hand if x is a CM point then $e(x) = \#(\mathcal{O}^\times)/2$. Thus $\mathcal{O}_{-3}$ gives 3-elliptic points, $\mathcal{O}_{-4}$ gives 2-elliptic points, and these are the only sources of elliptic points.

Often $\mathcal{X}^f$ is already compact:

Theorem 2.1.8 *$\mathcal{X}^f$ is compact iff $B \not\cong M_2(\mathbf{Q})$.*

In the exceptional case one adjoins cusps. The set of cusps is in natural bijection with the set $R^\times \backslash P^1(\mathbf{Q})$. In particular it has

$$\prod_{p|N^+}\left(\sum_{i=0}^{e_p}\phi(p^{\min(i,e_p-i)})\right)\prod_{p|N^-}\left(\begin{cases}\phi(p^{e_p/2}) & \text{if } e_p \equiv 0\ (2)\\ 0 & \text{else}\end{cases}\right)$$

elements (for example if $N^- = 1$ then it is natural to group the cusps into clumps indexed by divisors of N; representatives of the M-clump are (a, M) with $1 \le a \le M$ relatively prime to M).

In general for a compact Riemann surface with constant curvature K and singularities such as ours one has

$$\chi(\mathcal{X}) = K\frac{\text{area}(\mathcal{X})}{2\pi} + \sum_{x\in\mathcal{X}} \frac{e(x)-1}{e(x)}$$

by the Gauss-Bonnet theorem. Here if x is a cusp we put $e(x) = \infty$ and consider $(e(x)-1)/e(x)$ to be 1. Thus putting everything together we have a formula for the Euler characteristic of $\mathcal{X}$:

Corollary 2.1.9

$$\begin{aligned}
\chi(\mathcal{X}) \;=\; & (-1)^{\#(\Sigma(N^+,N^-))}\frac{N}{6}\prod_{p|N^+}(1+p^{-1})\prod_{p|N^-}(1-p^{-1}) \\
& +\frac{2}{3}\eta_{-3,N}\prod_{\substack{p|N^+\\ p\neq 3}}\left(1+\left\{\frac{-3}{p}\right\}\right)\prod_{\substack{p|N^-\\ p\neq 3}}\left(1-\left\{\frac{-3}{p}\right\}\right) \\
& +\frac{1}{2}\eta_{-4,N}\prod_{\substack{p|N^+\\ p\neq 2}}\left(1+\left\{\frac{-4}{p}\right\}\right)\prod_{\substack{p|N^-\\ p\neq 2}}\left(1-\left\{\frac{-4}{p}\right\}\right) \\
& +\prod_{p|N^+}\left(\sum_{i=0}^{e_p}\phi(p^{\min(i,e_p-i)})\right)\prod_{p|N^-}\left(\begin{cases}\phi(p^{e_p/2}) & \text{if } e_p\equiv 0\ (2)\\ 0 & \text{else}\end{cases}\right). \square
\end{aligned}$$

This formula, and its term-by-term interpretation, justifies the four pictures drawn in the introduction. We will use this formula often in the sequel, most especially in Section 7 in our computation of examples.

Our adelic definition of $\mathcal{X}^f$ makes it trivial to define Atkin-Lehner involutions and Hecke operators on $\mathcal{X}^f$. Namely if $m = \prod p^{f_p}||N$ then we have a natural automorphism

$$\begin{aligned}
w_m : \mathcal{Y}\times \text{Lat}(B)_R &\longrightarrow \mathcal{Y}\times \text{Lat}(B)_R\\
(y,\{L_p\}) &\longmapsto (y,\{w_{p^{f_p}}L_p\}).
\end{aligned}$$

This automorphism descends to an involution, also denoted w_m, on $\mathcal{X}^f$. Similarly if $m = \prod p^{f_p}$ is prime to N then we have a natural correspondence

$$\begin{aligned}
T_m : \mathcal{Y}\times \text{Lat}(B)_R &\longrightarrow \mathcal{Y}\times \text{Lat}(B)_R\\
(y,\{L_p\}) &\longmapsto (y,\{T_{p^{f_p}}L_p\})
\end{aligned}$$

of degree $\sigma_1(m)$. This correspondence descends to a correspondence, also denoted T_m, on $\mathcal{X}^f$. We use the same notation for the extension of these correspondences to all of $\mathcal{X}$.

Let $W(N)$ denote the group of symbols $\{w_m\}_{m||N}$, multiplication being defined by $w_{m_1}w_{m_2} = w_{m_1m_2/(m_1,m_2)}$. Then $W(N)$ acts faithfully on $\mathcal{X}$ (in fact the datum (N^+, N^-) alone, i.e. without any extra rigidifying choices, determines $\mathcal{X}_{N^+,N^-}$ up to the ambiguity of $W(N)$). We also denote by $W(N)$ the group algebra $\mathbf{Q}(W(N))$. So $W(N)$ in this sense is a commutative algebra of $\mathbf{Q}$-dimension 2^t. Similarly let $T(N) = \mathbf{Q}\langle\{T_m\}_{(m,N)=1}\rangle$ denote the usual Hecke algebra (relations generated by

$$\{T_{p^f} = T_{p^f-1}T_p - pT_{p^f-2}\}_{p\text{ prime}, f\in\mathbf{Z}_{\geq 2}}$$
$$\{T_{m_1m_2} = T_{m_1}T_{m_2}\}_{(m_1,m_2)=1}).$$

The $\mathbf{Q}$-algebra $W(N)\otimes T(N)$, for which $\{w_{m_1}T_{m_2}\}_{m_1||N,(m_2,N)=1}$ is a convenient basis, acts naturally on $\mathcal{X}$ by correspondences with coefficients in $\mathbf{Q}$.

2.2 Decomposition of cohomology $H^*(\mathcal{X}_{N^+,N^-}, \mathbf{C})$

In this subsection we let $W(N) := W(N)\otimes\mathbf{C}$ and $T(N) := T(N)\otimes\mathbf{C}$. The cohomology groups $H^i(\mathcal{X}_{N^+,N^-}, \mathbf{C})$ are naturally modules for the complex algebra $W(N)\otimes T(N)$. They are semisimple and hence determined up to isomorphism by their traces

$$\{\mathrm{Tr}(w_{m_1}T_{m_2}|H^i(\mathcal{X}, \mathbf{C}))\}_{m_1||N,(m_2,N)=1}.$$

In this subsection we explain how $H^i(\mathcal{X}, \mathbf{C})$ can be expressed as a sum of standard $W(N)\otimes T(N)$-modules constructed from the cohomology of $\mathcal{X}_0(M)$, $M|N$. The theorem has its origins in [Ei 56]. It is a specialization of one of the main theorems of [J-L 70].

First we decompose $H^*(\mathcal{X}, \mathbf{C})$ into a two-dimensional piece $H^*(\mathcal{X}, \mathbf{C})^{\mathrm{Eis}}$ and a complementary piece $H^*(\mathcal{X}, \mathbf{C})^{\mathrm{cusp}}$. If (N^+, N^-) is odd then we simply set

$$\begin{aligned} H^*(\mathcal{X}, \mathbf{C})^{\mathrm{Eis}} &= H^0(\mathcal{X}, \mathbf{C}) \oplus H^2(\mathcal{X}, \mathbf{C}) \\ H^*(\mathcal{X}, \mathbf{C})^{\mathrm{cusp}} &= H^1(\mathcal{X}, \mathbf{C}). \end{aligned}$$

If (N^+, N^-) is even then $H^0(\mathcal{X}, \mathbf{Z})$ and $H^2(\mathcal{X}, \mathbf{Z})$ are naturally identified with the group $\mathbf{Z}^{\pi_0(\mathcal{X})}$ of integer-valued functions on the set of components

$\pi_0(\mathcal{X})$. Assume for simplicity that $\mathcal{X}$ has no elliptic points. Then the actions of $W(N) \otimes T(N)$ on $H^0(\mathcal{X}, \mathbf{C})$ and $H^2(\mathcal{X}, \mathbf{C})$ induce the same action on $\mathbf{C}^{\pi_0(\mathcal{X})}$. The representation $T(N) \otimes W(N) \to \mathrm{End}(\mathbf{C}^{\pi_0(\mathcal{X})})$ is symmetric for the standard inner product. $(1, \ldots, 1)$ is an eigenvector and we put

$$\begin{aligned} H^*(\mathcal{X}, \mathbf{C})^{\mathrm{Eis}} &= \mathbf{C}(1, \ldots, 1)_0 \oplus \mathbf{C}(1, \ldots, 1)_2 \\ H^*(\mathcal{X}, \mathbf{C})^{\mathrm{cusp}} &= \mathbf{C}(1, \ldots, 1)_0^{\perp} \oplus \mathbf{C}(1, \ldots, 1)_2^{\perp}. \end{aligned}$$

First we treat the piece $H^*(\mathcal{X}, \mathbf{C})^{\mathrm{Eis}}$ which in our context should be regarded as trivial. Define a $W(N) \otimes T(N)$-module $\mathbf{C}_{\mathrm{Eis}}$ by putting $\mathbf{C}_{\mathrm{Eis}} := \mathbf{C}$ as a complex vector space and letting $w_{m_1} T_{m_2}$ act by the scalar $\sigma_1(m_2) := \sum_{d|m_2} d$. Then it is clear that

$$H^*(\mathcal{X}, \mathbf{C})^{\mathrm{Eis}} \cong 2\mathbf{C}_{\mathrm{Eis}}.$$

Now we treat the non-trivial part $H^*(\mathcal{X}, \mathbf{C})^{\mathrm{cusp}}$. It is traditional, although hardly necessary, to state the result in terms of normalized newforms. We will follow this tradition, referring to e.g. [B-SD 75] for definitions. The main point for us is that the space of holomorphic cusp forms of weight two on $\Gamma_0(N)$ has a canonical basis

$$\coprod_{M|N} \coprod_{d|N/M} \coprod_{f \in \mathrm{New}(M)} f(dz).$$

Here $\mathrm{New}(M)$ denotes the finite set of normalized newforms on $\Gamma_0(M)$. So if $f = \sum_{n=1}^{\infty} a_n q^n \in \mathrm{New}(M)$ then $a_1 = 1$ and $T_m f = a_m f$. We put $\mathrm{Norm}(N) = \coprod_{M|N} \mathrm{New}(M)$.

Let $M|N$ and let $f = \sum a_n q^n$ be a normalized newform on $\Gamma_0(M)$. Let $\mathbf{C}_f \subseteq H^0(\mathcal{X}_0(M), \Omega)$ be the corresponding one-dimensional eigenspace for $W(M) \otimes T(M)$. $\mathbf{C}_f$ is naturally a $T(N)$-module as $T(N) \subseteq T(M)$; in fact for all $(m, N) = 1$, T_m acts on $\mathbf{C}_f$ by a_m. $\mathbf{C}_f$ is not naturally a $W(N)$-module if $M \neq N$. Let $M = \prod p^{d_p}$, $N = \prod p^{e_p}$. We artificially turn $\mathbf{C}_f$ into a $W(N)$-module by declaring $w_{p^{e_p}}$ to act on $\mathbf{C}_f$ the way that $w_{p^{d_p}}$ does; in particular if $d_p = 0$ then $w_{p^{e_p}}$ acts as 1.

Some important twisting is associated with Atkin-Lehner operators. To treat this twisting we introduce the following notational convention. Let $p^e || N$. If H is a $W(N) \otimes T(N)$-module then we let $s_p H$ denote the same complex vector space with $W(N)\otimes T(N)$-module structure given by

$$\begin{aligned} T_m|s_pH &:= T_m|H \quad (m,N)=1 \\ w_{\ell^f}|s_pH &:= w_{\ell^f}|H \quad \ell^f||N, \ell\neq p \\ w_{p^e}|s_pH &:= -w_{p^e}|H. \end{aligned}$$

Theorem 2.2.1

$$H^*(\mathcal{X}_{N^+,N^-},\mathbf{C})^{\mathrm{cusp}} \cong 2\sum_{f\in \mathrm{Norm}(N)} \tilde{c}_{N^+,N^-;\mathrm{cond}(f)}\mathbf{C}_f$$

as $W(N)\otimes T(N)$*-modules where if* $N=\prod p^{n_p}$, $N/M=\prod p^{\ell_p}$,

$$\tilde{c}_{N^+,N^-;M} = \prod_{p|N^+}\left(\left\lceil\frac{\ell_p}{2}\right\rceil + \left\lfloor\frac{\ell_p}{2}\right\rfloor s_p\right)\prod_{p|N^-}\begin{cases} s_p^{n_p} & \text{if } \ell_p\equiv 0(2) \\ 0 & \text{if } \ell_p\not\equiv 0(2). \end{cases}$$

Proof. This theorem is proved in [Ei 56] when N is squarefree, (N^+,N^-) is odd, and one removes all reference to Atkin-Lehner operators. One computes the traces $\mathrm{Tr}(w_{m_1}T_{m_2}|H^i(\mathcal{X},\mathbf{C}))$ by using the Lefschetz fixed point theorem. One of several key points is Equations 1.5.3-4. The proof given there generalizes straightforwardly to our case; here however one has to the entire table of §1.5 (refined to give the appropriate information about Atkin-Lehner operators).

A much more general theorem is proved, in the very illuminating context of representation theory, in [J-L 70]. See also [Gr 88] where some of the details for deducing Theorem 2.2.3 from the main theorem of [J-L 70] are given. □

If one is interested only in Hecke operators one doesn't have to worry about twisting:

Corollary 2.2.2

$$H^*(\mathcal{X}_{N^+,N^-},\mathbf{C})^{\mathrm{cusp}} \cong 2\sum_{f\in \mathrm{Norm}(N)} c_{N^+,N^-;\mathrm{cond}(f)}\mathbf{C}_f$$

as $T(N)$*-modules where*

$$c_{N^+,N^-;M} = \prod_{p|N^+}(1+\mathrm{ord}_p(N/M))\prod_{p|N^-}(1+(-1)^{\mathrm{ord}_p(N/M)}). \quad \square$$

For example in the most familiar case $\mathcal{X}_0(N)$ the eigenspaces of $T(N)$ on $H^1(\mathcal{X}_0(N),\Omega)$ have $T(N)$-generators the normalized eigenforms $f\in \mathrm{Norm}(N)$.

2.3 Moduli interpretation

In this subsection we realize the curve $\mathcal{X}^f_{N^+,N^-}$ as the solution to a certain moduli problem. As a preliminary we define $\underline{I}$ to be the largest lattice in $\underline{R}$ which is stable under right multiplication by $\underline{R}_{\max}$. For example if $\underline{R} = R_0(N)$ then

$$\underline{I} = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbf{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \ (N) \right\}.$$

Definition 2.3.1 *Let $\mathcal{S}$ be an analytic space. An (N^+,N^-)-toroidal surface over $\mathcal{S}$ is a triple $(\mathcal{T},\mathcal{C},i)$ where*

1. *$\mathcal{T}$ is a toroidal surface over $\mathcal{S}$.*
2. *$\mathcal{C} \subset \mathcal{T}$ is a finite subgroup space of constant rank N/N^-_{max}.*
3. *i is an inclusion (of sheaves of rings over $\mathcal{S}$) $\underline{R}_{1,N^-_{max}} \hookrightarrow \mathrm{End}(\mathcal{T})^{\mathrm{opp}}$ such that $\underline{R}_{N^+,N^-}$ stabilizes $\mathcal{C}$ and as a right $\underline{R}_{N^+,N^-}$-module $\mathcal{C}$ is locally cyclic with annihilator $\underline{I}_{N^+,N^-}$.*

To lighten the notation we will henceforth drop explicit reference to the inclusion i. Consider the functor

$$\begin{aligned} \mathcal{F}^f_{N^+,N^-} : \text{Analytic spaces} &\rightarrow \text{Sets} \\ \mathcal{S} &\mapsto \{(N^+,N^-)\text{-toroidal surfaces over } \mathcal{S}\}/\sim . \end{aligned}$$

Here and later in similar situations "$\sim$" means "isomorphism". There is a natural identification $\mathcal{X}^f_{N^+,N^-} \rightarrow \mathcal{F}^f_{N^+,N^-}(\mathbf{C})$ namely $(y,L) \mapsto (\mathcal{T}_{y,L},\mathcal{C}_{y,L}) = (B_\infty/LI, L/LI)$. Here the real four-torus B_∞/LI is turned into a complex two-torus by declaring $i \in \mathbf{C}$ to act on B_∞ by y on the left. This identification, together with a standard deformation theory argument, gives the following proposition:

Proposition 2.3.2 *$\mathcal{F}^f_{N^+,N^-}$ is coarsely represented by the analytic space $\mathcal{X}^f_{N^+,N^-}$.* $\square$

Polarizations. Now we fix $y \in \mathcal{Y}$ and consider polarizations [Mu 70] on $\mathcal{T}_{y,L}$. Let $v \in B^0$. Then $E_v(l_1,l_2) := t_{B/\mathbf{Q}}(v\bar{l}_1, l_2)$ is a $\mathbf{Q}$-valued skew-symmetric pairing. The set of v such that $E_v(l_1,l_2) \in \mathbf{Z}$ for $l_1,l_2 \in L$ is a lattice $NS^0 \subset B$. Each $v \in NS^0$ corresponds to an algebraic equivalence class of line bundles $\mathcal{L}_v$ on $\mathcal{T}_{(y,L)}$ via the standard dictionary [Mu 70]. $\mathcal{L}_v$ is non-degenerate with index i (i.e. for all $\mathcal{L} \in \mathcal{L}_v$, $H^j(\mathcal{T}_{y,L}, \mathcal{L}) \neq 0$ iff $i = j$) iff the Hermitian form

$$H_v(l_1, l_2) := E_v(il_1, l_2) + iE_v(l_1, l_2)$$

on $\mathrm{Lie}(\mathcal{T}_{y,L}) = B_\infty$ is non-degenerate with i positive eigenvalues. But one can easily check that $H_v(\cdot, \cdot)$ is non-degenerate iff $n(v) \neq 0$ and furthermore that in this case

$$\#(\text{Pos. eigenvalues}(H_v)) = \begin{cases} 0 & n(v) > 0, v/n(v) \sim y \\ 1 & n(v) < 0 \\ 2 & n(v) > 0, v/n(v) \not\sim y \end{cases}$$

if B is indefinite and

$$\#(\text{Pos. eigenvalues}(H_v)) = 1$$

if B is definite. Here "$\sim$" means "lies on the same component of $\mathcal{Y}$". Thus the construction $v \mapsto E_v(\cdot, \cdot)$ gives us polarizations (:= non-degenerate line bundles with index 0) iff B is indefinite. We have the following proposition:

Proposition 2.3.3 *If (N^+, N^-) is odd then every (N^+, N^-)-toroidal surface is an (N^+, N^-)-abelian surface.* □

Finally if y is not a CM point on $\mathcal{Y}$ then every line bundle comes from an $E_v(\cdot, \cdot)$, i.e. NS^0 is the entire Neron-Severi group $NS(\mathcal{T}_{y,L})$; thus if B is definite the (N^+, N^-)-toroidal spaces $\mathcal{T}_{y,L}$ for y a non-CM point are not polarizable.

Connection with usual moduli interpretation of $\mathcal{X}_0(N)^f$. Suppose now that $(N^+, N^-) = (N, 1)$ so that $R_{N,1}$ is identified with the familiar Eichler order $R_0(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z}) : N|c \right\}$.

Definition 2.3.4 *Let $\mathcal{S}$ be an analytic space. An N-elliptic curve is a pair $(\mathcal{E}, \mathcal{Z})$ where*

1. *$\mathcal{E}$ is an elliptic curve over $\mathcal{S}$.*
2. *$\mathcal{Z}$ is a locally cyclic subgroup of order N.*

We can consider the functor

$$\begin{aligned}\mathcal{F}_0(N)^f : \text{Analytic spaces} &\longrightarrow \text{Sets}\\ \mathcal{S} &\longmapsto \{N\text{-elliptic curves over } \mathcal{S}\}/\sim\end{aligned}$$

We have a canonical isomorphism $\mathcal{F}^f_{N,1} \to \mathcal{F}_0(N)^f$. Namely given an (N^+, N^-)-abelian surface $(\mathcal{T}, \mathcal{C})$ over a space $\mathcal{S}$ let

$$\begin{aligned}\mathcal{E} &:= \text{Image}\left(\begin{pmatrix}0\,0\\0\,1\end{pmatrix} : \mathcal{T} \to \mathcal{T}\right) \subset \mathcal{T}\\ \mathcal{Z} &:= \mathcal{C}.\end{aligned}$$

Then $(\mathcal{E}, \mathcal{Z})$-is an N-elliptic curve over $\mathcal{S}$. Conversely given an N-elliptic curve $(\mathcal{E}, \mathcal{Z})$ define

$$\begin{aligned}\mathcal{T} &:= \mathcal{E} \oplus \mathcal{E}\\ \mathcal{Z} &:= 0 \oplus \mathcal{Z}\\ (e_1, e_2)\begin{pmatrix}a\,b\\c\,d\end{pmatrix} &:= (ae_1 + ce_2, be_1 + de_2).\end{aligned}$$

Compactifications. We recall that $\mathcal{X}^f$ is compact except in the exceptional case where $\Sigma(N^+, N^-) = \{\infty\}$ so $B = M_2(\mathbf{Q})$. In this case one can define a notion of generalized (N^+, N^-)-abelian surface so that the corresponding functor $\mathcal{F}_{N^+,N^-}$ is coarsely represented by the compactified curve $\mathcal{X}$. See [K-M 85] where this is done in the equivalent language of elliptic curves with appropriate level structure.

3 Algebraic curves X_{N^+,N^-}

We keep fixed the pair (N^+, N^-) fixed in Section 2. If (N^+, N^-) is even then one can trivially define a model X over $\mathbf{Q}$ for the complex analytic space $\mathcal{X}$. Namely seeing how

$$\mathcal{X} = B^\times \backslash \mathcal{Y} \times \hat{B}^\times / \hat{R}^\times$$

we simply define

$$X := B^\times \backslash Y \times \hat{B}^\times / \hat{R}^\times$$

Here Y is the genus zero curve canonically associated to B (the conic in the projective plane of trace zero elements given by the norm). The definition makes sense because the stabilizer in $B^\times$ of each component of $Y \times \hat{B}^\times / \hat{R}^\times$ is finite. We assume therefore that the fixed pair (N^+, N^-) is odd.

3.1 Definition via moduli

We define the notion of (N^+, N^-)-abelian scheme over a scheme S over $\mathbf{Z}[1/N]$ simply by replacing "complex analytic space" in Definition 2.3.1 with "scheme over $\mathbf{Z}[1/N]$" (and interpreting "toroidal" as "abelian"). The analytic moduli problem $\mathcal{F}^f$ naturally comes from an arithmetic moduli problem:

$$\begin{aligned} \underline{F}^f : \text{Schemes over } \mathbf{Z}[1/N] &\rightarrow \text{Sets} \\ S &\mapsto \{(N^+, N^-)\text{-Abelian surfaces over } S\}/\sim \end{aligned}$$

The analog of Proposition 2.3.2 holds here but lies considerably deeper.

Theorem 3.1.1 *$\underline{F}^f$ is coarsely represented by a regular, connected, two-dimensional scheme $\underline{X}^f_{N^+,N^-}$ smooth over* Spec $\mathbf{Z}[1/N]$. *If $\Sigma(N^+, N^-) \neq \{\infty\}$ then $\underline{X}^f_{N^+,N^-}$ is proper over $\mathbf{Z}[1/N]$.*

Proof. See e.g. [Dr 76]. □

An important fact, which will be used crucially in §3.3, is the Eichler-Shimura congruence. We recall that in general if C is a d-dimensional variety over $\mathbf{F}_p$ then one has the absolute Frobenius map $F_p : C \rightarrow C$ which is a finite map of degree p^d.

Theorem 3.1.2 *Let p be a prime not dividing N. On $\underline{X}_{\mathbf{F}_p}$, $T_p = F_p + F_p^t$.*

Proof. This relation follows from the description of the bad reduction of $\underline{X}_{N^+p,N^-}$ given in §4.3. □

3.2 Behavior of CM points

First we recall some facts from class field theory. Let $D = dC^2$ be an imaginary quadratic discriminant. Then one has the Hilbert ring class field $K_d \subseteq H_D \subset \mathbf{C}$. H_D is an abelian extension of K_d ramified only at primes dividing C. The Artin map

$$\begin{array}{rcl}(\text{Primes in } K_d \text{ prime to } C) & \longrightarrow & \mathrm{Gal}(H_D/K_d) \\ \mathcal{P} & \longmapsto & \sigma_{\mathcal{P}}\end{array}$$

factors precisely through $\mathrm{Cl}(\mathcal{O}_D)$ and this fact characterizes H_D. Thus we have a canonical isomorphism

$$\mathrm{Gal}(H_D/\mathbf{Q}) \xrightarrow{\sim} \mathrm{Cl}(\mathcal{O}_D) \rtimes \{1, \sigma_\infty\}. \tag{3.2.1}$$

Theorem 3.2.2 *$x_{\mathcal{O}_D,\epsilon} \cup x_{\mathcal{O}_D,-\epsilon}$ lies in the subset $X(H_D)$ of $X(\mathbf{C})$. Moreover the action of $\mathrm{Cl}(\mathcal{O}_D) \rtimes \{1, \sigma_\infty\}$ on $x_{\mathcal{O}_D,\epsilon} \cup x_{\mathcal{O}_D,-\epsilon}$ defined in §2.1 coincides with the natural action of $\mathrm{Gal}(H_D/\mathbf{Q})$ on $X(H_D)$ via 3.2.1.*

Proof. This identity follows from the theory of complex multiplication for abelian varieties which was developed in [S-T 61] (alternatively, it could be deduced from the corresponding theory for elliptic curves only). □

The theorem asserts in particular that the divisor $x_{\mathcal{O}_D,\epsilon} \oplus x_{\mathcal{O}_D,-\epsilon}$ on $\mathcal{X}$ is defined over $\mathbf{Q}$ and thus comes from a divisor on X; we will denote this divisor by $x_{D,\pm\epsilon}$. If $\epsilon \neq -\epsilon$ then $x_{D,\pm\epsilon}$ is a single scheme-theoretic point; the choice of a point in $x_{D,\pm\epsilon}(\mathbf{C})$ identifies the residue field with H_D. If $\epsilon = -\epsilon$ then $x_{D,\pm\epsilon}$ consists of a single scheme-theoretic point counted with multiplicity two; the choice of a point in $x_{D,,\pm\epsilon}(\mathbf{C})$ identifies the residue field of $x_{D,\pm\epsilon}$ with a subfield of index two in H_D. The former case is the more typical; however the latter case is perhaps more familiar as it is the only one which occurs on the j-line $X_{1,1}$.

3.3 Decomposition of Jacobian J_{N^+,N^-}

We will be considering both abelian varieties in the usual sense and abelian varieties defined up to isogeny. If A is an abelian variety then we let A^0 denote the corresponding abelian-variety-up-to-isogeny. In general all symbols used to represent abelian-varieties-up-to-isogeny will contain a superscript 0. As an example of our notation suppose that A is an abelian variety and $1 = e_1 + \cdots + e_n$ is a decomposition of the identity of $\mathrm{End}(A^0) = \mathrm{End}(A) \otimes \mathbf{Q}$ into orthogonal idempotents. Define A_{e_i} to be the largest abelian subvariety such that e_i acts by the identity and e_j, $j \neq i$ act by zero. This exact definition is partly just a convention as one could have alternatively defined A_{e_i} as a quotient variety. While one certainly doesn't have $A = \oplus A_{e_i}$ in general one does have $A^0 = \oplus A^0_{e_i}$.

Let $M|N$. Let $J_0(M)$ denote the Jacobian of $X_0(M)$. Let f be a normalized newform on $\Gamma_0(N)$. The Fourier coefficients of f generate a totally real number field $E_f \subset \mathbf{R}$ of finite degree over $\mathbf{Q}$. The q-expansions f^σ, $\sigma \in \mathrm{Hom}(E_f, \mathbf{R})$, are also normalized newforms on $\Gamma_0(M)$. Let $f^\bullet := \{f^\sigma\} \subset \mathrm{New}(M)$. Then $f^\bullet$ determines an idempotent in the Hecke algebra $T(M) \subseteq \mathrm{End}(J_0(M)^0)$ and hence a factor

$$A^0_{f^\bullet} \subseteq J_0(M)^0.$$

We have $\dim(A^0_{f^\bullet}) = [E_{f^\bullet} : \mathbf{Q}]$ and $\mathrm{End}(A^0_{f^\bullet}) \cong E_{f^\bullet}$ canonically.

Theorem 3.3.1

$$J^0_{N^+,N^-} \cong \bigoplus_{f^\bullet \subset \mathrm{Norm}(N)} c_{N^+,N^-;\mathrm{cond}(f^\bullet)} A^0_{f^\bullet}$$

where

$$c_{N^+,N^-;M} = \prod_{p|N^+} (1 + \mathrm{ord}_p(N/M)) \prod_{p|N^-} (1 + (-1)^{\mathrm{ord}_p(N/M)})$$

(as in Corollary 2.2.2).

Proof. First we recall some general facts. Let A_1 and A_2 be two abelian varieties each of dimension g. Let ℓ be a prime number. Consider the contravariant Tate modules

$$H^1(A_{i,\bar{\mathbf{Q}}}, \mathbf{Z}_\ell) := \mathrm{Hom}(A_i(\bar{\mathbf{Q}})[\ell^\infty], \mathbf{G}_m(\bar{\mathbf{Q}})[\ell^\infty]).$$

These are $\mathbf{Z}_\ell(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$-modules which are rank $2g$ as $\mathbf{Z}_\ell$-modules. If A_i has good reduction at a prime $p \neq \ell$ then any inertia group I_p at p acts trivially on $H^1(A_{i,\bar{\mathbf{Q}}}, \mathbf{Z}_\ell)$. The corresponding $\mathbf{Q}_\ell(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$-modules

$$H^1(A_{i,\bar{\mathbf{Q}}}, \mathbf{Q}_\ell) := H^1(A_{i,\bar{\mathbf{Q}}}, \mathbf{Z}_\ell) \otimes \mathbf{Q}_\ell$$

only depend on A_i^0. Faltings' semisimplicity theorem says that the two $\mathbf{Q}_\ell(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$-modules $H^1(A_{i,\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ are semisimple. His isogeny theorem says that if $H^1(A_{1,\bar{\mathbf{Q}}}, \mathbf{Q}_\ell) = H^1(A_{2,\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)$ as $\mathbf{Q}_\ell(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$-modules then in fact $A_1^0 \cong A_2^0$.

Still speaking generally, suppose that one is given two semisimple $\mathbf{Q}_\ell(\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}))$-modules M_1 and M_2, each unramified outside the set of primes dividing an integer N. Let p be a prime not dividing N. Let $\sigma_p \in \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$ be a geometric Frobenius element. Then, as the corresponding inertia group I_p acts trivially, σ_p acts on M_i. Furthermore the characteristic polynomials $P_{p,i}(x) := \det(1 - \sigma_p x | M_i)$ depend only on p. The Cebotarev density theorem implies that if $P_{p,1}(x) = P_{p,2}(x)$ for all $p \nmid N$, then in fact $M_1 \cong M_2$.

The last general fact we need is as follows. Let $\mathcal{P}$ be a place of $\bar{\mathbf{Q}}$ over the prime p not dividing $N\ell$. Let $\bar{\mathbf{F}}_p$ be the corresponding residue field $\bar{\mathbf{Z}}/\mathcal{P}$. Then there is a canonical identification $H^1(A_{\bar{\mathbf{Q}}}, \mathbf{Z}_\ell) = H^1(A_{\bar{\mathbf{F}}_p}, \mathbf{Z}_\ell)$. Furthermore the automorphism σ_p of $A_{\bar{\mathbf{Q}}}$ and the degree p^{2g} self-map F_p of $A_{\bar{\mathbf{F}}_p}$ induce the same endomorphism of this $\mathbf{Z}_\ell$-module.

Now we return to our particular setting. On the one hand we have

$$\begin{aligned}
\mathrm{Tr}(\sigma_p | H^1(J_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)) &= \mathrm{Tr}(F_p | H^1(J_{\bar{\mathbf{F}}_p}, \mathbf{Q}_\ell)) \\
&= \frac{1}{2}\mathrm{Tr}(T_p | H^1(J_{\bar{\mathbf{F}}_p}, \mathbf{Q}_\ell)) \\
&= \frac{1}{2}\mathrm{Tr}(T_p | H^1(J_{\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)) \\
&= \frac{1}{2}\mathrm{Tr}(T_p | H^1(\mathcal{J}, \mathbf{C})) \\
&= \frac{1}{2}\mathrm{Tr}(T_p | H^1(\mathcal{X}, \mathbf{C})).
\end{aligned}$$

On the other hand by a similar passing through characteristic p argument we have

$$\mathrm{Tr}(\sigma_p | H^1(\bigoplus_{f^\bullet} c_{N^+,N^-;\mathrm{cond}(f^\bullet)} A^0_{f^\bullet;\bar{\mathbf{Q}}}, \mathbf{Q}_\ell)) \;=\; \bigoplus_{f} c_{N^+,N^-;\mathrm{cond}(f)} \mathrm{Tr}(T_p | \mathbf{C}_f).$$

The result then follows the characteristic zero identity

$$H^1(\mathcal{X}_{N^+,N^-},\mathbf{C}) = \bigoplus_f c_{N^+,N^-;\mathrm{cond}(f)}\mathbf{C}_f$$

of Corollary 2.2.2. □

Corollary 3.3.2 *The abelian varieties $J^{\mathrm{new}}_{N^+,N^-}$ and $J_0(N)^{\mathrm{new}}$ are isogenous.* □

This corollary was in fact proved before Faltings' isogeny theorem in [Ri 80] under the assumption that there exists a prime p with $p||N$.

3.4 The Taniyama-Weil conjecture

Let A be an abelian variety over $\mathbf{Q}$. Then A has a conductor $C = \prod p^{c_p}$. c_p is defined as the Artin conductor of the associated representation of any inertia group I_p at p on $H^1_{\mathrm{et}}(\bar{A}, \mathbf{Q}_\ell)$, any $\ell \neq p$. $c_p = 0$ iff A has good reduction at p. C depends only on A^0. If $\mathrm{End}(A^0)$ contains a field of degree d then C is necessarily a d^{th} power.

The example of principal interest to us is the case where $\dim(A) = 1$, i.e. A is an elliptic curve. Then one has the following facts:

$$c_p \begin{cases} = 0 \\ = 1 \\ \geq 2 \end{cases} \text{if } A \text{ has } \begin{cases} \text{good} \\ \text{multiplicative} \\ \text{additive} \end{cases} \text{reduction at } p.$$

In fact if $p \neq 2,3$ then the last inequality is an equality, giving a completely geometric interpretation of the conductor.

Theorem 3.4.1 *Let $f^* = \{f, f^{\sigma_2}, \ldots, f^{\sigma_d}\} \subset \mathrm{New}(N)$. Then $A^0_{f^*}$ has conductor N^d.*

Proof. See [Ca 87] for a thorough discussion of this theorem and similar theorems. The basic point is that to determine $\mathrm{cond}(A^0_{f^*})$ at a prime p one has to thoroughly analyze the bad reduction of $X_0(N)$ at p. □

Conjecture 3.4.2 *Let A be a d-dimensional abelian variety over $\mathbf{Q}$ such that $\mathrm{End}(A^0)$ contains a totally real field of degree d. Let $N^d := \mathrm{cond}(A)$. Then there exists $f^* = \{f, f^{\sigma_2}, \ldots, f^{\sigma_d}\} \subseteq \mathrm{New}(N)$ such that $A^0 \cong A^0_{f^*}$.*

See e.g. [B-SD 75] or [Si 86] for a discussion of this conjecture (in the case of elliptic curves).

4 Arithmetic surfaces $\underline{X}_{N^+,N^-}$: facts

We continue to assume that our fixed pair (N^+, N^-) is odd. So as not to be distracted by elliptic points and cusps we make the following assumptions.

1. N^- is not a perfect square (= no cusps).

2. At least one of

$$\eta_{-3,N} \coprod \left\{1 + \left\{\frac{-3}{p}\right\}\right\}_{\substack{p|N^+ \\ p\neq 3}} \coprod \left\{1 + \left\{\frac{-3}{p}\right\}\right\}_{\substack{p|N^- \\ p\neq 3}}$$

   is zero and at least one of

$$\eta_{-4,N} \coprod \left\{1 + \left\{\frac{-4}{p}\right\}\right\}_{\substack{p|N^+ \\ p\neq 2}} \coprod \left\{1 + \left\{\frac{-4}{p}\right\}\right\}_{\substack{p|N^- \\ p\neq 2}}$$

   is zero (= no elliptic points in characteristic 0)

3. (Like 2 with "one" replaced by "two") (= no elliptic points in any characteristic)

However almost everything we say is literally true for (N^+, N^-) arbitrary and the remainder can be rather simply fixed up. In the sequel, most particularly in Section 7, we will apply the theorems of this section generalized to the arbitrary case without comment.

In §1 we define a proper scheme $\underline{X}_{N^+,N^-}$ over $\mathbf{Z}$ extending the scheme $\underline{X}_{N^+,N^-}$ over $\mathbf{Z}[1/N]$ defined in §3.1. For §§2–4 we fix a prime power p^e exactly dividing N. We describe the schemes $\underline{X}_{\mathbf{Z}_p}$ in the case $p \nmid N$, $p|N^+$, and $p||N^-$ respectively. The remaining case $p^2|N^-$ will be discussed in the next section.

Our object in this section is simply to describe the schemes $\underline{X}_{N^+,N^-}$, not to prove our descriptions are correct. We will give references as we go. These references certainly contain the main points. However we will omit deriving our specific statements from the general theorems of our references. The reader should note that were we to prove all the statements we would have to modify our order of presentation. In particular we would have to treat the deformation theory considered in §2-4 first and only then give the main existence theorem 4.1.1.

Our fundamental references for this and the next section are [K-M 85] for the case $p \notin \Sigma(N^+, N^-)$ and [Dr 76] for the case $p \in \Sigma(N^+, N^-)$. Strictly speaking, [K-M 85] treats only the case $\Sigma_f(N^+, N^-) = \emptyset$; however we will use it without comment as a reference for the general case: the situation above Spec $\mathbb{Z}_p$ for $p \notin \Sigma_f(N^+, N^-)$ is governed by p-divisible groups and for many questions it makes no difference what $\Sigma_f(N^+, N^-)$ is.

4.1 Definition via moduli

We define the notion of (N^+, N^-)-abelian scheme over an arbitrary scheme S by replacing "complex analytic space" in Definition 2.3.1 with "scheme" (and again interpreting "toroidal" as "abelian"). Even at the level of definitions there is subtlety. Namely here "locally cyclic" has to be interpreted in Drinfeld's sense; see [K-M 85].

The naive extension of $\underline{F}_{N^+,N^-}$ from schemes over $\mathbb{Z}[1/N]$ to all schemes is not what we want. One has to add an extra condition first introduced in [Dr 76]. Let (A, C) be an (N^+, N^-)-abelian surface over a scheme S. Suppose $p|N^-_{\max}$. Let F be a field containing $\mathbb{F}_{p^2}$. Let $s \in S(F)$. Consider the two-dimensional F-vector space Lie(A_s). It is naturally a right $\underline{R}_{1,N^-_{\max}}$-module, hence a right $\underline{R}_{1,N^-_{\max}}/p$-module. Now $\underline{R}_{1,N^-_{\max}}/p$ is a non-commutative rank four $\mathbb{F}_p$-algebra which sits in a short exact sequence:

$$\underline{J}_{1,N^-_{\max}} \subset \underline{R}_{1,N^-_{\max}}/p \to \underline{R}_{1,N^-_{\max}}/\underline{J}_{1,N^-_{\max}}.$$

Embed $\mathbb{F}_{p^2}$ into $\underline{R}_{1,N^-_{\max}}/p$ so that the projection onto $\underline{R}_{1,N^-_{\max}}/\underline{J}_{1,N^-_{\max}} = \mathbb{F}_{p^2}$ is the identity. For the moment we consider Lie(A) simply as a module over the subalgebra $\mathbb{F}_{p^2}$. There are three possibilities.

1. $\mathbb{F}_{p^2}$ acts through scalars through the inclusion $\mathbb{F}_{p^2} \subset F$ (pure of type 1).

2. $\mathbb{F}_{p^2}$ acts through scalars through the conjugate injection $\mathbb{F}_{p^2} \hookrightarrow F$ (pure of type -1).

3. Lie(A) is the direct sum of one-dimensional vector spaces Lie$(A)^1$ and Lie$(A)^{-1}$ on which $\mathbb{F}_{p^2}$ acts as scalars as indicated (mixed).

Here we follow the terminology of [Ri —] (mixed$_{[\text{Ri } -]}$ = special$_{[\text{Dr } 76]}$). If F is simply a field of characteristic p, rather than an extension field of $\mathbb{F}_{p^2}$, possibilities 1 and 2 are indistinguishable but we still have the distinction of pure versus mixed.

An (N^+, N^-)-abelian surface over a scheme S is called mixed if for all $p|N^-_{\max}$, all fields F of characteristic p, and all $s \in S(F)$, A_s is mixed. Consider the functor

$$\begin{aligned} \underline{F}_{N^+,N^-} : \text{Schemes} &\longrightarrow \text{Sets} \\ S &\longmapsto \{\text{mixed}(N^+, N^-)\text{-abelian surfaces over } S\}/\sim . \end{aligned}$$

It is this functor which is the good extension of F_{N^+,N^-}:

Theorem 4.1.1 *$\underline{F}_{N^+,N^-}$ is coarsely represented by an arithmetic surface $\underline{X}_{N^+,N^-}$.*

Proof. [Dr 76]. □

The fact that $\underline{X}$ represents $\underline{F}$ only coarsely presents an annoyance but not a serious problem. Namely our assumption excluding elliptic points implies that $\text{Aut}(A, C)_{\underline{R}} = \{\pm 1\}$ for any (N^+, N^-)-abelian surface (A, C) over a connected scheme S. Let S be a scheme over $\underline{X}$. Then descent theory says that there exists an (N^+, N^-)-abelian scheme over S inducing the given map $S \to \underline{X}$ iff a certain obstruction in $H^2(S, \pm 1)$ vanishes. Moreover, if this obstruction vanishes then the set of such abelian schemes up to isomorphism is naturally a principal homogeneous space over $H^1(S, \pm 1)$. The key example for us is when S is some closed point z on $\underline{X}$ or the completion $\underline{X}_z$ of $\underline{X}$ at such a point. Since the residue field F is finite $H^2(S, \pm 1) = H^2(\hat{\mathbf{Z}}, \pm 1) = 0$ and so the obstruction vanishes. Also $H^1(S, \pm 1) = \text{Hom}(\hat{\mathbf{Z}}, \pm 1) = \pm 1$, so there are two (N^+, N^-) abelian varieties over z inducing the identity map, each defined up to the ambiguity of ± 1. Instead of repeating awkward phrases like "Let $(A, C)_z$ be a semi-universal (N^+, N^-)-abelian surface over z and let $(A, C)_{\underline{X}_z}$ be its universal deformation" we will leave all this fussiness implicit and just write $(A, C)_z$. Thus we use abbreviated language which invites the reader to simply pretend that $\underline{X}$ is a fine moduli space.

4.2 $p \nmid N$

In this section we assume that $p \nmid N$ so that $\underline{X} := \underline{X}_{\mathbf{Z}_p}$ is smooth over $\mathbf{Z}_p$ by Theorem 3.1.1. We describe the closed points on $\underline{X}$ and their formal neighborhoods. The most basic of the several phenomena discussed is that there is a finite subscheme $z_{ss} \subset \underline{X}$ which is up to a quadratic twist a copy of the even curve $\mathcal{X}_{N^+,N^-p}$, components having been replaced by copies of Spec F_{p^2}.

Let $\bar{\mathsf{F}}_p$ be an algebraic closure of F_{p^2} and let (A, C) be an (N^+, N^-)-abelian surface over $\bar{\mathsf{F}}_p$. End(Lie$(A))_{\underline{R}}$ is then simply $\bar{\mathsf{F}}_p$. Hence in particular End$(A,C)_{\underline{R}}$ comes equipped with a map j : End$(A,C)_{\underline{R}} \rightarrow \bar{\mathsf{F}}_p$.

Theorem 4.2.1 *Either*

1. End$(A)_{\underline{R}}$ *is an Eichler order R of type (N^+, N^-p).*
2. End$(A)_{\underline{R}}$ *is an imaginary quadratic order, $\mathcal{O} \subset \mathsf{Z}_p$ maximal at p and embeddable in $\underline{R}$.*

Furthermore all possibilities admitted by 1 and 2 occur.

Proof. The necessary general theorems are in [Wa 69]. □

One says that A is supersingular in the first case and ordinary in the second case. One thereby obtains a partition

$$\underline{X} = z_{ss} \coprod \coprod_{\mathcal{O} \subset \mathsf{Z}_p} z_{\mathcal{O}}$$

in the sense of closed points. Here z_{ss} and each of the $z_{\mathcal{O}}$ are finite disjoint unions of spectra of finite fields.

Supersingular points. The points in z_{ss} have residue field either F_p or F_{p^2}. Instead of describing z_{ss} directly we will describe the base change $z_{ss} := (z_{ss})_{\mathsf{F}_{p^2}}$ together with the natural action of σ_p. This equivalent point of view is more convenient for several reasons — for starters, all points in z_{ss} now have the same residue field, namely F_{p^2}.

Theorem 4.2.2 *The map*

$$\begin{array}{rcl} z_{ss} & \longrightarrow & \overline{\mathrm{Ord}}_{N^+,pN^-} \\ z & \rightarrow & \mathrm{End}((A,C)_z)_{\underline{R}} \end{array}$$

is a bijection. Here End$((A,C)_z)_{\underline{R}}$ *is oriented by Remark 1.2.4.*

Proof. [Wa 69]. □

We will therefore write $z_{ss} = \coprod z_{\vec{R}}$. One of course has to carefully distinguish between the single fixed oriented order $\underline{\vec{R}}$ and the varying oriented orders $\vec{R}$. $\underline{R}$ is indefinite and plays a completely passive role; for example in the most classical case $(N^+, N^-) = (1,1)$ — technically excluded from our present discussion because of cusps and elliptic points — $\underline{R} = M_2(\mathbf{Z})$ and one can certainly discuss $X_{1,1}$ without mentioning $\underline{R}$ at all. The $\vec{R}$ are definite Eichler orders and in contrast play an active role in our considerations.

We now consider the completion $\underline{X}_{\vec{R}}$ of the scheme $\underline{X} := \underline{X}_{\mathbf{Z}_{p^2}}$ at the point $z_{\vec{R}}$. $\underline{X}_{\vec{R}}$ is identified with the universal deformation space of $(A,C)_{z_{\vec{R}}}$, considered with its right $\underline{R}$-module structure. By the Serre-Tate theorem it is further identified with the universal deformation space of the p-divisible group $A_{z_{\vec{R}}}[p^\infty]$, considered with its right $\underline{R}_p$-module structure. But now $\underline{R}_p = M_2(\mathbf{Z}_p)$ and so we have a decomposition

$$A_{\underline{X}_{\vec{R}}} = A_{\underline{X}_{\vec{R}}}[p^\infty]^1 \oplus A_{\underline{X}_{\vec{R}}}[p^\infty]^2$$

as in §2.3. For purely formal reasons $A_{\underline{X}_{\vec{R}}}[p^\infty]^1$ is identified with the universal deformation of $A_{z_{\vec{R}}}[p^\infty]^1$.

$A_{z_{\vec{R}}}[p^\infty]^1$ is a connected self-dual p-divisible group over $\mathbf{F}_{p^2}$ with height two. All such objects become isomorphic over $\bar{\mathbf{F}}_p$. Their deformation theory has been studied in detail in [Gr 86].

Before stating the result we need to make a few preliminary remarks. $\mathrm{End}(A_{z_{\vec{R}}}[p^\infty])_{\underline{R}_p}$ is the local Eichler order R_p of type $(1,p)$. If S is a nonempty closed subscheme of $\underline{X}_{\vec{R}}$ then the natural map $\mathrm{End}(A_S[p^\infty])_{\underline{R}_p} \to \mathrm{End}(A_{z_{\vec{R}}}[p^\infty])_{\underline{R}_p}$ is injective. We shall therefore view $\mathrm{End}(A_S[p^\infty])_{\underline{R}_p}$ as simply a suborder of R_p. As an example the theorem below implies that $\mathrm{End}(A_{\underline{X}_{\vec{R}}}[p^\infty])_{\underline{R}_p}$ consists only of the scalar endomorphisms $\mathbf{Z}_p$ in R_p.

Theorem 4.2.3 *Let $\mathcal{O}_p^c$ be a quadratic suborder of* $\mathrm{End}(A_{z_{\vec{R}}}[p^\infty])_{\underline{R}_p}$ *with conductor c. Then there is a unique closed subscheme $\underline{x}_{\mathcal{O}_p^c,\vec{R}}$ of $\underline{X}_{\vec{R}}$ having the following properties:*

1. *$\underline{x}_{\mathcal{O}_p^c,\vec{R}}$ is the spectrum of a discrete valuation ring.*
2. $\mathrm{End}(A_{\underline{x}_{\mathcal{O}_p^c,\vec{R}}}[p^\infty])_{\underline{R}_p} = \mathcal{O}_p^c$.

Also $\underline{x}_{\mathcal{O}^c_p,\tilde{R}}$ *is the spectrum of a ring class extension of* $\mathcal{O}_p$. *If* $\mathcal{O}_p$ *is inert then* $\underline{x}_{\mathcal{O}^c_p,\tilde{R}}/\mathrm{Spec}\ \mathcal{O}_p$ *is totally ramified of degree* $\begin{cases} 1 & \text{if } c = 0 \\ (p-1)p^{c-1} & \text{if } c \geq 1. \end{cases}$ *If* $\mathcal{O}_p$ *is ramified then* $\underline{x}_{\mathcal{O}^c_p,\tilde{R}}/\mathrm{Spec}\ \mathcal{O}_p$ *has inertia degree 2 and ramification degree* p^c.

Proof. [Gr 87]. □
If $c = 0$ then the p-divisible group $A_{x_{\mathcal{O}^c_p,\tilde{R}}}[p^\infty]$ is called a canonical lift of $A_{z_{\tilde{R}}}[p^\infty]$. If $c \geq 1$ then one uses the term quasi-canonical instead. As $\mathcal{O}_p$ runs through all inert and maximal quadratic orders in R_p, $x_{\mathcal{O}_p,\tilde{R}}$ runs over all points on $\underline{X}_{\tilde{R}}$ with residue field $\mathbf{Q}_{p^2}$ (this statement follows from the intersection formulas given below). On the other hand ramified canonical lifts and quasi-canonical lifts of both types are more thinly dispersed on $\underline{X}_{\tilde{R}}$. In general one has

$$\mathrm{End}(A_{x_{\mathcal{O}^c_p,\tilde{R}}})_{\underline{R}} = \mathrm{End}(A_{x_{\mathcal{O}^c_p,\tilde{R}}}[p^\infty])_{\underline{R}_p} \cap R$$

the intersection taking place in R_p. The intersection on the right is clearly either $\mathbf{Z}$ or an imaginary quadratic order $\mathcal{O}$ with p^c exactly dividing $\mathrm{cond}(\mathcal{O})$.

The key point for us is the following intersection formula. In general for two distinct curves $\underline{x}_0$ and $\underline{x}_1$ on $\underline{X}_{\tilde{R}}$ we put $\underline{x}_0 \cdot \underline{x}_1 = \mathrm{length}_{\mathbf{Z}_{p^2}}(\underline{x}_0 \cap \underline{x}_1) \in \mathbf{Z}_{\geq 1}$.

Theorem 4.2.4 *Let* $\mathcal{O}_{p,0}$ *and* $\mathcal{O}_{p,1}$ *be two distinct quadratic suborders of* R_p *both inert and maximal. Let* $S_p \subset R_p$ *be the quaternionic order they generate. Then*

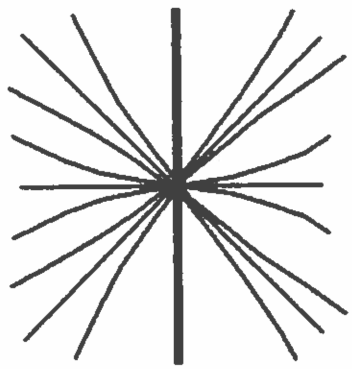
$$\underline{x}_{\mathcal{O}_{p,0},\tilde{R}} \cdot \underline{x}_{\mathcal{O}_{p,1},\tilde{R}} = (1 + \mathrm{level}(S_p))/2.$$

Proof. [Gr 87] again. □
Note that $\mathrm{level}(S_p)$ is odd since S_p is an Eichler order in a ramified quaternion algebra over $\mathbf{Q}_p$; thus the right side is integral for elementary reasons.

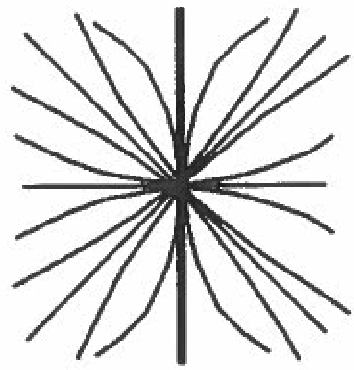
We will use another intersection formula as well. Namely let $\mathcal{O}_{p,0}$ be an inert and maximal suborder of R_p. Let $\mathcal{O}_{p,1}$ be a quadratic suborder which is either ramified or non-maximal. Then

$$\underline{x}_{\mathcal{O}_{p,0},\tilde{R}} \cdot \underline{x}_{\mathcal{O}_{p,1},\tilde{R}} = 1.$$

This formula follows formally from Theorem 4.2.3 since $\underline{x}_{\mathcal{O}_{p,0},\underline{R}} \cdot \underline{X}^0 = 1$ while $\underline{x}_{\mathcal{O}_{p,1},\underline{R}} \cdot \underline{X}^0 \geq 2$. Quite conveniently we have the following situation. We summarize our situation with two pictures of $\underline{X}_{\underline{R}}$.



$\underline{X}^0$



$\underline{X}^0$

Here the left picture shows some inert canonical lifts. The right diagram shows the same inert canonical lifts and some inert quasicanonical lifts as well.

Ordinary points. Let $\mathcal{O}$ be an imaginary quadratic order in $\mathbf{Z}_p$, maximal at p. So $\mathcal{O}$ is split at p. Let $\mathcal{P} = \mathcal{O} \cap (p) \subset \mathbf{Z}_p$ be the distinguished ideal above p in $\mathcal{O}$. Let $z \in z_{\mathcal{O}}(\bar{\mathbf{F}}_p)$. For $\ell \neq p$ consider the covariant Tate module $H_1(A_z/C_z, \mathbf{Z}_\ell)$. It is a right $\underline{R}_\ell$-module which is free of rank one. Hence $R_\ell := \mathrm{End}(H_1(A_z/C_z, \mathbf{Z}_\ell))_{\underline{R}_\ell}$ is an Eichler order, isomorphic to $\underline{R}_\ell$ canonically up to inner automorphisms. For $\ell | N$ the given orientations $\underline{f}_\ell$ on $\underline{R}_\ell$ thus give rise to orientations f_ℓ on R_ℓ. The embeddings $\mathcal{O}_\ell \hookrightarrow R_\ell$, ℓ varying, gives an element $\epsilon(z) \in \mathcal{E}_{N^+,N^-}(\mathcal{O})$. Thus we have a decomposition

$$z_{\mathcal{O}} = \coprod_{\epsilon \in \mathcal{E}_{N^+,N^-}(\mathcal{O})} z_{\mathcal{O},\epsilon}.$$

The scheme $z_{\mathcal{O},\epsilon}$ has a natural action of $\mathrm{Cl}(\mathcal{O})$ on it defined as in §2.1.

Theorem 4.2.5 *Let $f \in \mathbf{Z}_{\geq 1}$ be the order of $\mathcal{P}$ in the class group of $\mathcal{O}$. Then $z_{\mathcal{O},\epsilon}$ consists of h/f scheme-theoretic points, each with residue field having p^f elements.*

Proof. [Wa 69]. □

In distinction with the supersingular case there is no simple way of labeling the h/f points of $z_{\mathcal{O},\epsilon}$ (nor of identifying their residue fields up to unique isomorphism if $f \neq 1$). However if one chooses a point $z_{\mathcal{O},\epsilon,1} \in z_{\mathcal{O},\epsilon}$ then for any $a \in \mathrm{Cl}(\mathcal{O})/\langle \mathcal{P} \rangle$, $z_{\mathcal{O},e,a} = a z_{\mathcal{O},e,1}$ is well-defined.

We now consider the completion $\underline{X}_{\mathcal{O},e,a}$ of the scheme $\underline{X}$ at the closed point $z_{\mathcal{O},e,a}$. Thus $\underline{X}_{\mathcal{O},e,a}$ is the spectrum of a complete regular local ring with residue field F having p^f elements. It is identified with the universal deformation space of $(A,C)_{z_{\mathcal{O},e,a}}$ considered as a right $\underline{R}$-module. By the Serre-Tate theorem it is identified with the universal deformation space of the p-divisible group $A_{z_{\mathcal{O},e,a}}[p^\infty]$, considered as a right $\underline{R}_p$-module.

Theorem 4.2.6 *Let $\mathcal{O}_p^c$ be the quadratic suborder of $\mathcal{O}_p$ with conductor c. There is a unique subscheme $\underline{x}_{\mathcal{O}_p^c,e,a}$ of $\underline{X}_{\mathcal{O},e,a}$ satisfying*

1. *$\underline{x}_{\mathcal{O}_p^c,e,a}$ is the spectrum of a discrete valuation ring.*
2. $\mathrm{End}(A_{\underline{x}_{\mathcal{O}_p^c,e,a}}[p^\infty]) = \mathcal{O}_p^c$.

Moreover $\underline{x}_{\mathcal{O}_p^c,e,a}$ is the spectrum of a ring class extension of $\mathcal{O}_p$ with inertial degree f and ramification degree $\begin{cases} 1 & \text{if } c = 0 \\ (p-1)p^{c-1} & \text{if } c \geq 1. \end{cases}$

Proof. [L-S-T 64]. □

As in the supersingular case one has further that $\mathrm{End}((A,C)_{\underline{x}_{\mathcal{O}_p^c,e,j}})_{\underline{R}} = \mathcal{O}_p^c \cap \mathcal{O} \subseteq \mathcal{O}_p$. However here $\mathcal{O}^c := \mathrm{End}((A,C)_{\underline{x}_{\mathcal{O}_p,e,a}})_{\underline{R}}$ is guaranteed to be a quadratic order rather than $\mathbf{Z}$. In fact if $\mathcal{O}$ has discriminant D then $\mathcal{O}^c$ has discriminant Dp^{2c}.

The divisors $\underline{x}_{D,\pm\epsilon}$. Let D be an imaginary quadratic discriminant and write $D = dC^2$ with $p^c || C$. Let $\epsilon \in \mathcal{E}(\mathcal{O}_D)$ be an embedding invariant. It is now easy to describe the divisor $\underline{x}_{D,\pm\epsilon}$ on $\underline{X}_{\mathbf{Z}_p}$. Namely first suppose that $\left(\frac{D}{p}\right) = 0$ or 1. Then $\underline{x}_{D,\pm\epsilon}$ lies entirely on $\underline{X}_{ss}$. In fact

$$\underline{x}_{D,\pm\epsilon} = \bigoplus_{(\mathcal{O},\vec{R})} \underline{x}_{\mathcal{O}_p,\vec{R}}$$

on $\underline{X}_{\mathbf{Z}_{p^2}}$. Here the summation is over pairs $(\mathcal{O},\vec{R})$ with disc$\mathcal{O}) = D$ and $\prod_{\ell \neq p} \mathrm{inv}_\ell(\mathcal{O},\vec{R}) = \pm\epsilon$ in $\mathcal{E}_{N^+,N^-}(\mathcal{O}_D)$. The latter condition makes sense because $\mathcal{O}$ and $\mathcal{O}_D \subset \mathbf{C}$ are isomorphic, canonically up to sign. We recall that in the typical case $\epsilon \neq -\epsilon$ the divisor $x_{D,\pm\epsilon}$ on $\underline{X}_{\mathbf{Z}}$ consists of a single scheme-theoretic point with degree $2h(D)$. If $\left(\frac{D}{p}\right) = -1$, $x_{D,\pm\epsilon}$ consists of $h(D/p^{2c})$ scheme-theoretic points on $\underline{X}_{\mathbf{Z}_{p^2}}$. If $\left(\frac{D}{p}\right) = -1$, $x_{D,\pm\epsilon}$ consists of $h(D/p^{2c})/2$ scheme-theoretic points on $\underline{X}_{\mathbf{Z}_{p^2}}$.

Now suppose that $\left(\frac{D}{p}\right) = 1$. Then

$$\underline{x}_{D,\pm\epsilon} = \bigoplus_{a} \underline{x}_{\mathcal{O}^c_p,\pm\epsilon,a}.$$

Here $\mathcal{O}^c$ is the imaginary quadratic order in $\mathbf{Z}_p$ of discriminant D and $a \in \mathrm{Cl}(\mathcal{O})/\langle\mathcal{P}\rangle$.

4.3 $p|N^+$

This case is treated in considerable detail in [K-M 85]. One has a decomposition

$$\underline{X} = z_{ss} \coprod \coprod_{\mathcal{O}} z_{\mathcal{O}},$$

just as in the case $p \not| N$. The main new phenomenon here is that the special fiber $\underline{X}^0$ is not smooth as in fact it has $e+1$ components all meeting at each supersingular point.

Let (A, C) be an (N^+, N^-) abelian surface over a field F of characteristic p. Let $C[p^e]$ be the p-primary component of C. So $C[p^e]$ is a group scheme over F of rank p^e. If A is supersingular then $C[p^e]$ is in fact the unique subgroup scheme of $A[p^\infty]^1$ of rank p^e and for this reason the natural map $z_{ss} \to z_{N^+/p^e,N^-,ss}$ is an isomorphism. If A is ordinary then $C[p^e]$ is one of $e+1$ subgroup schemes of $A[p^\infty]^1$ of rank p^e. It is distinguished from the other such subgroup schemes by a simple numerical invariant. Namely consider the canonical decomposition

$$C[p^e]^{\mathrm{mult}} \to C[p^e] \to C[p^e]^{\mathrm{et}}.$$

Define $a, b \in \mathbf{Z}_{\geq 0}$ by $\mathrm{rank}(C[p^e]^{\mathrm{mult}}) = p^a$ and $\mathrm{rank}(C[p^e]^{\mathrm{et}}) = p^b$. Then $i := b - a \in \{-e, 2-e, \ldots e-2, e\}$ is the distinguishing invariant.

The invariant i gives us a decomposition of the special fiber $\underline{X}^0$ into its irreducible components:

$$\underline{X}^0 = \bigcup_i \underline{X}^0_i$$

We put C_i equal to the reduction of $\underline{X}_i$. We have

$$\underline{X}_i = \begin{cases} C_i & \text{if } i = \pm e \\ (p-1)p^{(e-2-|i|)/2}C_i & \text{if } i \neq \pm e \end{cases}$$

as divisors on $\underline{X}$. Also

$$C_i \to \underline{X}^0_{N^+/p^e,N^-} \text{ is } \begin{cases} \text{an isomorphism} & \text{if } i \geq 0 \\ \text{totally inseparable of degree } p^{|i|} & \text{if } i < 0 \end{cases}$$

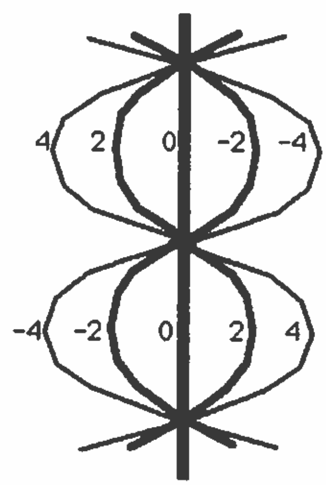
and so in particular the two curves have the same genus.

The local intersection number between C_i and C_j at a supersingular point $s \in \underline{X}_{\mathbf{Z}_{p^2}}$ is

$$C_i \cdot C_j = \begin{cases} 1 & \text{if } ij \leq 0 \\ p^{\min(|i|,|j|)} & \text{if } ij > 0 \end{cases}$$

Thus all intersections are transverse iff e equals one or two.

We summarize the above discussion with a picture of $\underline{X}_{\mathbf{F}_{p^2}}$, drawn for $e = 4$ and $\#(z_{ss}) = 3$.



4.4 $p||N^-$

This case is treated in [Dr 76]. See also [Jo-Li 85] and [Jo-Li 86]. The main phenomenon is that the choice of an (N^+, N^-)-abelian surface (A, C) over $\mathbf{F}_{p^2}$ gives a group theoretic description of the whole scheme $\underline{X}_{\mathbf{Z}_p}$ all at once, precisely analogous to the description of the analytic space $\mathcal{X}$

given in §2.1. Namely let $B = \text{End}(A, C)_{\underline{R}} \otimes \mathbf{Q}$. B is a quaternion algebra with ramification locus $\Sigma(N^+, N^-) - \{p\}$. Associated with B is the p-adic upper half plane $\underline{T}$ over $\mathbf{Z}_p$ discussed in §1.3. Put $\underline{Y} = \underline{T} \times \text{Spec } \mathbf{F}_{p^2}$. Here $B_p^\times$ acts on $\underline{T}$ as in §1.3 and on Spec $\mathbf{F}_{p^2}$ via $bf = \sigma_p^{\text{ord}_p(n(b))} f$. One has

$$\underline{X} = B^\times \backslash \underline{Y} \times B^{p\times} / R^{p\times}$$

over $\mathbf{Z}_p$. The theorems of this section are derived from this fundamental group theoretic description.

Let (A, C) be a mixed (N^+, N^-)-abelian surface over a field F containing $\mathbf{F}_{p^2}$. Consider the right action of $\underline{R}/p$ on the two-dimensional F-vector space Lie(A). Note that Lie$(A)\underline{J}$ is naturally a right $\underline{R}/\underline{J} = \mathbf{F}_{p^2}$ module, since Lie$(A)\underline{J}^2 = \text{Lie}(A)p = 0$. There are three possibilities:

1. $\dim_F(\text{Lie}(A)\underline{J}) = 1$ and $\mathbf{F}_{p^2}$ acts via the inclusion $\mathbf{F}_{p^2} \subset F$ (mixed of type 1).
2. $\dim_F(\text{Lie}(A)\underline{J}) = 1$ and $\mathbf{F}_{p^2}$ acts via the conjugate injection $\mathbf{F}_{p^2} \hookrightarrow F$ (mixed of type -1).
3. $\dim_F(\text{Lie}(A)\underline{J}) = 0$ (mixed exceptional).

Theorem 4.4.1 *a) The special fiber $\underline{X}^0$ is reduced. All irreducible components have genus zero. These components of $\underline{X}^0$ are in bijection with pairs $(i, \vec{R})$ with $i = \pm 1$ and $\vec{R}$ an oriented Eichler order of type $(N^+, N^-/p)$. Here $\underline{X}^0_{i,\vec{R}}$ is characterized by the properties*

1. *$(A, C)_{\xi_{i,\vec{R}}}$ is mixed of type i.*
2. *$\text{End}((A, C)_{\xi,\vec{R}})_{\underline{R}} \cong \mathbf{Z} + pR$.*

Here $\xi_{i,\vec{R}}$ is the generic point of the component $\underline{X}^0_{i,\vec{R}}$.

b) All singularities of $\underline{X}^0$ are ordinary nodes. The nodes are in bijection with oriented Eichler orders of type $(pN^+, N^-/p)$. Here the point $z_{\vec{R}}$ is characterized by

1. *$(A, C)_{z_{\vec{R}}}$ is mixed exceptional.*
2. *$\text{End}((A, C)_{z_{\vec{R}}})_{\underline{R}} = \vec{R}$.* □

All points in characteristic p should be considered supersingular. We call a point z in $\underline{X}^0$ a special supersingular point if it has residue field $\mathbf{F}_{p^2}$ and is not a crossing point. Special supersingular points are indexed by pairs $(i, \vec{R})$ where $i \in \{\pm 1\}$ and $\vec{R}$ is an oriented Eichler order of type (N^+, pN^-). Here $z_{i,\vec{R}}$ is characterized by

1. $(A, C)_{z_{i,\vec{R}}}$ has type i.

2. $\mathrm{End}((A, C)_{z_{i,\vec{R}}})_{\underline{R}} = \vec{R}$.

Let $z_{i,\vec{R}}$ be a special supersingular point. Let $\underline{X}_{i,\vec{R}}$ be the completion of $\underline{X}$ at the closed point $z_{i,\vec{R}}$. The deformation theory of $(A, C)_{z_{i,\vec{R}}}$ coincides with that of $A_{z_{i,\vec{R}}}[p^\infty]$ exactly as in the case $p \not| N$.

Theorem 4.4.2 *Let $\mathcal{O}_p$ be a maximal inert quadratic suborder of R_p. There is a unique subscheme $\underline{x}_{\mathcal{O}_p,i,\vec{R}}$ having the following properties:*

1. *$\underline{x}_{\mathcal{O}_p,i,\vec{R}}$ is the spectrum of a discrete valuation ring*
2. $\mathrm{End}(A_{\underline{x}_{\mathcal{O}_p}}[p^\infty]) = \mathcal{O}_p$.

Moreover $\underline{x}_{\mathcal{O}_p,i,\vec{R}} \cong \mathrm{Spec}\ \mathbf{Z}_{p^2}$. $\square$

This theorem is clearly analogous to Theorems 4.2.3 and 4.2.6. Moreover there is another analog, which we don't need, concerning canonical lifts of crossing points associated to maximal ramified quadratic orders. One should also note two differences between the present situation and that of §4.2. First there are no quasicanonical lifts here. Second, most of the closed points, namely all those with residue field $\neq \mathbf{F}_{p^2}$, do not have natural lifts at all. The abelian surface $(A, C)_{x_{\mathcal{O}_p,i,\vec{R}}}$ satisfies $\mathrm{End}((A, C)_{x_{\mathcal{O}_p,i,\vec{R}}}) = \mathcal{O}_p \cap R \subset R_p$.

Again a key point for us is an intersection formula.

Theorem 4.4.3 *Let $\mathcal{O}_{p,0}$ and $\mathcal{O}_{p,1}$ be two distinct quadratic suborders of R_p both inert and maximal. Let $S_p \subset R_p$ be the quaternionic order they generate. Then*

$$\underline{x}_{\mathcal{O}_{p,0},i,\vec{R}} \cdot \underline{x}_{\mathcal{O}_{p,1},i,\vec{R}} = \mathrm{level}(S_p)/2. \ \square$$

Note that here $\mathrm{level}(S_p)$ is even since S_p is an Eichler order in a split quaternion algebra over $\mathbf{Q}_p$; thus the right side is integral for elementary reasons.

5 Arithmetic surfaces $\underline{X}_{N^+,N^-}$: conjectures

We continue with the basic set-up of the previous section. Now we fix a prime power, p^e, $e \geq 2$, such that $p^e || N^-$. The schemes $\underline{X}_{\mathbf{Z}_p}$ are now normal but not regular, i.e. they have isolated singularities. We give a conjectural description of the minimal resolution of singularities $\underline{X}'_{\mathbf{Z}_p}$.

5.1 $p^2 | N^-$

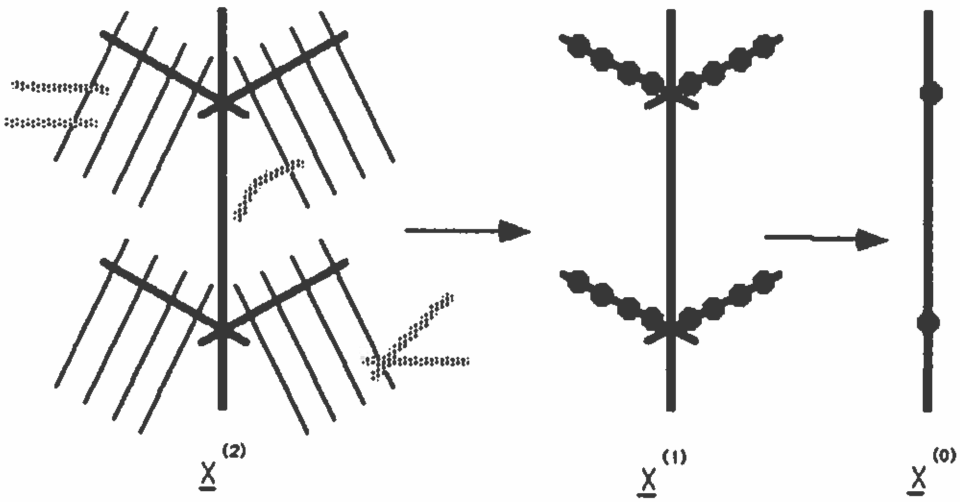
Instead of describing $\underline{X}_{\mathbf{Z}_p}$ directly we will describe $\underline{X} := \underline{X}_{\mathbf{Z}_{p^2}}$ and keep track of the Galois involution σ_p. We treat the two cases e even and e odd separately even though we conjecture that they look very much the same.

e **even.** First we consider the case where $e := \mathrm{ord}_p(N^-)$ is even. This case falls in the framework of [K-M 85]. However the treatment given there is not explicit enough for our purposes. In any case the special fiber $\underline{X}^0$ on $\underline{X}$ has just one irreducible component and this component has multiplicity $(p-1)p^{\frac{e-2}{2}}$. The natural map from the smooth curve $\underline{X}^0_{\mathrm{red}}$ to $\underline{X}^0_{N^+,N^-/p^e}$ is totally inseparable of degree $p^{e/2}$. Thus we have a canonical isomorphism $z_{ss} \to z_{N^+,N^-/p^e,ss}$, each scheme having $\frac{N}{12}\prod_{p|N^+}(1+p^{-1})\prod_{p|N^-}(1-p^{-1})$ points, all with residue field $\mathbf{F}_{p^2}$. As a scheme $\underline{X}$ is regular except at the supersingular points where it is singular. Let $\underline{X}'$ denote the minimal desingularization of $\underline{X}$. It is known that the reduced exceptional divisor above each supersingular point in $\underline{X}$ is a tree of genus zero curves (see §5.2 Point 2 below).

We conjecture that all singularities are removed by iterating the procedure "blow up all singular points" $\frac{e}{2}$ times:

$$\underline{X}' = \underline{X}^{(\frac{e}{2})} \to \underline{X}^{(\frac{e}{2}-1)} \to \cdots \to \underline{X}^{(1)} \to \underline{X}^{(0)} = \underline{X}.$$

The irreducible components of $\underline{X}'^0$, besides the non-exceptional component $\underline{X}'^0_0$, should be in canonical bijection with pairs $(i, \vec{R})$ with $i \in \{-e, 2-e, \cdots -2, 2, \cdots, e-2, e\}$ and $\vec{R}$ an oriented Eichler order of type $(N^+, N^-/p^{e+1-|i|})$. The reduced components $C_{i,\vec{R}}$ should have multiplicity $p^{(e-|i|)/2}$ in the full special fiber $\underline{X}'^0$. Thus the outer components $i = \pm e$ should have multiplicity one. The curves should be glued together according to the following diagram, drawn for $p = 2$, $e = 4$, and two supersingular points on $\underline{X}_{N^+,N^-/16}$, e.g. $(N^+, N^-) = (23, 16)$ (dotted curves to be explained momentarily).



Assuming these conjectures now, we call a point $z \in \underline{X}'$ special supersingular iff it lies on one of the extremal components $C_{i,\vec{R}}$, $i = \pm e$, has residue field $\mathbf{F}_{p^2}$, and is not the attachment point. The set of exceptional points should then be in bijection with pairs $(i, \vec{R})$ with $i \in \{\pm(e+2)\}$ and $\vec{R}$ an oriented Eichler order of type (N^+, N^-p).

There should be a theory of canonical lifts which applies to each special supersingular point $z_{i,\vec{R}}$. Namely for each quadratic suborder $\mathcal{O}_p \subset R_p$ isomorphic to $\mathbf{Z}_{p^2}$ there should be a subscheme $\underline{x}_{\mathcal{O}_p,i,\vec{R}}$ of $\underline{X}'_{i,\vec{R}}$ which is a section of the structure map $\underline{X}' \to \operatorname{Spec} \mathbf{Z}_{p^2}$. These are the dotted curves drawn on the figure above. Let $\mathcal{O}_{p,0}$ and $\mathcal{O}_{p,1}$ be two distinct such suborders. They generate a quaternionic suborder S_p which is a local Eichler order of type $(1, p^{e+1+2i})$ for some $i \in \mathbf{Z}_{\geq 0}$. In analogy with the known cases we should have

$$\underline{x}_{\mathcal{O}_{p,0},i,\vec{R}} \cdot \underline{x}_{\mathcal{O}_{p,1},i,\vec{R}} = (1 + \operatorname{level}(S_p) - e)/2. \qquad (5.1.1)$$

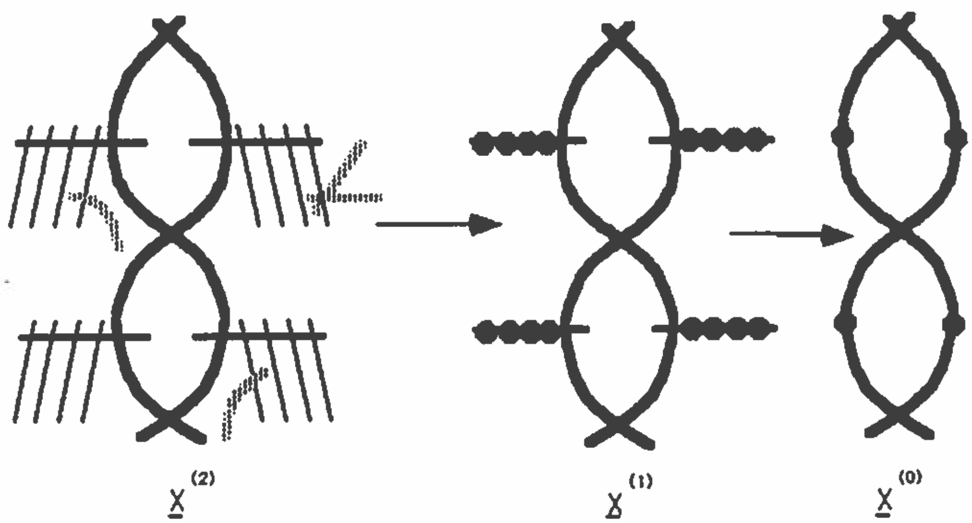
All this should apply in particular to the case when $\mathcal{O}_p \cap R$ is a quadratic $\mathbf{Z}$-order $\mathcal{O}$ (as opposed to $\mathbf{Z}$ itself). Then $x_{\mathcal{O}_p,i,\vec{R}} \in X$ would be a CM point.

e **odd.** Suppose now that e is odd and hence ≥ 3. This case is considered in [Dr 76] and involves uniformization by coverings of the p-adic upper half plane. However [Dr 76] is not explicit enough for our purposes. In any case the reduction $\underline{X}^0_{\text{red}}$ of the special fiber $\underline{X}^0$ is isomorphic to $\underline{X}_{N^+,N^-/p^{e-1}}$ and thus consists of genus zero curves glued together as in §4.4. Again let $\underline{X}'$ be the minimal desingularization of $\underline{X}$. The reduced exceptional fiber above each special supersingular point is a tree of genus zero curves again.

We expect that as a scheme $\underline{X}$ should be regular except at the points which have residue field F_{p^2} but are not crossings. We conjecture further that again all singularities are resolved by iterating the procedure "blow up all singular points" $\frac{e-1}{2}$ times:

$$\underline{X}' = \underline{X}^{(\frac{e-1}{2})} \to \underline{X}^{(\frac{e-3}{2})} \to \cdots \to \underline{X}^{(1)} \to \underline{X}^{(0)} = \underline{X}.$$

The irreducible components of $\underline{X}'^0$ should be in canonical bijection with pairs $(i, \vec{R})$ with $i \in \{-e, 2-e, \cdots, -1, 1, \cdots, e-2, e\}$, $\vec{R}$ an oriented Eichler order of type $(N^+, N^-/p^{e+1-|i|})$. The reduced components $C_{i,\vec{R}}$ should have multiplicity $p^{(e-|i|)/2}$ in the full special fiber $\underline{X}'^0$. Thus the outer components $i = \pm e$ should have multiplicity one. The curves should be glued together according to the following diagram, drawn for $p = 2$, $e = 5$, and two components on $\underline{X}_{N^+,N^-/32}$, e.g. $(N^+, N^-) = (1, 13 \cdot 32)$.



We call a point $z \in \underline{X}'$ special supersingular iff it lies on one of the extremal components $C_{i,\vec{R}}$, $i = \pm e$, has residue field F_{p^2}, and is not the attachment point. Just as in the case where e is even, the set of exceptional points should be in natural bijection with pairs $(i, \vec{R})$ with $i \in \{\pm(e+2)\}$

and $\vec{R}$ an oriented Eichler order of type (N^+, N^-p). There should be a theory of canonical lifts which applies to each special supersingular point $z_{i,\vec{R}}$ exactly as in the case e even, the analog of 5.1.1 being

$$\underline{x}_{\mathcal{O}_{p,0},i,\vec{R}} \cdot \underline{x}_{\mathcal{O}_{p,1},i,\vec{R}} = (\text{level}(S_p) - e)/2. \qquad (5.1.2)$$

5.2 Evidence for conjectures

Here are five reasons why we believe our conjectures:

1. The special fibers we describe have the right arithmetic genus. This assertion is explained in detail in the next subsection.

2. Let $g = g^{\text{good}} + g^{\text{mult}} + g^{\text{add}}$ be the canonical decomposition of g according to the special fiber of the Neron model $\underline{J}_{\mathbf{F}_p}$ (see [D-R 73]). Similarly we have $g = g^0 + \cdots + g^e$ where

$$g^i = \sum_{p^i \| \text{cond}(f)} c_{N^+,N^-,\text{cond}(f)}.$$

   Theorem 3.4.1 implies that $g^{\text{good}} = g^0$ and $g^{\text{mult}} = g^1$. These equalities imply that the resolutions of our singularities consist of trees of genus zero curves. Our conjectured resolutions do indeed consist of trees of genus zero curves.

3. We can prove the case $p^2|N^-$ starting from the description in [K-M 85] of the model over $\mathbf{Z}_p$ of a full level p structure $\underline{X}_{N^+,N^-/p^2}[p]$. This model is regular as a scheme; it is a (p^2-1)-fold covering of $\underline{X}$. Since (p^2-1) is prime to p this covering is tame and we can apply the theory of Hirzebruch-Jung singularities [BPV 84].

4. There have to be multiplicity one components as $\underline{X}_{\mathbf{Z}_{p^2}}$ has sections (namely closures of CM points with $p \nmid D$). Our proposed resolutions do indeed have multiplicity one components, namely the components in $C_{\pm e}$. More generally, our conjectures admit several natural-looking refinements such as the conjectural extension of the theory of canonical lifts which we have given.

5. The examples of $\underline{X}_{1,40}$ and $\underline{X}_{3,16}$ worked out in Section 7 do not lead to contradictions.

A final remark. We are concerned with the description of the formal neighborhood of the exceptional fibers of the minimal resolution $\underline{X}' \to \underline{X}$. This is purely a local problem concerning p-divisible groups. However the evidence we have collected, or rather 1,2,4, and 5, is all of a global nature. We should say moreover that we have not investigated the problem locally as one should, for lack of expertise. It is not even clear to us whether the problem is hard.

5.3 Arithmetic genus

In general let Z be a discrete valuation ring with field of fractions Q and residue field F. Let X be a smooth, proper, geometrically connected curve over Q of genus g. Let $\underline{X}$ be a regular model for X over Spec Z. Then the arithmetic genus of the special fiber $\underline{X}^0$ is necessarily g.

In this subsection we start with our description of $\underline{X}^0_{N^+,N^-}$ which we simply assume is correct in the unknown cases $p^3|N^-$. We then directly compute its arithmetic genus $g(\underline{X}^0_{N^+,N^-})$. We find that it coincides with $g(X_{N^+,N^-})$ in all cases. Our computations simply illustrate the general constancy-of-arithmetic-genus theorem in the known cases $p^3 \not| N^-$ while they provide evidence for our conjecture in the unknown cases $p^3|N^-$.

First we recall the exact definition of arithmetic genus. Let C be a proper, purely one-dimensional scheme over F which is geometrically connected. Then its arithmetic genus is defined by $g(C) := \dim_F H^1(C, \mathcal{O})$.

We will work not directly with $g(\cdot)$ but rather with a modified notion $h(\cdot)$ of arithmetic genus. Let C be a proper, purely one-dimensional scheme over F. We do not assume that C is geometrically connected. We define its modified arithmetic genus by

$$h(C) = \dim_F H^1(C, \mathcal{O}) - \dim_F H^0(C, \mathcal{O}).$$

C is geometrically connected iff $\dim_F H^0(C, \mathcal{O}) = 1$ in which case of course $h(C) = g(C) - 1$. The advantage of $h(\cdot)$ is that it behaves well with respect to disjoint unions $h(C_1 \coprod C_2) = h(C_1) + h(C_2)$ and etale covers $h(\tilde{C}) = \deg(\tilde{C} : C)h(C)$. Of course if C is smooth then $h(C) = -\frac{1}{2}\chi(C)$ and so these nice properties should come as no surprise.

We now return to our original situation where C is given as the special fiber $\underline{X}_F$ of the map $\underline{X} \to$ Spec Z. Suppose that as a divisor C can be decomposed

$$C = \sum_i m_i C_i,$$

with i running over a finite subset of $\mathbf{Z}$, $m_i \in \mathbf{Z}_{\geq 1}$ and C_i (not necessarily reduced or irreducible) effective subdivisors of C. Then the adjunction formula says

$$h(\sum_i m_i C_i) = \sum_i m_i h_i + \frac{1}{2}\sum_i m_i(m_i - 1) C_i \cdot C_i + \sum_{i_1 < i_2} m_{i_1} m_{i_2} C_{i_1} \cdot C_{i_2}. \quad (5.3.1)$$

We now begin our calculations, splitting into the four cases

$$\begin{array}{ll} p^e \| N^+, & e \text{ even} \\ p^e \| N^+, & e \text{ odd} \\ p^e \| N^-, & e \text{ even} \\ p^e \| N^-, & e \text{ odd.} \end{array}$$

In all four cases we use the natural decomposition

$$\underline{X}^0 = \sum_{\substack{-e \leq i \leq e \\ i \equiv e\,(2)}} m_i C_i.$$

We know the multiplicities m_i and the intersection matrix $I_{i_1,i_2} := C_{i_1} \cdot C_{i_2}$ and so it is simply a question of plugging into 5.3.1. Note that C_{-e} and C_e each have multiplicity one and so do not contribute to the self-intersection term.

In cases 1 and 2 we put $h := h(X_{N^+/p^e, N^-})$. In case 3 we put $h := h(X_{N^+, N^-/p^e})$. In case 4 we put $h = \#(\text{components of } \mathcal{X}_{N^+, N^-/p^e})$. So as not to be encumbered by the completely passive constant factor h we introduce the convention $a' := a/h$ where a is any real number. Also for convenience we write $e = 2f$ if e is even and $e = 2f - 1$ if e is odd.

$\underline{p | N^+, e \text{ even}}$

We need to show that $h'(X_{N^+, N^-}) = p^e + p^{e-1}$.

label	i	$\neq \pm e$	$\pm e$
#(components)	n_i	1	1
multiplicity	m_i	$(p-1)p^{f-\|i\|/2-1}$	1
modified genus	h_i	h	h
self-intersection	$C_i \cdot C_i$	$-2p^{\|i\|}h$	$-(p-1)p^{2f-1}h$

If $i_1 < i_2$ then

$$C_{i_1} \cdot C_{i_2} = \begin{cases} (p-1)h & \text{if } i_1 i_2 \leq 0 \\ (p-1)hp^{\min(|i_1|,|i_2|)} & \text{if } i_1 i_2 > 0. \end{cases}$$

First term:

$$\begin{aligned}
\sum_i m_i h'_i &= m_0 h'_0 + 2\sum_{j=1}^{f-1} m_{2j} h'_{2j} + m_e h'_e \\
&= (p-1)p^{f-1} + 2\sum_{j=1}^{f-1}(p-1)p^{f-j-1} + 2 \\
&= (p^f - p^{f-1}) + (2p^{f-1} - 2) + 2 \\
&= p^f + p^{f-1}.
\end{aligned}$$

Second term:

$$\begin{aligned}
\frac{1}{2}\sum_i m_i(m_i - 1)I'_{i,i} &= \frac{1}{2}m_0(m_0-1)I'_{0,0} + \sum_{j=1}^{f-1} m_{2j}(m_{2j}-1)I'_{2j,-2j} \\
&= \frac{1}{2}(p-1)p^{f-1}[(p-1)p^{f-1} - 1](-2) \\
&\quad + \sum_{j=1}^{f-1}(p-1)p^{f-j-1}[(p-1)p^{f-j-1} - 1](-2p^{2j}) \\
&= -p^{2f} + 2p^{2f-1} - p^{2f-2} + p^f - p^{f-1} \\
&\quad -2(p-1)\left[\sum_{j=1}^{f-1} p^{2f-1} - p^{2f-2} - p^{f+j-1}\right] \\
&= -p^{2f} + 2p^{2f-1} - p^{2f-2} + p^f - p^{f-1} \\
&\quad -2(f-1)(p^{2f} - 2p^{2f-1} + p^{2f-2}) + 2p^{2f-1} - 2p^f \\
&= (-2f+1)p^{2f} + 4fp^{2f-1} + (-2f+1)p^{2f-2} \\
&\quad -p^f - p^{f-1}.
\end{aligned}$$

Third term:

$$\begin{aligned}
\sum_{i_1 < i_2} m_{i_1} m_{i_2} I'_{i_1,i_2} &= \sum_{j=1}^{f}\sum_{k=1}^{f} m_{-2j} m_{2k} I'_{-2j,2k} \\
&\quad +2\sum_{j=0}^{f-1}\sum_{k=j+1}^{f} m_{2j} m_{2k} I'_{2j,2k}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{f} m_{-2j} \sum_{k=1}^{f} m_{2k}(p-1) \\
&\quad +2\sum_{j=0}^{f-1}(p-1)p^{f-j-1}p^{2j}p^{f-j-1}(p-1) \\
&= p^{f-1}p^{f-1}(p-1) + 2\sum_{j=0}^{f-1}(p^{2f} - 2p^{2f-1} + p^{2f-2}) \\
&= 2fp^{2f} + (-4f+1)p^{2f-1} + (2f-1)p^{2f-2}.
\end{aligned}$$

Combining the three terms,

$$\begin{aligned}
h'(\underline{X}^0) &= (p^f + p^{f-1}) \\
&\quad +((-2f+1)p^{2f} + 4fp^{2f-1} + (-2f+1)p^{2f-2} - p^f - p^{f-1}) \\
&\quad +(2fp^{2f} + (-4f+1)p^{2f-1} + (2f-1)p^{2f-2}) \\
&= p^{2f} + p^{2f-1}. \quad \square
\end{aligned}$$

$\underline{p|N^+, e \text{ odd}}$

We need to show that $h'(X_{N^+,N^-}) = p^e + p^{e-1}$.

label i	$\neq \pm e$	$\pm e$
#(components) n_i	1	1
multiplicity m_i	$(p-1)p^{f-(\lvert i\rvert+1)/2}$	1
modified genus h_i	h	h
self-intersection $C_i \cdot C_i$	$-2p^{\lvert i\rvert}h$	$-(p-1)p^{2f}h$

If $i_1 < i_2$ then

$$C_{i_1} \cdot C_{i_2} = \begin{cases} (p-1)h & \text{if } i_1 i_2 < 0 \\ (p-1)hp^{\min(|i_1|,|i_2|)} & \text{if } i_1 i_2 > 0. \end{cases}$$

First term:

$$\begin{aligned}
\sum_i m_i h'_i &= 2\sum_{j=0}^{f-1} m_{2j+1}h'_{2j+1} + m_e h'_e \\
&= 2\sum_{j=0}^{f-1}(p-1)p^{f-j-1} + 2 \\
&= (2p^f - 2) + 2 \\
&= 2p^f.
\end{aligned}$$

Second term:

$$\begin{aligned}
\frac{1}{2}\sum_i m_i(m_i-1)I'_{i,i} &= \sum_{j=0}^{f-1} m_{2j+1}(m_{2j+1}-1)I'_{2j+1,-2j+1} \\
&= \sum_{j=0}^{f-1}(p-1)p^{f-j-1}[(p-1)p^{f-j-1}-1](-2p^{2j+1}) \\
&= -2(p^2-2p+1)\sum_{j=0}^{f-1} p^{2f-1} + 2(p-1)\sum_{j=0}^{f-1} p^{f+j} \\
&= -2fp^{2f+1} + 4fp^{2f} - 2fp^{2f-1} + 2p^{2f} - 2p^f.
\end{aligned}$$

Third term:

$$\begin{aligned}
\sum_{i_1<i_2} m_{i_1}m_{i_2}I'_{i_1,i_2} &= \sum_{j=0}^{f}\sum_{k=0}^{f} m_{-2j-1}m_{2k+1}I'_{-2j-1,2k+1} \\
&\quad +2\sum_{j=0}^{f-1}\sum_{k=j+1}^{f} m_{2j+1}m_{2k+1}I'_{2j+1,2k+1} \\
&= (p-1)p^{2f} + 2\sum_{j=0}^{f-1}(p-1)p^{f-j-1}p^{2j+1}p^{f-j-1}(p-1) \\
&= (p-1)p^{2f} + 2(p^2-2p+1)\sum_{j=0}^{f-1} p^{2f-1} \\
&= (2f+1)p^{2f+1} + (-4f-1)p^{2f} + 2fp^{2f-1}.
\end{aligned}$$

Combining the three terms,

$$\begin{aligned}
h(\underline{X}^0) &= 2p^f + (-2fp^{2f+1} + 4fp^{2f} - 2fp^{2f-1} + 2p^{2f} - 2p^f \\
&\quad +((2f+1)p^{2f+1} + (-4f-1)p^{2f} + 2fp^{2f-1} \\
&= p^{2f+1} + p^{2f}. \quad \square
\end{aligned}$$

<u>$p|N^-$, e even</u>

We need to show that $h'(X_{N^+,N^-}) = p^e - p^{e-1}$.

label i	0	$\neq 0, \pm e$	$\pm e$
#(components) n_i	1	$(p-1)p^{\lvert i\rvert-2}h$	$(p-1)p^{e-2}h$
multiplicity m_i	$(p-1)p^{f-1}$	$p^{f-\lvert i\rvert/2}$	1
modified genus h_i	h	$-n_i$	$-n_i$
self-intersection $C_i \cdot C_i$	$-2h$	$-2pn_i$	$-pn_i$

If $i_1 < i_2$ then

$$C_{i_1} \cdot C_{i_2} = \begin{cases} n_{\max\{|i_1|,|i_2|\}} & \text{if } |i_1 - i_2| = 2 \text{ or } (i_1, i_2) = (-2, 2) \\ 0 & \text{else.} \end{cases}$$

First term:

$$\begin{aligned} \sum_i m_i h'_i &= m_0 h'_0 + 2\sum_{j=1}^{f} m_{2j} h'_{2j} \\ &= (p-1)p^{f-1} + 2\sum_{j=1}^{f} p^{f-j}[-(p-1)p^{2j-2}] \\ &= (p^f - p^{f-1}) + 2\sum_{j=1}^{f}(-p^{f+j-1} + p^{f+j-2}) \\ &= (p^f - p^{f-1}) + 2(-p^{2f-1} + p^{f-1}) \\ &= = 2p^{2f-1} + p^f + p^{f-1}. \end{aligned}$$

Second term:

$$\begin{aligned} \frac{1}{2}\sum_i m_i(m_i - 1)I'_{i,i} &= \frac{1}{2}m_0(m_0 - 1)I'_{0,0} + 2 \cdot \frac{1}{2}\sum_{j=1}^{f-1} m_{2j}(m_{2j} - 1)I'_{2j,2j} \\ &= \frac{1}{2}(p-1)p^{f-1}[(p-1)p^{f-1} - 1](-2) + \\ &\quad \sum_{j=1}^{f-1} p^{f-j}(p^{f-j} - 1)[-2p(p-1)p^{2j-2}] \\ &= p^{2f} - 2p^{2f-1} + p^{2f-2} - p^f + p^{f-1} \\ &\quad -2\sum_{j=1}^{f-1}[p^{2f} - p^{2f-1} - p^{f+j} + p^{f+j-1}] \\ &= p^{2f} - 2p^{2f-1} + p^{2f-2} - p^f + p^{f-1} \\ &\quad -2(f-1)(p^{2f} - p^{2f-1}) + 2p^{2f-1} - 2p^f \\ &= (-2f+1)p^{2f} + (2f+2)p^{2f-1} \\ &\quad -p^{2f-2} - p^f - p^{f-1}. \end{aligned}$$

Third term:

$$\sum_{i_1 < i_2} m_{i_1} m_{i_2} I'_{i_1,i_2} = m_{-2} m_2 I'_{-2,2} + 2\sum_{j=0}^{f-1} m_{2j} m_{2j+2} I'_{2j,2j+2}$$

$$\begin{aligned}
&= p^{f-1}p^{f-1}(p-1) + 2(p-1)p^{f-1}p^{f-1}(p-1) + \\
&\quad 2\sum_{j=1}^{f-1} p^{f-j}p^{f-j-1}(p-1)p^{2j} \\
&= (p^{2f-1} - p^{2f-2}) + (2p^{2f} - 4p^{2f-1} + 2p^{2f-2}) \\
&\quad +2\sum_{j=1}^{f-1}(p^{2f} - p^{2f-1}) \\
&= (2f)p^{2f} + (-2f+3)p^{2f-1} + p^{2f-2}.
\end{aligned}$$

Combining the three terms,

$$\begin{aligned}
h'(\underline{X}^0) &= 2p^{2f-1} + p^f + p^{f-1} \\
&\quad +(-2f+1)p^{2f} + (2f+2)p^{2f-1} - p^{2f-2} - p^f - p^{f-1} \\
&\quad +(2f)p^{2f} + (-2f+3)p^{2f-1} + p^{2f-2} \\
&= p^{2f} - p^{2f-1} \quad \square
\end{aligned}$$

$p|N^-$, e odd

We need to show that $h'(X_{N^+,N^-}) = p^e - pe - 1$.

label i	± 1	$\neq \pm 1, \pm e$	$\pm e$
#(components) n_i	h	$(p-1)p^{\lvert i\rvert-2}h$	$(p-1)p^{e-2}h$
multiplicity m_i	p^f	$p^{f-(i-1)/2}$	1
modified genus h_i	$-n_i$	$-n_i$	$-n_i$
self-intersection $C_i \cdot C_i$	$-2pn_i$	$-2pn_i$	$-pn_i$

If $i_1 < i_2$ then

$$C_{i_1} \cdot C_{i_2} = \begin{cases} (p+1)h & \text{if } (i_1, i_2) = (-1, 1) \\ n_{max\{\lvert i_1\rvert, \lvert i_2\rvert\}} & \text{if } \lvert i_1 - i_2\rvert = 2 \text{ and } (i_1, i_2) \neq (-1, 1) \\ 0 & \text{else.} \end{cases}$$

First term:

$$\begin{aligned}
\sum_i m_i h_i' &= 2m_1h_1' + 2\sum_{j=1}^{f} m_{2j+1}h_{2j+1}' \\
&= 2p^f(-1) + 2\sum_{j=1}^{f} p^{f-j}(-(p-1)p^{2j-1}) \\
&= -2p^f - 2p^{2f} + 2p^f \\
&= -2p^{2f}.
\end{aligned}$$

Second term:

$$
\begin{aligned}
\frac{1}{2}\sum_i m_i(m_i-1)I'_{i,i} &= m_1(m_1-1)I'_{1,1} + \sum_{j=1}^{f-1} m_{2j+1}(m_{2j+1}-1)I'_{2j+1,2j+1} \\
&= p^f(p^f-1))(-2p) \\
&\quad + \sum_{j=1}^{f-1} p^{f-j}(p^{f-j}-1)(-2p(p-1)p^{2j-1}) \\
&= -2p^{2f+1} + 2p^{f+1} \\
&\quad +2\sum_{j=1}^{f-1}(-p^{2f+1}+p^{2f}) + 2\sum_{j=1}^{f-1}(p^{f+j+1}-p^{f+j}) \\
&= -2p^{2f+1} + 2p^{f+1} \\
&\quad -(2f-2)p^{2f+1} + (2f-2)p^{2f} + 2p^{2f} - 2p^{f+1} \\
&= -2fp^{2f+1} + 2fp^{2f}.
\end{aligned}
$$

Third term:

$$
\begin{aligned}
\sum_{i_1<i_2} m_{i_1}m_{i_2}I'_{i_1,i_2} &= m_{-1}m_1I'_{-1,1} + 2\sum_{j=0}^{f-1} m_{2j+1}m_{2j+3}I'_{2j+1,2j+3} \\
&= p^fp^f(p+1) \\
&\quad +2\sum_{j=0}^{f-1} p^{f-j}p^{f-j-1}(p-1)p^{2j+1} \\
&= p^{2f+1} + p^{2f} + 2\sum_{j=0}^{f-1}(p^{2f+1}-p^{2f}) \\
&= (2f+1)p^{2f+1} - (2f-1)p^{2f}.
\end{aligned}
$$

Combining the three terms,

$$
\begin{aligned}
h(\underline{X}^0) &= (-2p^{2f}) + (-2fp^{2f+1} + 2fp^{2f}) + (2f+1)p^{2f+1} - (2f-1)p^{2f} \\
&= p^{2f+1} - p^{2f}. \quad \square
\end{aligned}
$$

6 An intersection formula

In this section we allow the pair (N^+, N^-) to be of either parity. §1 – §3 consist in preliminaries. In §4 we compute some coincidence numbers and in §5 some intersection numbers as defined in the introduction. The special case of the j-line — i.e. $X_{1,1}$ in our notation — was treated very explicitly in [G-Z 85]. The case of $X_{1,N}$, N prime, was treated in [Gr 87]. Finally the case of $X_0(N)$ was treated in [G-K-Z 87].

6.1 Set-up

Let $D_0 = d_0 C_0^2$ and $D_1 = d_1 C_1^2$ be imaginary quadratic discriminants and abbreviate $\mathcal{O}_i := \mathcal{O}_{D_i} \subseteq \mathbf{C}$. Let $\epsilon_0 \in \mathcal{E}(\mathcal{O}_0)$ and $\epsilon_1 \in \mathcal{E}(\mathcal{O}_1)$. We make the following simplifying assumptions.

1. $x_{\mathcal{O}_0,\epsilon_0}$ and $x_{\mathcal{O}_1,\epsilon_1}$ both consist of Heegner points, i.e. C_0 and C_1 are each relatively prime to N.

2. D_0 and D_1 are relatively prime.

Here Assumption 1 is particularly important as it allows us to explicitly identify the embedding invariants ϵ_i with some concrete objects. Also, in the odd case, it keeps the local computations at primes $p|N$ manageable. Assumption 2 is less important; it cuts down the number of possible phenomena and correspondingly keeps the final formula rather simple.

So our first task is to explicitly identify the sets $\mathcal{E}_{N^+,N^-}(\mathcal{O}_i)$. Since we are in the Heegner situation we have identifications

$$\begin{array}{rcl} \mathcal{E}_{N^+,N^-}(\mathcal{O}_i) & = & \mathrm{Hom}(\mathcal{O}_i, A_{N^+,N^-}) \\ (\mathcal{O}_i \xrightarrow{h} R \xrightarrow{f} A_{N^+,N^-}) & \longmapsto & (\mathcal{O}_i \xrightarrow{f\circ h} A_{N^+,N^-}) \end{array}$$

mentioned in §1.5 Point 3. As motivation let's assume for a moment that $\mathrm{ord}_2(N) \neq 1$. Then we have the further identification

$$\begin{array}{rcl} \mathrm{Hom}(\mathcal{O}_i, A_{N^+,N^-}) & = & (\text{Square roots of } D_i \text{ in } A_{N^+,N^-}) \\ \mathcal{O}_i \xrightarrow{g} A_{N^+,N^-} & \longmapsto & g(\sqrt{D_i}), \end{array}$$

$\sqrt{D_i} \in \mathcal{O}_i$ by convention lying in the upper half plane. Thus we can identify ϵ_0 and ϵ_1 with elements r_0 and r_1 of A_{N^+,N^-} respectively. We need to fix up this simple construction in order to treat the case $2||N$. In fact the modifications necessary are useful in the case $2 \not| N$.

Define $A'_{2^e,1}$ by the Cartesian diagram of rings

$$\begin{array}{ccc} A'_{2^e,1} & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ A_{2^{e+1},1} & \longrightarrow & \mathbb{Z}/2 \oplus \mathbb{Z}/2 \end{array}$$

and similarly define $A'_{1,2^e}$ by

$$\begin{array}{ccc} A'_{1,2^e} & \longrightarrow & \mathbb{Z}/2 \\ \downarrow & & \downarrow \\ A_{1,2^{e+1}} & \longrightarrow & \mathbb{F}_4. \end{array}$$

We have just given two definitions of $A'_{1,1}$; however they coincide: $A'_{1,1} = \mathbb{Z}/2$.

We need the analogous construction on the level of quaternionic orders:

Definition 6.1.1 *Let R_2 be an Eichler order over $\mathbb{Z}_2$. Then its derived order is*

$$R'_2 := \mathbb{Z}_2 + 2R_2.$$

Suppose now that R_2 has type $(2^e, 1)$ and comes equipped with an orientation $f_2 : R_2 \to A_{2^e,1}$. Then there is a natural map $f'_2 : R'_2 \to A'_{2^e,1}$. To define this map it suffices to define it on our standard oriented order $(\underline{R}_{2^e,1}, \underline{f}_2)$ since (R_2, f_2) is isomorphic to $(\underline{R}_{2^e,1})$ canonically up to inner automorphisms and $A_{2^e,1}$ is commutative. Very simply we define

$$\begin{array}{rcl} \underline{f}'_2 : \underline{R}'_{2^e,1} & \longrightarrow & A'_{2^e,1} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \longmapsto & (a, d). \end{array}$$

Similarly we treat the twisted case by defining

$$\begin{array}{rcl} \underline{f}'_2 : \underline{R}'_{1,2^e} & \longrightarrow & A'_{1,2^e} \\ \begin{pmatrix} a & b \\ pb^\sigma & a^\sigma \end{pmatrix} & \longmapsto & a. \end{array}$$

Thus the formulas are exactly those of §1.5; its just that the smaller domain allows us to take a bigger range. The upshot of this discussion is that we can identify ϵ_0 and ϵ_1 with traceless elements r_0 and r_1 of A'_{N^+,N^-}. In particular we can form the product $r_0 r_1$ which is a squareroot of $D_0 D_1$ lying in the subring $\mathbf{Z}/2N\mathbf{Z}$ of the ring A'_{N^+,N^-}.

We are interested in the divisors $x_{D_i,\pm r_i}$. For convenience we consider the following linear combination:

$$c_{D_i,\pm r_i} := \sum_{a|C_i} u(D_i/a^2)^{-1} x_{D_i/a^2,\pm r_i/a}. \qquad (6.1.2)$$

The main theorems of this section consist in formulas for the coincidence number $(c_{D_0,\pm r_0}, c_{D_1,\pm r_1})$ if (N^+, N^-) is even and the intersection number $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})$ if (N^+, N^-) is odd. The inverse equation to 6.1.2 is

$$x_{D_i,\pm r_i} = u(D_i) \sum_{a|C_i} \mu(a) c_{D_i/a^2,\pm r/a},$$

μ being as usual the Mobius function. We thus obtain formulas for the geometrically more appealing quantity $(x_{D_0,\pm r_0}, x_{D_1,\pm r_1})$.

6.2 Explicit Eichler orders

Let n be an integer with the same parity as $D_0 D_1$ and satisfying $n^2 < D_0 D_1$. Let $[D_0, n, D_1]$ denote the binary quadratic form $D_0 x^2 + 2nxy + D_1 y^2$. Put A equal to the discriminant of this form, i.e. put $A := (n^2 - D_0 D_1)/4$. So A is a negative integer.

Let $p|A$. Our hypothesis $(D_0, D_1) = 1$ says that at least one of the $\mathcal{O}_i$ is maximal and unramified at p. We need to exclude one more possibility.

Lemma 6.2.1 *Suppose that both $\mathcal{O}_0$ and $\mathcal{O}_1$ are maximal and unramified at p. Then either both $\mathcal{O}_0$ and $\mathcal{O}_1$ are split or they are both inert at p.*

Proof. First consider $p = 2$ so that

$$\left(\frac{D_i}{2}\right) = \begin{cases} 1 & \text{if } D_i \equiv 1\ (8) \\ -1 & \text{if } D_i \equiv 5\ (8). \end{cases}$$

But now

$$\begin{aligned} 2|(n^2 - D_0 D_1)/4 &\Rightarrow 8|(n^2 - D_0 D_1) \\ &\Rightarrow D_0 D_1 \equiv 1\ (8) \\ &\Rightarrow D_0 \equiv D_1\ (8). \end{aligned}$$

Now consider the remaining case p odd.

$$\begin{aligned} p|(n^2 - D_0D_1)/4 \;&\Rightarrow\; n^2 \equiv D_0D_1\ (p) \\ &\Rightarrow\; \left(\frac{D_0D_1}{p}\right) = 1 \\ &\Rightarrow\; \left(\frac{D_0}{p}\right) = \left(\frac{D_1}{p}\right). \; \square \end{aligned}$$

We define a relatively prime factorization $-A = A^+A^-$ by the following prescription

$$\left\{ \begin{array}{l} p|A^+ \\ p|A^- \end{array} \right. \text{ iff at least one of the } \mathcal{O}_i \text{ is } \left\{ \begin{array}{l} \text{split} \\ \text{inert} \end{array} \right. \text{ and maximal at } p.$$

Of course here we are using Lemma 6.2.1 to ensure that our two conditions are mutually exclusive.

Let $B := B_{D_0,n,D_1} := \mathbf{Q}[e_0, e_1]/(e_0^2 = D_0, e_1^2 = D_1, e_0e_1 + e_1e_0 = 2n)$. Let $S := S_{D_0,n,D_1}$ be the lattice generated by

$$b_1 := 1 \;\; b_2 := (e_0 + \bar{D}_0)/2$$

$$b_3 := (e_1 + \bar{D}_1)/2 \;\; b_4 := (e_0e_1 + e_0\bar{D}_1 + e_1\bar{D}_0 + \bar{D}_0\bar{D}_1)/4.$$

Here $\bar{D}_i = 0, 1$ is as defined in 2.1.4.

Proposition 6.2.2 *S_{D_0,n,D_1} is an Eichler order of type (A^+, A^-).*

Proof. One can easily verify that S is closed under multiplication; for example this is obvious in the excluded case where both D_i are even so both $\bar{D}_i$ are zero. With respect to the basis $\{b_1, b_2, b_3, b_4\}$ the form $(x, y) := \mathrm{Tr}(x\bar{y})$ has matrix representation

$$\begin{bmatrix} 2 & \bar{D}_0 & \bar{D}_1 & (n + \bar{D}_0\bar{D}_1)/2 \\ \bar{D}_0 & (\bar{D}_0 - D_0)/2 & (\bar{D}_0\bar{D}_1 - n)/2 & \bar{D}_1(\bar{D}_0 - D_0)/4 \\ \bar{D}_1 & (\bar{D}_1 - D_1)/2 & (\bar{D}_1\bar{D}_0 - n)/2 & \bar{D}_0(\bar{D}_1 - D_1)/4 \\ (n + \bar{D}_0\bar{D}_1)/2 & \bar{D}_1(\bar{D}_0 - D_0)/4 & \bar{D}_0(\bar{D}_1 - D_1)/4 & (\bar{D}_0 - D_0)(\bar{D}_1 - D_1)/8 \end{bmatrix}$$

One can compute directly that the determinant of this matrix is A^2. This shows that the reduced discriminant is $-A$. Let $p|A$. Choose i such that $\mathcal{O}_i$ is maximal and unramified at p. If $\left(\frac{D_i}{p}\right) = 1$ then S_p contains a copy of $\mathbf{Z}_p \oplus \mathbf{Z}_p$, namely $\mathbf{Z}_p[(e_i + D_i)/2]$. Similarly if $\left(\frac{D_i}{p}\right) = -1$ then S_p contains a copy of the inert quadratic algebra $\mathbf{Z}_{p^2}$, again $\mathbf{Z}_p[(e_i + D_i)/2]$. $\square$

6.3 A finite Dirichlet series

Here we define some finite Dirichlet series which will allow us to give a concise formulation of the two main formulas.

Definition 6.3.1 *Let p be a prime number and let $e \in \mathbb{Z}_{\geq 0}$. Then*

$$\begin{aligned} L_{p^e,1}(s) &:= 1 + \cdots + p^{-is} + \cdots + p^{-es} \\ L_{1,p^e}(s) &:= 1 + \cdots + (-1)^i p^{-is} + \cdots + (-1)^e p^{-es}. \end{aligned}$$

Clearly we have the functional equations

$$\begin{aligned} L_{p^e,1}(s) &= p^{-es} L_{p^e,1}(-s) \\ L_{1,p^e}(s) &= (-1)^e p^{-es} L_{1,p^e}(-s). \end{aligned}$$

Evaluating at the central point,

$$L_{p^e,1}(0) = e + 1.$$

Also if e is even then

$$L_{1,p^e}(0) = 1$$

while if e is odd

$$\begin{aligned} L_{1,p^e}(0) &= 0 \\ L'_{1,p^e}(0) &= \frac{1}{2}(1+e)\log p. \end{aligned}$$

Finally:

Definition 6.3.2 *Let (M^+, M^-) be a pair of relatively prime positive integers. Then*

$$L_{M^+,M^-}(s) = \prod_{p^e \| M^+} L_{p^e,1}(s) \prod_{p^e \| M^-} L_{1,p^e}(s).$$

Thus $L_{M^+,M^-}(s)$ has a zero of order $\#(\Sigma_f(M^+, M^-))$ at $s = 0$.

6.4 A coincidence formula

Here we assume that (N^+, N^-) is even. We repeat the definition of coincidence number given in the introduction, now taking elliptic points into account:

Definition 6.4.1 *Let x_0 and x_1 be two points on $\mathcal{X}_{N^+,N^-}$. Let $\mathcal{X}_i$ denote the component of $\mathcal{X}$ containing x_i and let $4\pi/w_i$ denote its volume. Then*

$$(x_0, x_1) = \begin{cases} w_0 & \text{if } \mathcal{X}_0 = \mathcal{X}_1 \\ 0 & \text{if } \mathcal{X}_0 \neq \mathcal{X}_1. \end{cases}$$

By linearity the coincidence pairing is defined on arbitrary divisors.

Theorem 6.4.2

$$(C_{D_0,r_0}, C_{D_1,r_1}) = \sum_{\substack{n^2 < D_0 D_1 \\ n \equiv r_0 r_1 \ (2N)}} L_{A^+/N^+, A^-/N^-}(0).$$

Here (A^+, A^-) is by definition the type of S_{D_0,n,D_1}.

Proof. The quantity $(C_{D_0,r_0}, C_{D_1,r_1})$ in question is the cardinality of a set Q which we define as follows. Q is the set of isomorphism classes of quadruples (R, f, u_0, u_1) where (R, f) is an oriented order of type (N^+, N^-) and $u_i \in R$ with $f'(u_i) = r_i \in A'_{N^+,N^-}$.

To determine the cardinality of Q we first consider an integral invariant of its elements, namely the function

$$\begin{aligned} i : Q &\longrightarrow \mathbb{Z} \\ (R, f, u_0, u_1) &\longmapsto (u_0 u_1 + u_1 u_0)/2. \end{aligned}$$

Let n be in the image of i. Then

1. $n^2 < D_0 D_1$ because R is definite.
2. $n \equiv r_0 r_1 \ (2N)$.

Thus we have

$$Q = \coprod_{\substack{n^2 < D_0 D_1 \\ n \equiv r_0 r_1 \ (2N)}} Q_n.$$

The final step is to determine $\#(Q_n)$. Consider the order S_{D_0,n,D_1} of §6.2. It comes equipped with preferred elements e_i and a preferred map $f'_n : S'_{D_0,n,D_1} \to A'_{N^+,N^-}$ characterized by $f'_n(e_i) = r_i$. A complete set of representatives for Q_n consists of all quadruples (R, f, e_0, e_1) where $S_{D_0,n,D_1} \subset R \subset B_{D_0,n,D_1}$ and f extends f_n. We must simply count the number of such orders R as the extension from f_n to f exists and is unique. But if $p|A^+/N^+$ then S_p has exactly

$$L_{p^{a_p/e_p},1}(0) = a_p - e_p + 1$$

overorders R_p of type $(p^e, 1)$. Similarly if $p|N^-$ then S_p has exactly

$$L_{1,p^{a_p/e_p}}(0) = \begin{cases} 1 & \text{if } a_p \equiv e_p\ (2) \\ 0 & \text{if } a_p \not\equiv e_p\ (2) \end{cases}$$

overorders R_p of type $(1, p^e)$. Thus $\#(Q_n) = L_{A^+/N^+,A^-/N^-}(0)$ giving the formula. □

6.5 The intersection formula

Here we assume that (N^+, N^-) is odd. We assume the conjectures of §5.1 if N^- is not squarefree. We repeat the definition of intersection number given in the introduction, now paying attention to elliptic points.

Definition 6.5.1 *Let x_0 and x_1 be two distinct scheme-theoretic points on X with closures $\underline{x}_0$ and $\underline{x}_1$ on $\underline{X}'$. Suppose that $\underline{x}_0 \cap \underline{x}_1$ is disjoint from the singular locus of $\underline{X}'$. Then*

$$(\underline{x}_0, \underline{x}_1) := \log \#(A)$$

where A is the ring of functions on the scheme-theoretic intersection $\underline{x}_0 \cap \underline{x}_1$.

Again by linearity $(\cdot,\cdot)$ is defined on arbitrary divisors with disjoint support and missing the singular locus. Here by singular locus we mean the set of scheme-theoretic points having Zariski tangent space of dimension ≥ 3, i.e. the non-regular locus. We remark that there is a reasonable way to define $(\underline{x}_0, \underline{x}_1)$ even when $\underline{x}_0 \cap \underline{x}_1$ intersects the singular locus; see [Mu 60].

Before proceeding we describe the singular locus of $\underline{X}'$. Under the assumptions made in Sections 4 and 5 the singular locus is empty. In the general case a point z in characteristic p is singular iff z is an isolated ramification point for some cover $\underline{X}'_{\ell N^+,N^-} \to \underline{X}'$, $\ell \neq p$. These occur precisely in the following situations.

1. $p^e||N^+$. Here singularities can occur both at supersingular points and at ordinary points. The supersingular point $z_{\underline{R}}$ is singular iff

$$R^\times/\mathbb{Z}^\times \text{ contains 3-torsion and } p^e \neq 3$$

   or

$$R^\times/\mathbb{Z}^\times \text{ contains 2-torsion and } p^e \neq 2.$$

   An ordinary point $z_{i,\mathcal{O},\epsilon,a}$ is singular iff $\mathcal{O}^\times/\mathbb{Z}^\times$ has torsion and $i \neq \pm e$. Thus singularities can occur at ordinary points only if $e \geq 2$.

2. $p^e||N^-$. All singularities of $\underline{X}'_{\mathbb{Z}_{p^2}}$ occur at crossing points, i.e. points of the form $z_{i,\underline{R}}$, $i \in \{-e+1, -e+3, \ldots, e-3, e-1\}$. The point $z_{i,\underline{R}}$ is singular iff

$$R^\times/\mathbb{Z}^\times \text{ contains 3-torsion and } p^e \neq 3$$

   or

$$R^\times/\mathbb{Z}^\times \text{ contains 2-torsion and } p^e \neq 2.$$

In particular all non-crossings points on the components C_{-e} and C_e are regular. As will be clear in the proof of the next theorem, our assumptions imply that all the points in $\underline{c}_{D_0,\pm r_0} \cap \underline{c}_{D_1,\pm r_1}$ are regular and so the definition applies.

Theorem 6.5.2

$$(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1}) = \sum_{\substack{n^2 < D_0 D_1 \\ n \equiv \pm r_0 r_1 \ (2N)}} L'_{A^+/N^+, A^-/N^-}(0).$$

Here (A^+, A^-) is by definition the type of S_{D_0,n,D_1}.

Proof. The condition that $(D_0, D_1) = 1$ ensures that $c_{D_0,\pm r_0}$ and $c_{D_1,\pm r_1}$ are disjoint so that $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})$ is well-defined. The equality to be proved is of the form $\sum a_p = \sum b_p$, the sums being over the set of primes with all but finitely many addends being zero. Namely on the left side we have

$$(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1}) = \sum_p (\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p$$

$(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p$ being the intersection number on $\underline{X}_{\mathbb{Z}_p}$. On the right side we have

$$\sum_{\substack{n^2 < D_0 D_1 \\ n \equiv \pm r_0 r_1 \,(2N)}} L'_{A^+/N^+, A^-/N^-}(0) =$$

$$\sum_p \sum_{\substack{n^2 < D_0 D_1 \\ n \equiv \pm r_0 r_1 \,(2N) \\ p | A^-/N^-}} L_{A^+/N^+, A^-/N^- p}(0)(1 + \mathrm{ord}_p(A^-/N^-))/2 \log p.$$

We will prove $\sum a_p = \sum b_p$ by proving $a_p = b_p$ for all p.

Fix a prime p for the rest of the proof. By our relative primality assumption at least one of the D_0 and D_1 is prime to p. By relabeling if necessary we assume that D_0 is prime to p. Write $D_1 = d_1 C_1^2$ and define c by $p^c || C_1$.

$\underline{p \nmid N}$

There are six possibilities to consider,

$$\left(\left(\frac{D_0}{p} \right), \left(\frac{D_1}{p} \right) \right) = (1,1), (1,0), (1,-1), (-1,1), (-1,0), (-1,-1),$$

all of which can certainly occur. In the case $(1,1)$, $\underline{c}_{D_0,\pm r_0}$ and $\underline{c}_{D_1,\pm r_1}$ reduce to ordinary points $z_{D_0,\pm r_0}$ and $z_{D_1/p^{2c},\pm r_1/p^c}$ respectively. However since we are assuming $(D_0, D_1) = 1$ we must have $z_{D_0,\pm r_0} \cap z_{D_1/p^{2c},\pm r_1/p^{2c}} = \emptyset$ and so certainly $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p = 0$ here. In the next three cases $(1,0)$,$(1,-1)$,$(-1,1)$ one of $\underline{c}_{D_0,\pm r_0}$ and $\underline{c}_{D_1,\pm r_1}$ reduces to ordinary points and the other reduces to supersingular points. Thus $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p = 0$ here as well.

It is the last cases $(-1,0)$ and $(-1,-1)$ which are non-trivial. We will compute the local intersection number $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_{p^2}$ on the base-changed curve $\underline{X}_{\mathbb{Z}_{p^2}}$ and divide the final result by two. We have

$$\underline{c}_{D_i,\pm r_i} = \sum_{(\mathcal{O}_i, \bar{R})} \underline{x}_{\mathcal{O}_i, \bar{R}}.$$

Here the summation runs over pairs $(\mathcal{O}_i, \vec{R})$ with $\vec{R} = (R, f)$ an oriented Eichler order of type (N^+, pN^-) and $\mathcal{O}_i$ a quadratic suborder containing one of discriminant D and such that $f(\pm\sqrt{D_i}) = \pm r_i$. Thus we have expressed $\underline{c}_{D_0,\pm r_0}$ as a sum of canonical lifts and $\underline{c}_{D_1,\pm r_1}$ as a sum of canonical and quasi-canonical lifts. It is convenient to rewrite the latter sum as a double sum

$$\underline{c}_{D_1,\pm r_1} = \sum_{(\mathcal{O}_1,\vec{R})} \left(\sum_{a=0}^{c_1} \underline{x}_{\mathcal{O}_1^a,\vec{R}} \right)$$

Here $(\mathcal{O}_1, \vec{R})$ is as above except that $\mathcal{O}_1$ is restricted to be maximal at p. $\mathcal{O}_1^a$ then denotes the suborder $\mathcal{O}_1 \cap \mathcal{O}_{1,p}^a$. Thus we are grouping all quasi-canonical lifts which may occur with their associated canonical lifts.

Now we have the formula

$$\left(\underline{x}_{\mathcal{O}_0,\vec{R}}, \left(\sum_{a=0}^{c} \underline{x}_{\mathcal{O}^a,\vec{R}} \right) \right)_{p^2} = \delta_{\vec{R}_0,\vec{R}_1} (1 + \text{level}(S_{\mathcal{O}_0,\mathcal{O}_1}))/2 \ \log p^2$$

Together with the computation of the previous section, this gives the desired relation.

$\underline{p|N^+}$

Here we must have $\left(\frac{D_0}{p}\right) = 1$ as otherwise $\underline{c}_{D_0,\pm r_0}$ is empty. Moreover we must have $c = 0$ by the Heegner condition $(C_1, N) = 1$. Thus there are only two possibilities,

$$\left(\left(\frac{D_0}{p}\right), \left(\frac{D_1}{p}\right) \right) = (1,1), (1,0),$$

the latter being actually possible only when $p||N^+$. In the first case $\underline{c}_{D_0,\pm r_0}$ and $\underline{c}_{D_1,\pm r_1}$ reduce to collections of ordinary points, but these collections are disjoint again by our relative primality assumption $(D_0, D_1) = 0$. In the second case $\underline{c}_{D_0,\pm r_0}$ reduces to ordinary points while $\underline{c}_{D_1,\pm r_1}$ reduces to supersingular points. Thus in both cases $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p = 0$.

$\underline{p \mid N^-}$

Here we must have $\left(\frac{D_0}{p}\right) = -1$ as otherwise $\underline{c}_{D_0,\pm r_0}$ is empty. Moreover we must have $c = 0$ by the Heegner condition $(C_1, N) = 1$. Thus there are only two possibilities,

$$\left(\left(\frac{D_0}{p}\right), \left(\frac{D_1}{p}\right) \right) = (-1,0), (-1,-1),$$

the former being actually possible only when $p||N^-$. In the first case $\underline{c}_{D_0,\pm r_0}$ reduces to non-crossing points while $\underline{c}_{D_1,\pm r_1}$ reduces to crossing points. Thus $(\underline{c}_{D_0,\pm r_0}, \underline{c}_{D_1,\pm r_1})_p = 0$.

The second case is the non-trivial one. However, given the intersection formulae 4.4.3, 5.1.1, and 5.1.2, this is exactly like the case $p \nmid N$ save for the fact that one doesn't have to contend with quasicanonical lifts. $\square$

7 Examples of uniformizations

In this section we give a sequence of examples which illustrate

1. The description given in Sections 4 and 5 of the bad reduction of $\underline{X}_{N^+,N^-}$ at primes $p|N$.

2. The fact, Corollary 3.3.2, that $J^{\text{new}}_{N^+,N^-}$ is isogenous to $J_0(N)^{\text{new}}$.

We make extensive use of the tables [SD 75].

7.1 Computation of reductions

We consider all pairs (X_{N^+,N^-}, W) where $N \leq 60$ and W is a minimal subgroup of the Atkin-Lehner group such that $X_{N^+,N^-}/W$ has genus one and has Jacobian in $J^{new}_{N^+,N^-}$ (not just in J_{N^+,N^-}). There are exactly 72 such pairs, 30 with $\#(W) = 1$, 39 with $\#(W) = 2$ and 3 with $\#(W) = 4$. Our aim is to compute the Kodaira symbol (see e.g. [Ta 75]) of the reduction of $\text{Jac}(X_{N^+,N^-}/W)$ at primes p dividing N. While it is often not true that $X_{N^+,N^-}/W \cong \text{Jac}(X_{N^+,N^-}/W)$, it is always true that $(X_{N^+,N^-}/W)_{\mathbf{Q}_{p^2}} \cong (\text{Jac}(X_{N^+,N^-}/W))_{\mathbf{Q}_{p^2}}$. Thus for the purposes of computing Kodaira symbols we can work directly on the curve $(\underline{X}_{N^+,N^-}/W)_{\mathbf{Z}_{p^2}}$.

Of course the basic procedure is to take our description of the bad reduction of X_{N^+,N^-}, mod out by W if necessary, and see where the resulting curve configuration fits into the Kodaira-Neron classification. The computations are complicated — i.e. made interesting! — by the presence of singularities. Singularities can arise at three stages of the computation:

1. Of course if $p^2|N^-$ then we begin our computation by replacing the singular scheme $\underline{X}_{\mathbf{Z}_{p^2}}$ by the less-singular scheme $\underline{X}'_{\mathbf{Z}_{p^2}}$. We simply assume that the descriptions given in §5.1 are correct.

2. If $\underline{X}'_{\mathbf{Z}_{p^2}}$ has elliptic singularities then we must resolve them to obtain a regular scheme $\underline{X}''_{\mathbf{Z}_{p^2}}$. The singularity types (except in a few wild cases in which we will simply stop our computation) are given as

follows:

Tame 2-elliptic:	A_1-singularity	
Tame 3-elliptic:	A_2-singularity	iff $\mathrm{ord}_p(N)$ is odd and the 3-elliptic point is a crossing point
	$A_{3,1}$-singularity	otherwise
Wild 2-elliptic:	A_1-singularity	if $2\|\|N$
	D_4-singularity	if $2^2\|\|N$
	??	if $2^3\|N$
Wild 3-elliptic:	A_1-singularity	if $3\|\|N$
	A_5-singularity	if $3^2\|\|N$
	??	if $3^3\|N$
Wild 6- or 12-elliptic:	??.	

The resolutions of these singularities are given as follows:

A_1  A_2  $A_{3,1}$ 

D_4  A_5 

All components have genus zero and all self-intersection numbers are -2 except the one indicated to be -3. Here we do not know direct local proofs of our assertions in the wild cases.

3. The fixed point set of w_m in general consists of both divisors and isolated fixed points. The singularities of the divisors and the isolated fixed points each contribute to singularities on $(\underline{X}''_{\mathbb{Z}_p})/w_m$. If $p \neq 2$ then these new singularities do not pose a serious problem. If $p = 2$ then things can be quite a mess and so we just stop.

7.2 Comparison with Swinnerton-Dyer's table

Altogether we compute 108 reductions. We then compare our results with Table 1 of [SD 75] which was constructed by Swinnerton-Dyer and others. It is here where things become particularly interesting. This table gives a Weierstrass equation for every elliptic curve which up to isogeny is a factor of $J_0(N)^{\mathrm{new}}$ for $N \leq 200$. Conveniently for us this table also lists the Kodaira symbol for the reductions of all these elliptic curves. By way of comparison, these Kodaira symbols were computed there not by using the geometry of $\underline{X}_0(N)$, but rather by applying Tate's algorithm directly to the Weierstrass equation.

The fact that $J^{\mathrm{new}}_{N^+,N^-}$ is isogenous to $J_0(N)^{\mathrm{new}}$ implies that the Jacobians of all 72 curves we consider appear in [SD 75]. We verify that in each of the 72 cases there is indeed a curve in [SD 75] having the same reduction as our curve at all the places at which the computation went through. In 59 of these 72 cases this curve is unique and we thus obtain a Weierstrass equation for the Jacobian of our curve.

7.3 Four explicit examples

In the following four computations we use the formula in Theorem 2.1.9, and its term-by-term interpretation, repeatedly. Also we use Table 5 of [SD 75] to determine the eigenvalues of $W(N)$; here one must also take twisting into account as in Theorem 2.2.1.

All components which appear have multiplicity one or two. We indicate the distinction by using thin lines for the former and thick lines for the latter. When a scheme is regular we indicate the self-intersection number of each of the components; the lack of a label means that the component in question has self-intersection number one. The symbol Z_{M^+,M^-} means

1. The connected curve $(\underline{X}_{M^+,M^-})_{\mathbf{F}_{p^2}}$ if (M^+, M^-) is odd.

2. A copy of $\mathcal{X}_{M^+,M^-}$, genus zero curves over $\mathbf{C}$ having been replaced by genus zero curves over $\mathbf{F}_{p^2}$, if (M^+, M^-) is even.

Similarly z_{M^+,M^-} means a copy of $\mathcal{X}_{M^+,M^-}$, components having been replaced by copies of Spec $\mathbf{F}_{p^2}$. If (M^+, M^-) is even we define $\mathrm{mass}(Z_{M^+,M^-}) := \mathrm{mass}(z_{M^+,M^-}) := \mathrm{area}(\mathcal{X}_{M^+,M^-})/4\pi$.

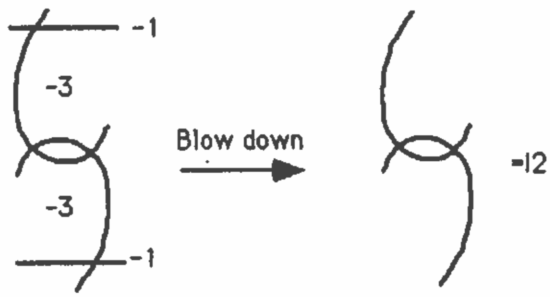
$X_{1,34}$ (genus 1)

We begin with an example that presents no problems at either prime. Many of the 72 curves we treat share this property.

Mod 2: The symbolic dual graph is

$$Z_{1,17} \overset{z_{2,17}}{\longleftrightarrow} Z_{1,17}.$$

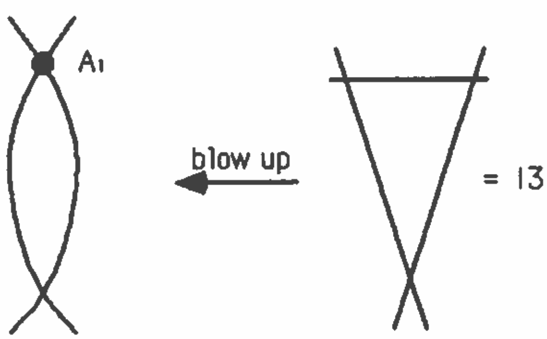
$Z_{1,17}$ consists of two components, one with mass one, the other with mass $\frac{1}{3}$. $z_{2,17}$ consists of four points, all with mass 1.



Mod 17: The symbolic dual graph is

$$Z_{1,2} \overset{z_{17,2}}{\longleftrightarrow} Z_{1,2}.$$

$Z_{1,2}$ consists of a single curve of mass $\frac{1}{12}$. $z_{17,2}$ consists of two points, one with mass 1, the other with mass $\frac{1}{3}$. The A_1-singularity is at the point with mass $\frac{1}{3}$.



There are four *a priori* possibilities for $J_{1,34}$, namely

	2	17
34A	$I6$	$I1$
34B	$I3$	$I2$
34C	$I2$	$I3$
34D	$I1$	$I6$

Hence each computation alone shows that $J_{1,34} = 34C$.

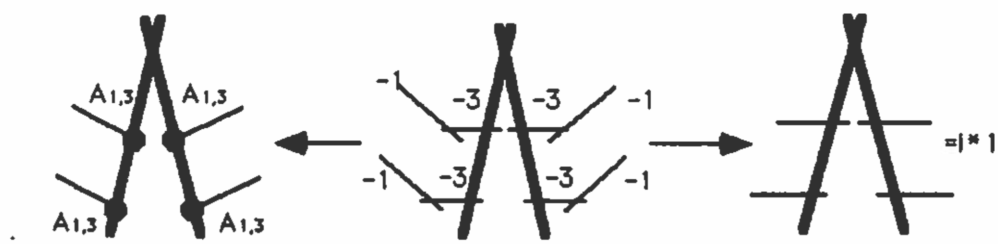
<u>$X_{1,40}$ (genus 1)</u>

Here are mod 2 computation begins with our conjectural description for $p^3 || N^-$. The mod 5 computation is completely straightforward.

Mod 2: The symbolic dual graph is

$$Z_{1,20} \overset{z_{1,20}}{\longleftrightarrow} 2Z_{1,5} \overset{z_{2,5}}{\longleftrightarrow} 2Z_{1,5} \overset{z_{1,20}}{\longleftrightarrow} Z_{1,20}.$$

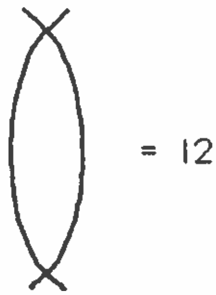
$Z_{1,20}$ consists of two components, each with mass $\frac{1}{3}$. $Z_{1,5}$ consists of a single component, also with mass $\frac{1}{3}$. Finally, $z_{2,5}$ is a single point with mass one.



Mod 5: The symbolic dual graph is

$$Z_{1,8} \overset{z_{5,8}}{\longleftrightarrow} Z_{1,8}.$$

$Z_{1,8}$ consists of a single component of mass $\frac{1}{3}$. $z_{5,8}$ consists of two points, each of mass one.



The four *a priori* possibilities for $J_{1,40}$ are

	2	5
40A	III	$I1$
40B	I^*1	$I2$
40C	III^*	$I4$
40D	III^*	$I1$

Hence each computation alone shows that $J_{1,40} = 40B$.

<u>$X_{3,16}$ (genus 1)</u>

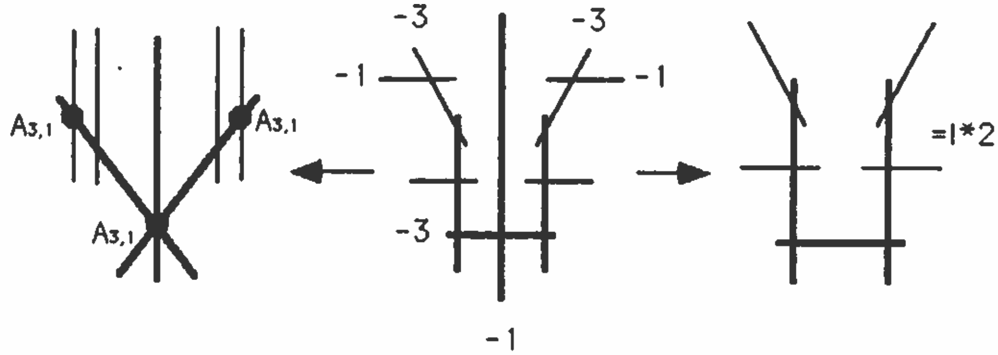
Here our mod 2 computation begins with our conjectural description of bad reduction for $p^4||N^-$. The mod 3 computation is completely straightforward.

Mod 2: The symbolic dual graph is

$$Z_{1,8} \overset{z_{1,8}}{\longleftrightarrow} Z_{1,2} \overset{z_{1,2}}{\top} Z_{1,2} \overset{z_{1,2}}{\longleftrightarrow} Z_{1,8}$$

$$Z_{3,1}.$$

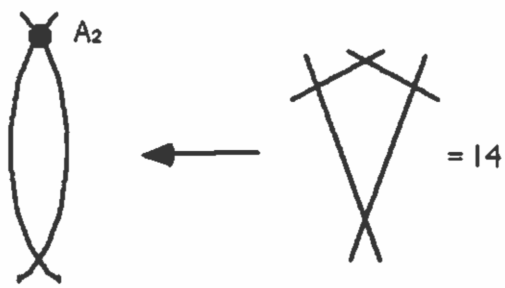
$Z_{3,1}$ is a genus zero curve. $Z_{1,2}$ has mass $\frac{1}{12}$. $Z_{1,8}$ has mass $\frac{1}{3}$.



Mod 3 (wild): The symbolic dual graph is

$$Z_{1,16} \overset{z_{3,16}}{\longleftrightarrow} Z_{1,16}.$$

$Z_{1,16}$ is a genus zero curve with two supersingular points. One of the two supersingular points in $z_{3,16}$ has mass one, the other $\frac{1}{3}$.



The six *a priori* possibilities for $J_{3,16}$ are

	2	3
$48A$	II	$I1$
$48B$	$I^{*}0$	$I2$
$48C$	$I^{*}2$	$I4$
$48D$	$I^{*}2$	$I1$
$48E$	$I^{*}3$	$I8$
$48F$	$I^{*}3$	$I2$

Hence the mod 2 computation alone shows that $J_{3,16} = 48C$ or $48D$ while the mod 3 computation alone shows that $J_{3,16} = 48C$.

$\mathcal{X}_{1,57} \rightarrow \mathcal{X}_{1/57}/w_{57}$ (genus 3 $\rightarrow$ genus 1)

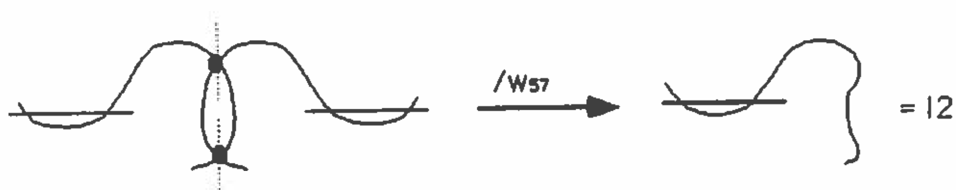
Finally we give an example where the group W is non-trivial. This example will be developed in more detail in the next section.

On $\mathcal{X}_{1,57}$, w_{57} has four fixed points, namely the four CM points with discriminant $D = -228$ ($h_{-228} = 4$).

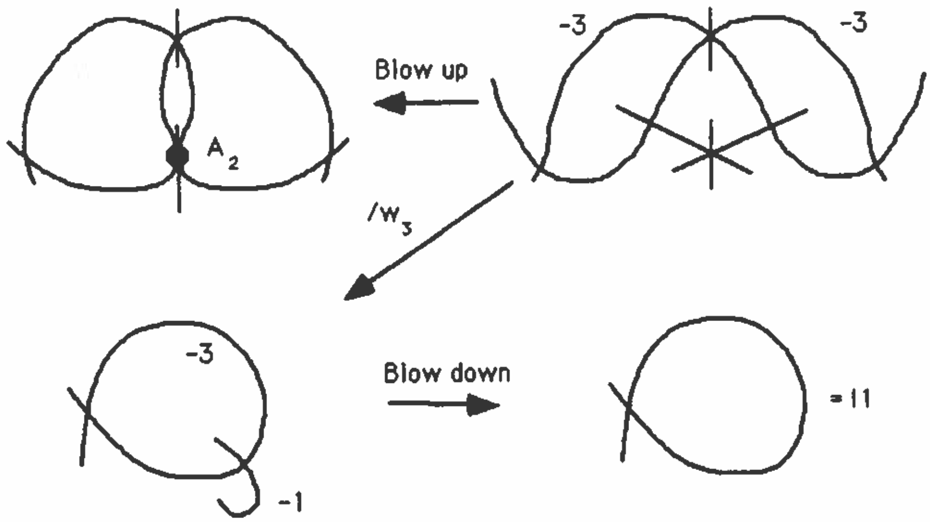
Mod 3: The symbolic dual graph is

$$Z_{1,19} \overset{z_{3,19}}{\longleftrightarrow} Z_{1,19}.$$

$Z_{1,19}$ has two components, one with mass one, the other with mass $\frac{1}{3}$. $z_{3,19}$ has four points, all of mass one.



Mod 19:



The only possibility for $\mathrm{Jac}(X_{1,57}/w_{57})$ is 57E which has reduction $I2$ at 3 and $I1$ at 19. Each of the two computations simply confirms this fact.

7.4 Table of reductions

There are 45 isogeny classes of modular elliptic curves with conductor ≤ 60. For each of these 45 classes our table has a block of rows.

To explain how to read the table we consider the typical case of 24A–F. $-+$ means that w_8 acts as -1 and w_3 acts as $+1$ on the corresponding factor $H^1(\mathcal{X}_0(24), \mathbb{C})^{f_{24A-F}}$. In general this information can be deduced from the fixed point formula for $w_{m_1}T_{m_2}$ or — more easily! — read off from Table 5 of [SD 75]. Here it isn't useful because both $\mathcal{X}_{24,1}$ and $\mathcal{X}_{1,24}$ have genus one. The entries I^*1. and $I2$ on the first row means that we compute the reduction at 2 to be I^*1 and at 3 to be $I2$; the '.' after I^*1 means that only one of 24A–F has this type of reduction at 2 and hence this computation alone suffices to determine $X_{1,24}$. In fact the table is

	2	3
$24A$	III	$I1$
$24B$	I^*1	$I2$
$24C$	III^*	$I4$
$24D$	III^*	$I1$
$24E$	II^*	$I8$
$24F$	II^*	$I2$

The last entry is the answer: $X_{24,1} = 24B$.

Isogeny class	ϵ_{p_i}	Shimura curve	Red_{p_1}	Red_{p_2}	Red_{p_3}	Curve
$11A - F$	$-$	$X_{11,1}$	$I5$.			$11B$
$14A - F$	$+-$	$X_{14,1}$	$I6$.	$I3$.		$14C$
		$X_{1,14}$	$I3$.	$I6$.		$14D$
$15A - H$	$+-$	$X_{15,1}$	$I4$	$I4$.		$15C$
		$X_{1,15}$	$I4$	$I4$.		$15C$
$17A - D$	$-$	$X_{17,1}$	$I4$.			$17C$
$19A - C$	$-$	$X_{19,1}$	$I3$.			$19B$
$20A - D$	$-+$	$X_{20,1}$	IV^*	$I2$.		$20B$
		$X_{5,4}$	IV^*	$I6$.		$20D$
$21A - F$	$-+$	$X_{21,1}$	$I4$.	$I2$		$21B$
		$X_{1,21}$	$I2$	$I4$.		$21D$
$24A - F$	$-+$	$X_{24,1}$	I^*1.	$I2$		$24B$
		$X_{1,24}$	??	$I4$.		$24C$
$26A - C$	$+-$	$X_{26,1}/w_2$	??	$I3$.		$26B$
		$X_{1,26}/w_{13}$	??	$I3$.		$26B$
$26D - E$	$-+$	$X_{26,1}/w_{13}$	??	$I1$.		$26D$
		$X_{1,26}/w_2$	??	$I7$.		$26E$
$27A - D$	$-$	$X_{27,1}$	??			??
$30A - H$	$+-+$	$X_{30,1}/w_5$	??	$I3$	$I1$	$30A$
		$X_{2,15}/w_3$	??	$I1$	$I3$	$30C$
		$X_{3,10}$	$I2$.	$I6$.	$I2$.	$30B$
		$X_{5,6}$	$I6$.	$I2$.	$I6$.	$30F$
$32A - D$	$-$	$X_{32,1}$	??			??
$33A - D$	$+-$	$(X_{33,1})$				
		$X_{1,33}$	$I6$.	$I2$.		$33B$
$34A - D$	$-+$	$X_{34,1}/w_{17}$	??	$I1$.		$34A$
		$X_{1,34}/w_2$	??	$I3$.		$34C$
$35A - C$	$+-$	$X_{35,1}/w_5$	$I3$.	$I3$.		$35B$
		$X_{1,35}/w_7$	$I3$.	$I3$.		$35B$

Isogeny class	ϵ_{p_i}	Shimura curve	Red_{p_1}	Red_{p_2}	Red_{p_3}	Curve
$36A-D$	$-+$	$X_{36,1}$	IV	III		$36A$
		$X_{4,9}$	IV^*	III		$36B$
		$X_{9,4}$	IV	III^*		$36C$
		$X_{1,36}$	IV^*	III^*		$36D$
$37A$	$+$	$X_{37,1}/w_{37}$	$I1.$			$37A$
$37B-D$	$-$	$(X_{37,1})$				
$38A-B$	$-+$	$X_{38,1}/w_{19}$	??	$I1.$		$38A$
		$X_{1,38}/w_2$	??	$I5.$		$38B$
$38C-E$	$+-$	$(X_{38,1})$				
		$X_{1,38}/w_{19}$	??	$I3.$		$38D$
$39A-D$	$+-$	$X_{39,1}/w_3$	$I2.$	$I2.$		$39B$
		$X_{1,39}/w_{13}$	$I2.$	$I2.$		$39B$
$40A-D$	$+-$	$(X_{40,1})$				
		$X_{1,40}$	$I^*1.$	$I2.$		$40B$
$42A-F$	$-++$	$X_{42,1}/\{w_3, w_7\}$	??	$I2$	$I1$	$42A$ or $42D$
		$X_{2,21}/w_{21}$	??	$I2$	$I2$	$42A$ or $42D$
		$X_{3,14}/w_2$	??	$I8.$	$I4.$	$42C$
		$X_{7,6}$	$I2$	$I8.$	$I4.$	$42C$
$43A$	$+$	$X_{43,1}/w_{43}$	$I1.$			$43A$
$44A-B$	$-+$	$X_{44,1}/w_{11}$	??	$I1.$		$43A$
		$X_{11,4}/w_{11}$	??	$I3.$		$44B$
$45A-H$	$-+$	$X_{45,1}/w_5$	I^*1	$I1$		$45A$ or $45D$
		$X_{5,9}$	$I2$	$I2$		$45B$
$46A-B$	$+-$	$(X_{46,1})$				
		$X_{1,46}$	$I5.$	$I2.$		$46B$
$48A-F$	$+-$	$X_{48,1}$				
		$X_{3,16}$	I^*2	$I4.$		$48C$
$49A-D$	$-$	$X_{49,1}$	III			$49A$ or $49B$
		$X_{1,49}$	III			$49A$ or $49B$

Isogeny class	ϵ_{p_i}	Shimura curve	Red_{p_1}	Red_{p_2}	Red_{p_3}	Curve
$50A - D$	$-+$	$X_{50,1}/w_{25}$	??	II		$50A$ or $50B$
		$X_{2,25}/w_{25}$	??	II		$50A$ or $50B$
$50E - H$	$+-$	$X_{50,1}/w_2$	??	IV		$50E$ or $50F$
		$X_{2,25}/w_2$	??	IV		$50E$ or $50F$
$51A - B$	$-+$	$X_{51,1}/w_{17}$	$I3.$	$I1.$		$51A$
		$X_{1,51}/w_3$	$I1.$	$I3.$		$51B$
$52A - B$	$-+$	$(X_{52,1})$				
		$X_{13,4}$	??	$I2.$		$52B$
$53A$	$+$	$X_{53,1}/w_{53}$	$I1.$			$53A$
$54A - C$	$-+$	$(X_{54,1})$				
		$X_{1,54}/w_2$	??	??		??
$54D - F$	$+-$	$X_{54,1}/w_2$	??	$IV^*.$		$54E$
		$X_{1,54}/w_{27}$	??	??		??
$55A - D$	$-+$	$X_{55,1}/w_{11}$	$I2.$	$I2.$		$55B$
		$X_{1,55}/w_5$	$I2.$	$I2.$		$55B$
$56A - B$	$-+$	$(X_{56,1})$				
		$(X_{1,56})$				
$56C - F$	$-+$	$X_{56,1}/w_7$	??	$I1$		$56E$ or $56F$
		$X_{1,56}/w_2$	??	$I4.$		$56E$
$57A - D$	$-+$	$(X_{57,1})$				
		$(X_{1,57})$				
$57E$	$++$	$X_{57,1}/\{w_3, w_{19}\}$	$I2.$	$I1.$		$57E$
		$X_{1,57}/w_{57}$	$I2.$	$I1.$		$57E$
$57F - G$	$-+$	$(X_{57,1})$				
		$(X_{1,57})$				
$58A$	$++$	$X_{58,1}/\{w_2, w_{29}\}$	??	$I1.$		$58A$
		$X_{1,58}/w_{58}$	??	$I1.$		$58A$
$58B - C$	$-+$	$(X_{58,1})$				
		$X_{1,58}/w_2$	??	$I5.$		$58C$

It is clear that all but a very few of the ?? can be filled in by working backwards. Thus for example we know Jac($X_{1,24}$) is 24C so its reduction at 2 must be III^*. Similarly we are obstructed by singularities from determining $X_{27,1}$ and $X_{32,1}$. However by other methods one knows that $X_{27,1} \cong 27B$ and $X_{32,1} \cong 32B$ [SD 75]; hence $X_{27,1}$ has bad reduction IV^* and $X_{32,1}$ and bad reduction I^*3 again by [SD 75]. It would be interesting to treat all these very wild cases directly.

7.5 Relations to the Birch-Swinnerton-Dyer conjecture

Let E be a modular elliptic curve over $\mathbf{Q}$ with conductor N. For each relatively prime factorization $N = N^+N^-$ we define an invariant $i_{N^+,N^-}(E) \in \mathbf{Z}_{\geq 1}/\mathbf{Z}_{\geq 1}^2$. Here $\mathbf{Z}_{\geq 1}/\mathbf{Z}_{\geq 1}^2$ is the group of positive integers under multiplication modulo squares; we will usually identify an element of $\mathbf{Z}_{\geq 1}/\mathbf{Z}_{\geq 1}^2$ with its unique square-free representative in $\mathbf{Z}_{\geq 1}$.

The bad reduction of E figures into our invariants. There is a small subtlety here and to make it clear we introduce more notation than we need. Let $\bar{\mathbf{F}}_p$ be an algebraic closure of $\mathbf{F}_{p^2}$. Consider the group of multiplicity one components $C = \underline{E}_{\bar{\mathbf{F}}_p}^{\text{smooth}}/\underline{E}_{\bar{\mathbf{F}}_p}^{\text{smooth},0}$. Let $A(C)$ denote the group of automorphisms of C which come from automorphisms of the entire singular fiber $E_{\bar{\mathbf{F}}_p}$. $\text{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_p)$ acts naturally on C through $A(C)$. We put $\bar{m}_p = \#(C)$ and for $n \in \mathbf{Z}_{\geq 1}$ we put $m_{p^n} = \#(C/\text{Gal}(\bar{\mathbf{F}}_p/\mathbf{F}_{p^n}))$. Thus the possibilities are

Type	C	$A(C)$	m_p	m_{p^2}	$\bar{m}_p$
I_1	$\mathbf{Z}/1$	1	1	1	1
I_2	$\mathbf{Z}/2$	1	2	2	2
I_n , $n = 2k-1$	$\mathbf{Z}/n$	± 1	k or n	n	n
I_n , $n = 2k-2$	$\mathbf{Z}/n$	± 1	k or n	n	n
I_0^*	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	$GL_2(\mathbf{Z}/2)$	2, 3, or 4	2 or 4	4
I_n^* , n odd	$\mathbf{Z}/4$	± 1	3 or 4	4	4
I_n^* , n even	$\mathbf{Z}/2 \oplus \mathbf{Z}/2$	± 1	3 or 4	4	4
II or II^*	$\mathbf{Z}/1$	1	1	1	1
III or III^*	$\mathbf{Z}/2$	1	2	2	2
IV or IV^*	$\mathbf{Z}/3$	± 1	2 or 3	3	3

Here we care only about m_{p^2}. The point to be made is that, with the exception of I^*0, $m_{p^2} = \bar{m}_p$ and hence m_{p^2} can be read off from the tables of [SD 75]. The exceptional case certainly can occur as indeed 32C and 32D each have I^*0 reduction at 2 with $m_4 = 2$ and $\bar{m}_2 = 4$. However conveniently it will not occur for us.

To define our invariant for an odd factorization $N = N^+N^-$ we need one more preliminary. Let d_{N^+,N^-} be the degree of any parametrization $\pi_{N^+,N^-} : X_{N^+,N^-} \to E$, considered as an element of $\mathbb{Z}_{\geq 1}/\mathbb{Z}^2_{\geq 1}$. d_{N^+,N^-} is well-defined (because we are over $\mathbb{Q}$ and so any complex multiplications E may have over some larger field don't figure in).

To define our invariant for an even factorization $N = N^+N^-$ we need the corresponding preliminary. Consider the one-dimensional subspace $H^0(\mathcal{X}_{N^+,N^-}, \mathbb{C})^{f_E} \subseteq H^0(\mathcal{X}_{N^+,N^-}, \mathbb{C})$. This vector space naturally comes from a rank one $\mathbb{Z}$-module, namely $H^0(\mathcal{X}_{N^+,N^-}, \mathbb{Z})^{f_E} := H^0(\mathcal{X}_{N^+,N^-}, \mathbb{Z}) \cap H^0(\mathcal{X}_{N^+,N^-}, \mathbb{C})^{f_E}$. Let e be one of the two bases and put d_{N^+,N^-} equal to the coincidence number (e, e) defined as in §4. Again we consider d_{N^+,N^-} as an element of $\mathbb{Z}_{\geq 1}/\mathbb{Z}^2_{\geq 1}$.

Definition 7.5.1

$$i_{N^+,N^-} = d_{N^+,N^-} \prod_{p|N^-} m_{p^2} \in \mathbb{Z}_{\geq 1}/\mathbb{Z}^2_{\geq 1}.$$

One reason that i_{N^+,N^-} is a convenient invariant is that it behaves simply under isogeny.

Proposition 7.5.2 *Let $E \to E'$ be an isogeny of degree d. Then*

$$i_{N^+,N^-}(E) = d\, i_{N^+,N^-}(E') \in \mathbb{Z}_{\geq 1}/\mathbb{Z}^2_{\geq 1}.$$

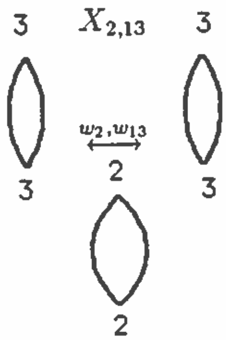
Proof. In fact we have the stronger statement $m_{p^2}(E) = d^{\mathrm{ord}_p(N)} m_{p^2}(E')$ for all p. In the odd case there are an even number of p with $\mathrm{ord}_p(N)$ odd and also $d_{N^+,N^-}(E) = d\,d_{N^+,N^-}(E')$. In the even case there are an odd number of such p but now $d_{N^+,N^-}(E) = d_{N^+,N^-}(E')$ as $e_E = e_{E'}$. □

In the following table we calculate $i_{N^+,N^-}(E)$ for one curve in each isogeny class of modular elliptic curves witn $N \leq 37$. The curve we use is the elliptic curve with a strong parametrization $X_0(N) \to E$.

The way our computations are presented in the table is best explained by an example. So consider 26D which has $m_{2^2} = 7$ and $m_{13^2} = 1$. Look at the second row which contains the even curve $X_{13,2}$ and the odd curve $X_{1,26}$.

The even curve has three components, say C_1, C_2, and C_3 with areas $4\pi/3$, $4\pi/3$, and $4\pi/2$ respectively. This fact follows from the term-by-term interpretation of 2.1.9. e_{26D} as a function on $\{C_1, C_2, C_3\}$ is $(1, 1, -2)$

as more generally



$$\begin{aligned} e_{\text{Eis}} &= (2,2,3) \\ e_{26B} &= (1,-1,0) \\ e_{26D} &= (1,1,-2) \end{aligned}$$

In the range of the table it is necessary to consider only Atkin-Lehner operators to determine e_E; [SD 75,Table 5] is extremely helpful. In general one would have to consider Hecke operators as well. $d_{13,2}(26D)$ is thus $3 \cdot 1^2 + 3 \cdot 1^2 + 2 \cdot 2^2 = 14$ as recorded. $i_{13,2}(26D)$ is thus $14 \cdot 7 = 2$ as recorded.

From the table in §7.4 we have $X_{1,26}/w_2 \sim 26E$. From the isogeny diagram in [SD 75,Table 1] we know that there is a degree seven isogeny $26E \rightarrow 26D$ as indicated. Thus $d_{1,26}(26D)$ is $2 \cdot 7 = 14$. $i_{1,26}(26D)$ is thus $14 \cdot 7 = 2$ as recorded.

E	$m_{p_i^2}$	Even curves					d	i	Odd curves	d	i
$11B$	5	$X_{1,11}$	1(3)	−1(2)			5	1	$X_{11,1} = 11B$	1	1
$14C$	6,3	$X_{7,2}$	1(3)	−1(3)			6	1	$X_{14,1} = 14C$	1	1
		$X_{2,7}$	1	−1(2)			3	1	$X_{1,14} \sim 14D \xrightarrow{2} 14C$	2	1
$15C$	4,4	$X_{5,3}$	1(2)	−1(2)			4	1	$X_{15,1} = 15C$	1	1
		$X_{3,5}$	1	−1(3)			4	1	$X_{1,15} \sim 15C$	1	1
$17C$	4	$X_{1,17}$	1	−1(3)			4	1	$X_{17,1} = 17C$	1	1
$19B$	3	$X_{1,19}$	1	−1(2)			3	1	$X_{19,1} = 19B$	1	1
$20B$	3,2	$X_{4,5}$	1	−1			2	1	$X_{20,1} = 20B$	1	1
		$X_{1,20}$	1(3)	−1(3)			6	1	$X_{5,4} = 20D \xrightarrow{3} 20B$	3	1
$21B$	4,2	$X_{3,7}$	1	−1			2	1	$X_{21,1} = 21B$	1	1
		$X_{7,3}$	1	−1(3)			4	1	$X_{1,21} \sim 21D \xrightarrow{2} 21B$	2	1
$24B$	4,2	$X_{8,3}$	1	−1			2	1	$X_{24,1} = 24B$	1	1
		$X_{3,8}$	1	−1(3)			4	1	$X_{1,24} \sim 24C \xrightarrow{3} 24B$	2	1
$26B$	3,3	$X_{2,13}$	1	1	−2		6	2	$X_{26,1}/w_2 = 26B$	2	2
		$X_{13,2}$	1(3)	−1(3)	0(2)		6	2	$X_{1,26}/w_{13} \sim 26B$	2	2
$26D$	7,1	$X_{2,13}$	1	−1	0		2	2	$X_{26,1}/w_{13} = 26D$	2	2
		$X_{13,2}$	1(3)	1(3)	−2(2)		14	2	$X_{1,26}/w_2 \sim 26E \xrightarrow{7} 26D$	14	2
$27B$	3	$X_{1,27}$	1	−1(2)			3	1	$X_{27,1} = 27B$	1	1
$30A$	4,3,1	$X_{1,30}$	1(3)	−1(3)			6	2	$X_{30,1}/w_5 = 30A$	2	2
		$X_{15,2}$	1	−1			2	2	$X_{2,15}/w_3 \sim 30C \xrightarrow{3} 30A$	6	2
		$X_{10,3}$	1	1	−1(2)	−1(2)	6	2	$X_{3,10} \sim 30B \xrightarrow{2} 30A$	2	2
		$X_{6,5}$	0	0	1	−1	2	2	$X_{5,6} \sim 30F \xrightarrow{6} 30A$	6	2
$32B$	4	$X_{1,32}$	1	−1(3)			4	1	$X_{32,1} = 32B$	1	1
$33B$	6,2	$X_{3,11}$	−1	1	1	−1(3)	6	3	$X_{33,1} \xrightarrow{??} 33B$	??	??
		$X_{11,3}$	1	−1			2	3	$X_{1,33} \sim 33B$	1	3
$34A$	6,1	$X_{2,17}$	1	−1	0	0	2	2	$X_{34,1}/w_{17} = 34A$	2	2
		$X_{17,2}$	1	−1(2)			3	2	$X_{1,34} \sim 34C \xrightarrow{3} 34A$	3	2
$35B$	3,3	$X_{5,7}$	1	1	−1(2)	−1(2)	6	2	$X_{35,1}/w_5 = 35B$	2	2
		$X_{7,5}$	0	0	1(3)	−1(3)	6	2	$X_{1,35}/w_7 \sim 35B$	2	2
$36A$	3,2								$X_{36,1} = 36A$	1	1
									$X_{4,9} \sim 36B \xrightarrow{2} 36A$	2	1
									$X_{9,4} = 36C \xrightarrow{3} 36A$	3	1
									$X_{1,36} \sim 36D \xrightarrow{6} 36A$	6	1
$37A$	1	$X_{1,37}$	0	1	−1		2	2	$X_{37,1}/w_{37} = 37A$	2	2
$37C$	3	$X_{1,37}$	2	−1	−1		6	2	$X_{37,1} \to 37C$	2	2

The point to be made is that for E in the above tables, $i_{N^+,N^-}(E)$ is independent of (N^+, N^-). In fact this independence result is implied by a generalization of the Gross-Zagier theorem together with the Birch-Swinnerton-Dyer conjecture. We will not describe this implication save to say that a crucial point is that the order of the Shafarevich-Tate group is a square.

8 A detailed example

In this section we consider the behavior of some of the CM points on $X_{1,57}$ under a suitably normalized isomorphism $X_{1,57}/w_{57} \xrightarrow{\sim} 57E$. There are exactly 13 CM points on $X_{1,57}/w_{57}$ with residue field $\mathbf{Q}$ and in §1 we use Theorem 6.5.2 to calculate the intersections of their closures on $\underline{X}_{1,57}/w_{57}$. On the other hand the elliptic curve $E := 57E : y^2 + y = x^3 - x^2 - 2x + 2$ has an infinite cyclic Mordell-Weil group $E(\mathbf{Q})$ with generator $P = (2,1)$ and in §2 we explicitly calculate $(\underline{mP} \cdot \underline{nP})$ for $|m - n|$ small. Finally in §3 we compare our two calculations to determine where the 13 CM points sit in the Mordell-Weil group.

8.1 $X_{1,57}/w_{57}$

The CM points corresponding to the 13 discriminants with class number one are distributed among the four Riemann surfaces $\mathcal{X}_{N^+,N^-}$ with $N = 57$ as follows:

$\mathcal{X}_{57,1}$	$\mathcal{X}_{19,3}$	$\mathcal{X}_{3,19}$	$\mathcal{X}_{1,57}$
2(−3)	2(−3)	4(−11)	4(−4)
4(−8)	2(−12)		4(−7)
2(−12)	2(−19)		4(−16)
4(−27)	4(−67)		2(−19)
			4(−28)
			4(−43)
			4(−163)

We are interested in the case $(N^+, N^-) = (1, 57)$. To ease notation we upt $a := i_3 \in \mathbf{Z}_{3^2}$ and $b = i_{19} \in \mathbf{Z}_{19^2}$ so that $a^2 = -1$ and $b^2 = -1$ by the conventions of §1.1. Thus the four CM points with $D = -4$ are $x_{-4;a,2b}$, $x_{-4;a,-2b}$, $x_{-4,-a,2b}$, and $x_{-4;-a,-2b}$. The 26 points on $\mathcal{X}_{1,57}$ correspond to 13 scheme-theoretic points $x_{D;\pm(e_3,e_{19})}$ on $X_{1,57}$. There are only two possibilities for e_3, namely a and $-a$; we will notationally take advantage of this fact by abbreviating $x_{D;e_{19}} := x_{D;\pm(a,e_{19})}$.

The residue field $\mathbf{Q}(x_{D;e_{19}})$ is of course imaginary quadratic with discriminant D. w_{57} fixes each of these 13 points but acts non-trivially on the residue field. Thus on the quotient curve $X_{1,57}/w_{57}$ the 13 points remain distinct and now have residue field $\mathbf{Q}$.

On the following table we use Theorem 6.5.2 to compute $\underline{x}_{-163;7b} \cdot \underline{x}_{D,e_{19}}$

on $\underline{X}_{1,57}$. In the second column we list n satisfying $n^2 < -163D$, $n \equiv -163D$ (2), $n \equiv 1$ (3), $n \equiv 7be_{19}$ (19). Thus here we are essentially following the suggestion made in §6.5 and not bothering to list the remaining allowable n, namely the negatives of those listed. However here we do not have to correspondingly multiply the result by two at the end because we are working on $X_{1,57}/w_{57}$ rather than $X_{1,57}$ itself.

Divisor	n	Type of $S_{-163,n,D}$	$e^{\text{intersection number}}$
$c_{-43;10b}$	-13	$(1, 2\cdot 3\cdot 5)$	
$c_{-28;16b}$	2	$(1, 2^2\cdot 5)$	5
$c_{-16;4b}$	-28	$(1, 2^3)$	2^2
$c_{-4;2b}$			
$c_{-7;8b}$			
$c_{-19;0}$	-19	$(1, 2^2\cdot 3)$	3
$c_{-7;-8b}$	-1	$(1, 5)$	5
$c_{-4;-2b}$	14	$(1, 2)$	2
$c_{-16;-4b}$	-10	$(1, 11)$	11
$c_{-28;-16b}$	74	$(1, 13)$	13
$c_{-43;-10b}$	-25	$(1, 2^2\cdot 7)$	7
$c_{-163;-7b}$	125	$(1, 2^4\cdot 3)$	3
	11	$(1, 2^2\cdot 29)$	29
	-103	$(1, 2\cdot 5\cdot 7)$	

Note that on the last row we have included the case $(D, e_{19}) = (-163, -7b)$, despite the fact that $D_0 = -163$ and $D_1 = -163$ are not relatively prime. Theorem 6.5.2 actually applies here as well because all the $S_{-163,n,D}$ appearing here are in fact Eichler orders. Proceeding similarly, and remembering

$$\begin{aligned} c_{-28;e_{19}} &= x_{-28;e_{19}} + x_{-7;e_{19}/2} \\ c_{-16;e_{19}} &= x_{-16;e_{19}} + \frac{1}{2} x_{-4;e_{19}/2}, \end{aligned}$$

we obtain the entire (entry-wise exponentiated) intersection matrix.

	-163 $7b$	-43 $10b$	-28 $16b$	-16 $4b$	-4 $2b$	-7 $8b$	-19 0	-7 $-8b$	-4 $-2b$	-16 $-4b$	-28 $-16b$	-43 $-10b$	-163 $-7b$
$x_{-163;7b}$	*			2			3	5	2^2	11	13	7	$3\cdot 29$
$x_{-43;10b}$		*					2			3	5	2^2	7
$x_{-28;16b}$			*					2^*			3	5	13
$x_{-16;4b}$	2			*					2			3	11
$x_{-4;2b}$					*					2			2^2
$x_{-7;8b}$						*					2^*		5
$x_{-19;0}$	3	2					*					2	3
$x_{-7;-8b}$	5		2^*					*					
$x_{-4;-2b}$	2^2			2					*				
$x_{-16;-4b}$	11	3			2					*			2
$x_{-28;-16b}$	13	5	3			2^*					*		
$x_{-43;-10b}$	7	2^2	5	3			2					*	
$x_{-163;-7b}$	$3\cdot 29$	7	13	11	2^2	5	3			2			*

Here the starred entries correspond to an intersection at an ordinary point. We note that at this juncture we have no right to expect any symmetry in the intersection matrix other than the four-fold symmetry

$$x_{D_0,r_0} \cdot x_{D_1,r_1} = x_{D_0,-r_0} \cdot x_{D_1,-r_1} = x_{D_1,r_1} \cdot x_{D_0,r_0} = x_{D_1,-r_1} \cdot x_{D_0,-r_0}.$$

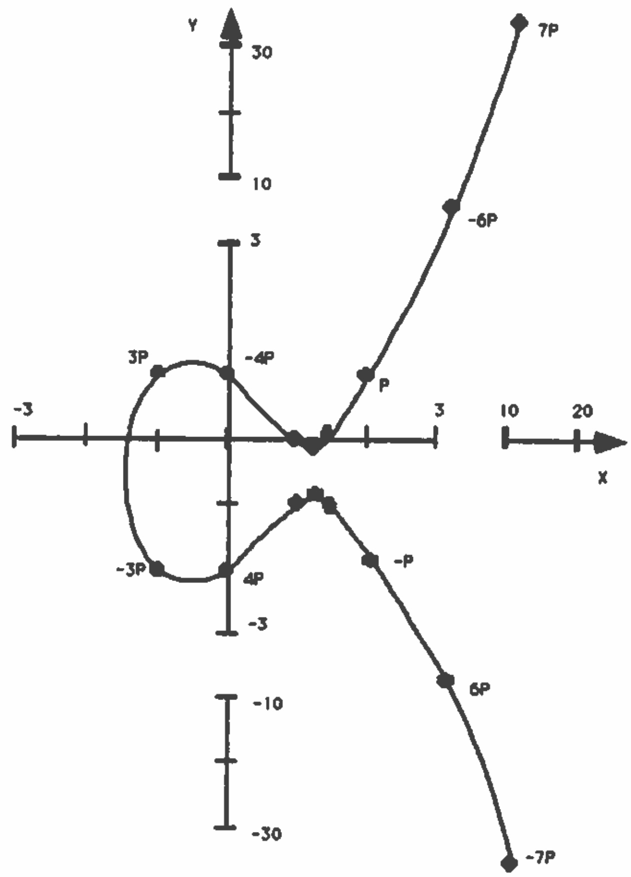
Our matrix does have more symmetry and we have presented things (i.e. labeled the 13 points, ordered the 13 points, and included the gaps between rows/columns 1-2 and 12-13) to make this extra symmetry apparent.

Similarly we compute that the seven sections $\underline{x}_{-4;\pm 2b}$, $\underline{x}_{-19,0}$, $\underline{x}_{-28,\pm 16b}$ and $\underline{x}_{-163,\pm 7b}$ all reduce to one of the two components at 3 and the remaining six sections $\underline{x}_{-7,\pm 8b}$, $\underline{x}_{-16,\pm 4b}$, $\underline{x}_{-43,\pm 10b}$ reduce to the other component.

8.2 $57E$

Now we temporarily forget what we've done in §8.1 and consider the elliptic curve $57E : y^2 + y = x^3 - x^2 - 2x + 2$. $P = (x_1, y_1) = (2, 1)$ is clearly a solution and we apply the addition formula to obtain others:

P	$(2,1)$
$2P$	$(1,0)$
$3P$	$(-1,1)$
$4P$	$(0,-2)$
$5P$	$(\frac{5}{4}, -\frac{7}{8})$
$6P$	$(4,-7)$
$7P$	$(11,34)$
$8P$	$(\frac{13}{9}, -\frac{1}{27})$
$9P$	$(\frac{14}{25}, \frac{62}{125})$
$10P$	$(-\frac{23}{16}, -\frac{51}{64})$
$11P$	$(\frac{86}{121}, -\frac{1765}{1331})$
$12P$	$(\frac{261}{169}, -\frac{2588}{2197})$
$13P$	$(\frac{995}{49}, -\frac{30701}{343})$
$14P$	$(\frac{1660}{529}, \frac{44122}{12167})$
$15P$	$(\frac{8789}{7396}, -\frac{80647}{636056})$
$16P$	$(-\frac{1976}{7569}, \frac{750007}{658503})$



If $nP = (x_n, y_n)$ then $-nP = (x_n, -1 - y_n)$ giving us yet more solutions. We have drawn $\{nP\}_{-8 \le n \le 8} \subset E(\mathbf{R})$.

One can prove by descent that in fact $E(\mathbf{Q})$ is cyclic with generator $(2, 1)$. As for computing the intersection numbers $(\underline{nP} \cdot \underline{mP})$, the group law simplifies things considerably:

$$(\underline{mP} \cdot \underline{nP}) = (\underline{(m-n)P} \cdot \underline{\text{Origin}}) = \log d_{m-n}.$$

Here d^2_{m-n} is the denominator of x_{m-n} (or equivalently d^3_{m-n} is the denominator of y_{m-n}). We have $d_i = d_{-i}$. On the table below we list those d_i, $1 \leq |i| \leq 16$, for which $d_i \neq 1$.

i	1	2	3	4	5	6	7	8
d_i					2			3

i	9	10	11	12	13	14	15	16
d_i	5	2^2	11	13	7	23	$2 \cdot 43$	$3 \cdot 29$

8.3 Comparison

We now compare our two approaches. On $X_{1,57}/w_{57}$ the involutions w_3 and w_{19} coincide. $w_3 = w_{19}$ has four fixed points in $(X_{1,57}/w_{57})(\bar{\mathbf{Q}})$, one being $x_{-19,0}$ and the other three forming one Galois orbit, namely $x_{-76,0}(\bar{\mathbf{Q}})$. We turn the genus one curve $X_{1,57}/w_{57}$ into an elliptic curve by taking $x_{-19,0}$ as the origin; $w_3 = w_{19}$ then acts as -1. We are still left with the choice of one of the two isomorphisms $X_{1,57}/w_{57} \xrightarrow{\sim} E$; to eliminate this ambiguity we arbitrarily require that $x_{-163,7b}$ be identified with nP for some $n > 0$.

Proposition 8.3.1 *Assume that all 13 of the CM points we are considering are in $\{nP\}_{-8\leq n\leq 8}$. Then*

$$
\begin{aligned}
x_{-19;0} &= 0 \\
x_{-7;\pm 8b} &= \pm P \\
x_{-4;\pm 2b} &= \pm 2P \\
x_{-16;\pm 4b} &= \pm 3P \\
x_{-28;\pm 16b} &= \pm 4P \\
x_{-43;\pm 10b} &= \pm 5P \\
x_{-163;\pm 7b} &= \pm 8P
\end{aligned}
$$

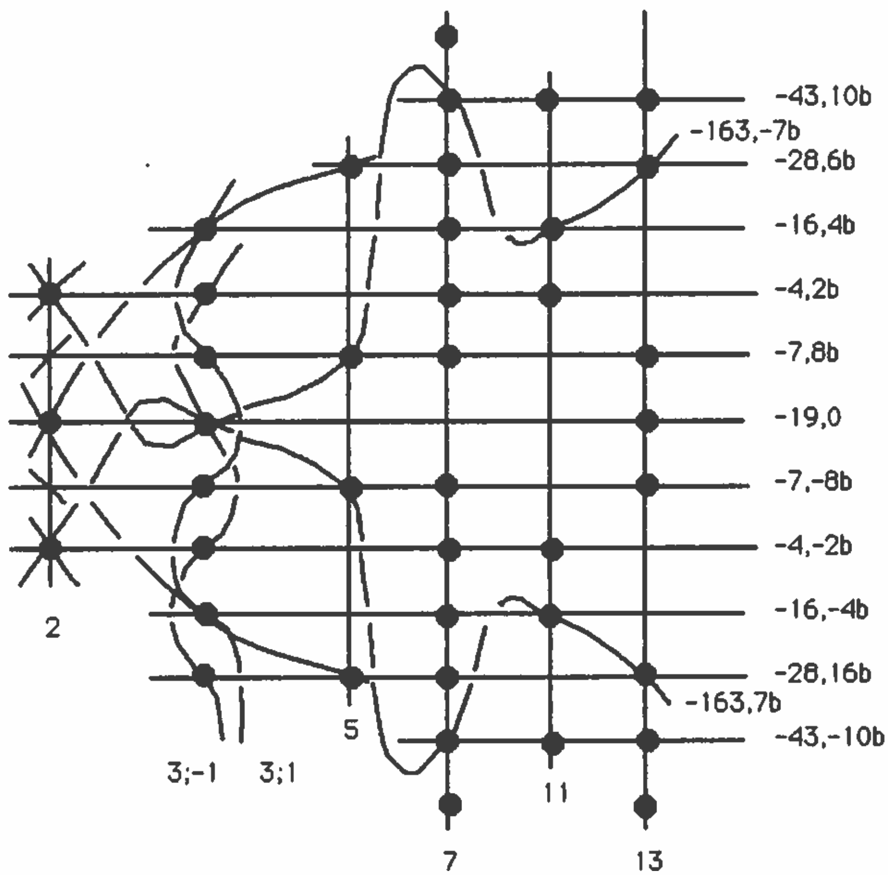
Proof. Comparing the matrix in §8.1 with the table in §8.2 one immediately finds that the given identification is the only possibility. Indeed there is a lot of redundant information all of which is consistent with the conclusion. □

There is a lot of flexibility in the hypothesis of Proposition 8.3.1. For example, one could weaken it by replacing "$-8 \leq n \leq 8$" by "$-20 \leq n \leq 20$" say. Alternatively one could replace it by the much stronger but more natural sounding hypothesis "Assume that there are no points in $E(\mathbf{Q})$ with integral coordinates other than nP for $|n| \in \{1,2,3,4,6,7\}$"; with a good effective version of Siegel's theorem (see e.g. [Si 86]) one might be able to establish this stronger hypothesis but for now it's not even clear that it's true. In any case it follows from a generalization of the Gross-Zagier theorem to be proved in a later paper.

We summarize our situation by a picture. From e.g. Table 3 of [SD 75] we have

p	2	3	5	7	11	13
$\#(E(\mathsf{F}_p))$	5	8	9	13	11	12

Conveniently for the purposes of drawing the picture, $(2,1)$ is a generator for $E(\mathsf{F}_p)$ for these first six primes.



The picture shows

1. the 13 horizontal divisors discussed above;
2. the 7 vertical divisors F_2, $F_{3;-1}$, $F_{3;1}$, F_5, F_7, F_{11}, and F_{13};
3. the 46 points with residue field a prime field F_p, $p \leq 13$.

Note that all but three of these points lie on one of the 13 horizontal divisors. If z is one of these 43 points with $z \in F_p$ and $z \in \underline{x}_{D;e_{19}}$ then we can immediately determine the type of z: if $\left(\frac{D}{p}\right) = 1$, then z is ordinary while if $\left(\frac{D}{p}\right) = 0$ or -1, then z is supersingular. The three remaining points are also supersingular. We have drawn the supersingular points slightly larger on the figure.

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