Wild Ramification in moduli fields of three-point covers

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1. Moduli algebras

2. Ramification

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4. Evidence for the truth and sharpness of the conjecture

1. Moduli algebras. Let $n \geq 1$ and $g \geq 0$ be integers. Let $(\lambda_0, \lambda_1, \lambda_\infty)$ be a triple of partitions of n with all together $n + 2 - 2g$ parts. Then the theory of *three-point covers* (= *dessins d'enfants* = *Belyi maps*) gives an associated *moduli algebra* $K(\lambda_0, \lambda_1, \lambda_\infty)$. The degree N of $K(\lambda_0, \lambda_1, \lambda_\infty)$ can be computed by diagrammatic or group-theoretic techniques.

Example with $(n, g) = (7, 0)$. Let $(\lambda_0, \lambda_1, \lambda_\infty) = (61, 421, 3211)$. Consider rational functions

$$f(x) = \frac{\kappa x^6 (x - a)}{(x - 1)^2 (x^2 + bx + c)}$$

viewed as maps from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$. The fibers $f^{-1}(0)$ and $f^{-1}(\infty)$ have types $\lambda_0 = 61$ and $\lambda_\infty = 3211$ respectively. There are $N = 12$ values for (a, b, c, κ) which make the fiber $f^{-1}(1)$ have the desired type 421. The a's are the roots of

$F(a) =$

$$256a^{12} - 7680a^{11} - 1620a^{10} + 1519268a^9$$
$$-457995a^8 - 197818644a^7 - 1135592364a^6$$
$$+1323901404a^5 + 29033249406a^4$$
$$+88620573860a^3 + 117954887400a^2$$
$$+74118870000a + 18015003125$$

The moduli algebra is $\mathbb{Q}[a]/F(a)$ and depends only on $(61, 421, 3211)$. Choosing a better generator gives

$K(61, 421, 3211) =$

$$\mathbb{Q}[x]/(x^{12} - 3x^{11} + x^9 + 21x^8 - 12x^7 - 92x^6$$
$$+132x^5 + 24x^4 - 120x^3 - 36x^2 + 180x - 100)$$

In general, the embeddings of $K(\lambda_0, \lambda_1, \lambda_\infty) \to \mathbb{C}$ index three-point covers with ramification type $(\lambda_0, \lambda_1, \lambda_\infty)$ and monodromy group all of A_n or S_n. Experimentally, like in this case, $K(\lambda_0, \lambda_1, \lambda_\infty)$ tends to be a field with Galois group all of S_N. Typically, $N >> n$.

2. Ramification. Ramification in number algebras is measured by discriminants, e.g.

$$\mathrm{disc}(K(61, 421, 3211)) = -2^{19}3^{12}5^{2}7^{5}.$$

On a more refined level, ramification at a given prime p is measured by factoring over $\mathbb{Q}_p^{\mathrm{un}}$. For $K(61, 421, 3211)$, this yields

p	Factorization	Type of ramification
2	$8_{16}\, 2_3\, 1_0\, 1_0$	Wild, top slope = 3
3	$3_3\, 3_3\, 3_3\, 3_3$	Wild, top slope = 1.5
5	$7_0\, 2_1\, 2_1\, 1_0$	Tame
7	$6_5\, 6_0$	Tame
≥ 11	$1_0 \cdots 1_0$	Un

Always m_c is a factor of degree m and discriminant p^c. Factors come in three types:

Unramified:	1_0	
Tamely ramified:	m_{m-1}	$(p \nmid m)$
Wildly ramified:	m_c	$(p \mid m$ and $c \geq m)$

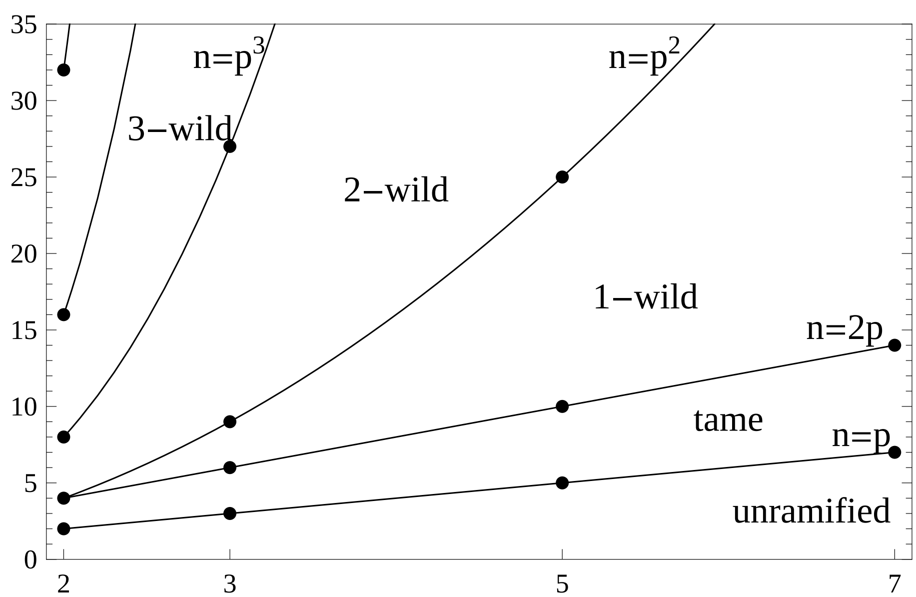
Wild factors can be complicated, and a measure of their wildness is their largest p-adic slope. The simplest example is $m = p$. Then the possibilities for c are p, . . . , $2p+1$, and the largest p-adic slope is $s = c/(p-1)$.

3. Upper bounds on ramification. Let $K = K(\lambda_0, \lambda_1, \lambda_\infty)$ be the moduli algebra of a degree n partition triple.

Theorem (Grothendieck 1961; Beckmann 1989). *If $p \in (n, \infty)$, then p is unramified in K.*

Theorem (Wewers 2003). *If $p \in (n/2, n]$ then p is at most tamely ramified in K.*

Conjecture. *If $p \in [2, n/2]$, then K's largest p-adic slope s satisfies $s \leq \lfloor \log_p n \rfloor + \frac{1}{p-1}$.*



4. Evidence for the truth and sharpness of the conjecture. Thousands of moduli algebras have been computed with $n \leq 12$. Because of computational difficulties, almost all have $g = 0$, and most have degree $N \leq 40$. The conjecture holds for all of them, with the top slope often near or at its conjectural upper bound $\lfloor \log_p n \rfloor + \frac{1}{p-1}$.

A systematic supply of degree N moduli algebras for varying n has been investigated theoretically. Namely for fixed N let a and b be positive integers and put $n = N + a + b + 1$. Define

$$\begin{aligned} \lambda_0 &= (N+1)1^{n-N} \\ \lambda_1 &= (a+1)(b+1)1^{n-a-b-1} \\ \lambda_\infty &= n \end{aligned}$$

Then the moduli algebra is given by a Jacobi polynomial $P_N^{a+b-1,-b-N-1}(x)$. Specialize to the case $N = p^j$. Then the conjecture holds and moreover the upper bound is obtained immediately upon entering the j-wild region, for all p and all j.