Hurwitz Belyi maps

David P. Roberts
University of Minnesota, Morris

1. Background on Belyi maps via a very unusual example

2. A conjecture on the existence of certain Belyi maps in arbitrarily large degree

3. Hurwitz Belyi maps: numerical examples and how we expect they possibly can be used to prove the conjecture

Parts 2 and 3 are analogs for Belyi maps of work with Akshay Venkatesh on number fields.

The Belyi map case is similar to the number field case but more geometric.

1A. Generic vs. Belyi maps. Any degree n function $F : \mathsf{Y} \to \mathsf{P}^1$ has

$$2n + 2\,\text{genus}(Y) - 2$$

critical points in Y, counting multiplicity.

Generically, the critical points $y_i \in \mathsf{Y}$ are distinct; also the critical values $F(y_i) \in \mathsf{P}^1$ are distinct.

Class of examples. Consider $F : \mathsf{P}^1_y \to \mathsf{P}^1_t$ given by $F(y) = f(y)/g(y)$. Then

$$F'(y) = \frac{f'(y)g(y) - f(y)g'(y)}{g(y)^2}.$$

If $f(y)$, $g(y)$ are "random" degree n polynomials in $\mathbb{C}[y]$, then the y_i are the $2n-2$ distinct roots of the numerator.

Definition. *F is called a Belyi map if its critical values are within $\{0, 1, \infty\}$.*

So Belyi maps are as far from generic as possible, and moreover their critical values are normalized.

1B. A sample Belyi map. Define $\beta : \mathsf{P}^1_y \to \mathsf{P}^1_t$ by

$$\beta(y) = \frac{(y+2)^9 y^{18} (y^2-2)^{18} (y-2)}{(y+1)^{16} (y^3-3y+1)^{16}}.$$

so that $F(\infty) = 1$. Where are the

$$2n + 2g - 2 = 2 \cdot 64 + 2 \cdot 0 - 2 = 126$$

critical points?

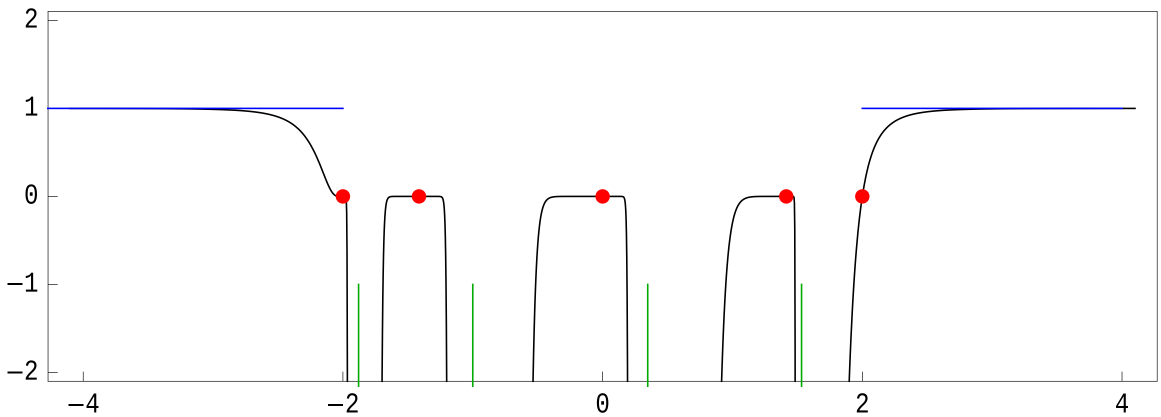
0: From numerator, $8 + 3 \cdot 17 = \boxed{59}$ points with crit value 0.

∞**:** From denominator, $4 \cdot 15 = \boxed{60}$ points with crit value ∞.

1: From degree(numerator − denominator) = 56, ∞ is a crit point with multiplicity $\boxed{7}$.

Since $59 + 60 + 7 = 126$, β is indeed a Belyi map.

1C. A real-variable visualization of β (fun but not so useful):



Clearly visible:

0: Zeros with multiplicity 9, 18, 18, 18, 1.

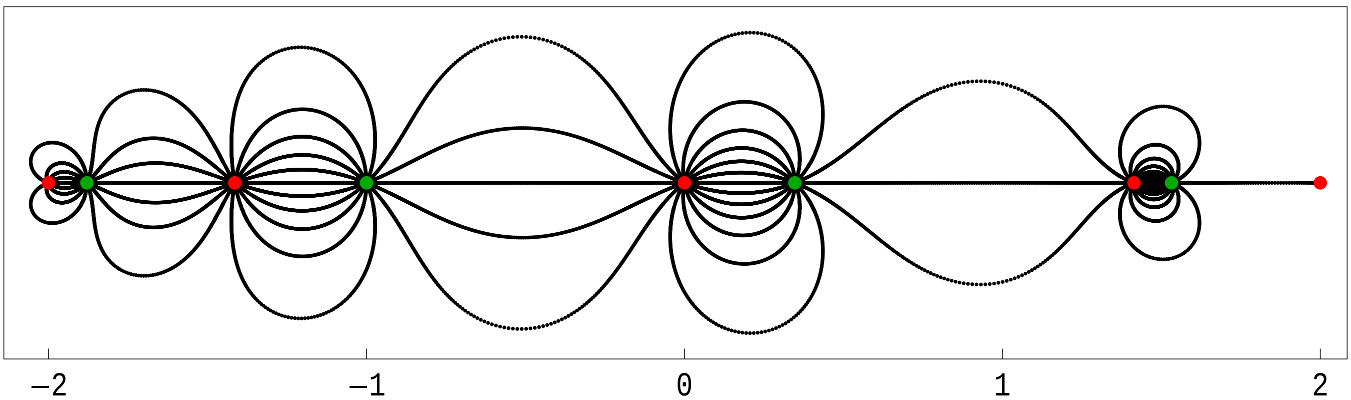
1: Horizontal asymptote at level $t = 1$ with multiplicity 8.

∞: Poles with multiplicity 16, 16, 16, 16.

Also there are 56 non-critical non-real preimages of 1. This makes the ramification triple

$$(\lambda_0, \lambda_1, \lambda_\infty) = (18^3\,9\,1,\ 8\,1^{56}, 16^4)$$

1D. A complex-variable visualization of β (extremely useful):



The *dessin* $\beta^{-1}([-\infty, 0]) \subset \mathsf{P}^1_y$.

Monodromy operators on the set of edges can be read off:

g_∞ = rotate counterclockwise minimally about •

g_0 = rotate counterclockwise minimally about •

$g_1 = g_0{}^{-1} g_\infty{}^{-1}$ so that $g_0 g_1 g_\infty = 1$.

Each g_i has cycle type λ_i and $\langle g_0, g_1, g_\infty \rangle = S_{64}$.

1E. Computation of β. To obtain β, first consider

$$\frac{(y+2)^9\left(y^3+ay^2+by+c\right)^{18}(y-2)}{\left(y^4+dy^3+ey^2+fy+g\right)^{16}}$$

such that Degree(Num−Denom)=56. There are seven equations in the seven unknowns a, b, c, d, e, f, g.

There are 35 solutions (a,b,c,d,e,f,g) corresponding to 35 Belyi maps. The a-values are the roots of

$$\begin{aligned} &a\,(8096790625a^{34} - 1360260825000a^{33} + \cdots \\ &-12940132959358757742449293935866601984000a^2 \\ &+1444543635586477445099159157466988544000a \\ &-633054568549175937272241607139131392000) . \end{aligned}$$

The first factor a gives our β via the solution $(a,b,c,d,e,f,g) = (0,-2,0,1,-3,-2,1)$. The second factor has Galois group S_{34} and field discriminant

$$2^{71}3^{44}5^{27}7^{27}11^{23}13^{19}19^{15}23^{10}29^{11}31^{8}37^{4}47^{3}.$$

1F. Three invariants of a general degree n Belyi map illustrated by our β with comments:

• The monodromy group $\Gamma \subseteq S_n$. For β, it's $\boxed{S_{64}}$. It's very easy to make Γ *full*, meaning $\Gamma \in \{A_n, S_n\}$.

• The field of definition $F \subset \mathbb{C}$. For our β, it's $\boxed{\mathbb{Q}}$, despite

$$\text{degree}(18^3\,9\,1,\ 8\,1^{56}, 16^4) = 35.$$

When degree$(\lambda_0, \lambda_1, \lambda_\infty)$ is large, very commonly all of the corresponding Belyi maps are conjugate.

• The bad reduction set $\mathcal{P}$, consisting of certain primes $\leq n$. For β, it's just $\boxed{\{2,3\}}$ (because for other p the numerator and denominator are coprime in $\mathbb{F}_p[y]$). Typically, $\mathcal{P}$ is close to being all primes $\leq n$.

2. Two expectations. Call a finite set of primes $\mathcal{P}$ *anabelian* if it contains the set of primes dividing the order of a finite nonabelian simple group, and *abelian* otherwise.

Examples. The set $\mathcal{P} = \{2, 3, p\}$ is anabelian for $p \in \{5, 7, 13, 17\}$. All other $\mathcal{P}$ with $|\mathcal{P}| \leq 3$ are abelian.

Conjecture. *Let $\mathcal{P}$ be an anabelian set of primes. Then there exist* $\boxed{\textit{full}}$ *Belyi maps,* $\boxed{\textit{defined over } \mathbb{Q}}$*, and* $\boxed{\textit{ramified within } \mathcal{P}}$*, of arbitrarily large degree n.* (Reason for believing: the existence of Hurwitz Belyi maps)

Personal guess. *Let $\mathcal{P}$ be an abelian set of primes. Then there exist* $\boxed{\textit{full}}$ *Belyi maps,* $\boxed{\textit{defined over } \mathbb{Q}}$*, and* $\boxed{\textit{ramified within } \mathcal{P}}$*, only for finitely many degrees n.* (Reason for believing: all examples with n large for a given $\mathcal{P}$, like β from before, seem "accidental")

3A. Hurwitz parameters. Consider again a general degree n map $F : \mathsf{Y} \to \mathsf{P}^1_t$. Three invariants are:

- Its global monodromy group $G \subseteq S_n$
- The list $C = (C_1, \ldots, C_s)$ of distinct conjugacy classes arising as non-identity local monodromy operators.
- The corresponding list $(D_1, \ldots, D_s)$ of disjoint finite subsets $D_i \subset \mathsf{P}^1_t$ over which these classes arise.

To obtain a single discrete invariant, we write $\nu = (\nu_1, \ldots, \nu_s)$ with $\nu_i = |D_i|$. We then form the *Hurwitz parameter* $h = (G, C, \nu)$.

In our computational examples, we will typically take G closely related to one of the twenty smallest simple groups, and $r := \sum |\nu_i| \in \{4, 5\}$.

3B. Hurwitz maps. An r-point Hurwitz parameter $h = (G, C, \nu)$ determines a cover of r-dimensional complex varieties:

$$\pi_h : \mathsf{Hur}_h \to \mathsf{Conf}_\nu.$$

A point $x \in \mathsf{Hur}_h$ indexes an isomorphism class of covers

$$\mathsf{Y}_x \to \mathsf{P}^1_t \tag{1}$$

of type h. The base Conf_ν is the space of possible branch loci

$$(D_1, \ldots, D_s). \tag{2}$$

The map π_h sends a cover (1) to its branch locus (2).

When G is simple and C contains sufficiently many classes,

$$\text{degree}(\pi_h) = \frac{1}{|G|^2} \prod_{i=1}^{s} |C_i|^{\nu_i}.$$

3C. Known facts about Hurwitz maps

A new group-theoretic fact: With Venkatesh, we have necessary and sufficient conditions on a pair (G, C) for $\pi_h : \mathsf{Hur}_h \to \mathsf{Conf}_\nu$ to have $\boxed{\text{full monodromy}}$ for sufficiently large $\min_i \nu_i$. Every simple G gives rise to infinitely many full Hurwitz maps.

Older arithmetic facts: If all C_i are rational, then $\pi_h : \mathsf{Hur}_h \to \mathsf{Conf}_\nu$ is $\boxed{\text{defined over } \mathbb{Q}}$. Moreover, π_h then has all its bad reduction $\boxed{\text{within the set } \mathcal{P}_G}$ of primes dividing G.

3D. Belyi pencils. A Belyi pencil of type ν is an embedding $u : \mathsf{P}^1 - \{0, 1, \infty\} \to \mathsf{Conf}_\nu$.

Examples (with $k = j - 1$):

$$\begin{array}{rcl} u_4 : \mathsf{P}^1_j - \{0,1,\infty\} & \to & \mathsf{Conf}_{3,1}, \\ j & \mapsto & ((t^3 - 3jt + 2j), \{\infty\}), \end{array}$$

$$\begin{array}{rcl} u_5 : \mathsf{P}^1_j - \{0,1,\infty\} & \to & \mathsf{Conf}_{4,1}, \\ j & \mapsto & k^2t^4 - 6jkt^2 - 8jkt - 3j^2. \end{array}$$

Polynomial discriminants are

$$\begin{array}{rcl} D_4(j) & = & 2^2 3^3 j^2 (j-1), \\ D_5(j) & = & -2^{12} 3^3 j^4 (j-1)^6. \end{array}$$

So the bad reduction sets are $\mathcal{P}_{u_4} = \mathcal{P}_{u_5} = \{2, 3\}$.

It is easy to get Belyi pencils into infinitely many Conf_ν defined over $\mathbb{Q}$, all with bad reduction set in any given non-empty $\mathcal{P}$.

3E. Hurwitz Belyi maps. Suppose given

- A Hurwitz parameter $h = (G, C, \nu)$.

- A Belyi pencil $u : \mathsf{P}^1 - \{0, 1, \infty\} \to \mathsf{Conf}_\nu$.

Definition. *The Hurwitz Belyi map $\beta_{h,u}$ is obtained by by pulling back and canonically completing:*

$$\begin{array}{rccccc} \mathsf{X} & \supset & \mathsf{X}^0 & \to & \mathsf{Hur}_h \\ \beta_{h,u}\downarrow & & \downarrow & & \downarrow \pi_h \\ \mathsf{P}^1 & \supset & \mathsf{P}^1 - \{0,1,\infty\} & \stackrel{u}{\to} & \mathsf{Conf}_\nu. \end{array}$$

If h and u are defined over $\mathbb{Q}$, then $\beta_{h,u}$ is likewise $\boxed{\text{rational}}$.

The bad reduction set of $\mathcal{P}_{h,u}$ is contained in $\boxed{\mathcal{P}_G \cup \mathcal{P}_u}$.

If π_h is full then one would generally expect $\beta_{h,u}$ to be full too. However it's possible that the monodromy group becomes smaller.

3F. Four full examples. All examples are presented by giving $f(j,x)$, where $f(j,x)=0$ describes $\beta_{h,u} : \mathsf{P}^1_x \to \mathsf{P}^1_j$,

Example 1. $h = (S_5, (41, 2111), (3, 1))$ **and** $u = u_4$ **giving degree** $n = 32$ **and** $\mathcal{P} = \{2, 3, 5\}$**.**

$$
\begin{aligned}
f(j,x) = &\\
&\Big(x^{10} - 38x^9 + 591x^8 - 4920x^7 + \\
&\quad 24050x^6 - 71236x^5 + 125638x^4 \\
&\quad -124536x^3 + 40365x^2 + 85050x \\
&\quad -91125\Big)^3 \left(x^2 - 14x - 5\right) \\
&+2^{20}3^3 j x^6 (x-5)^5 \left(x^2 - 4x + 5\right)^4 (x-9)^3,
\end{aligned}
$$

$$
\begin{aligned}
&\mathrm{disc}_x(f(j,x)) = \\
&\quad -2^{1032}3^{261}5^{289} j^{20} (j-1)^{16}.
\end{aligned}
$$

One of ≈ 16 full Belyi maps with ramification triple $(3^{10}\,1^2,\ 2^{16}, 10\ 6\ 5\ 4^2\ 3)$.

Example 2. $h = (A_6, (3111, 2211), (4, 1))$ **and** $u = u_5$ **giving degree** $n = 192$ **and** $\mathcal{P} = \{2, 3, 5\}$**.**

$$\begin{aligned}
&f(j,x) = \\
&\quad \big(14659268544x^{64} - 1012884030720x^{63} \\
&\qquad \cdots - 1245316608x^4 + 28200960x^3 \\
&\qquad -569088x^2 + 11008x - 64\big)^3 \\
&\quad -2^4 3^6 j \left[-3x^3 + 7x^2 - 11x + 1\right]^{15} \\
&\qquad \left(-6x^5 + 36x^4 - 72x^3 + 64x^2 - 23x + 4\right)^{12} \cdot \\
&\qquad \left[-3x^3 + 9x^2 - 3x - 1\right]^9 \cdot \\
&\qquad \big(9x^8 - 72x^7 + 240x^6 - 444x^5 + 474x^4 \\
&\qquad\qquad -280x^3 + 72x^2 - 12x + 1\big)^6 \cdot \\
&\qquad [3-x]^5\,[x]^4\,[1-x]^3,
\end{aligned}$$

$$\begin{aligned}
&\mathrm{disc}_x(f(j,x)) = \\
&\quad -2^{6028} 3^{9585} 5^{10525} j^{128} (j-1)^{84}.
\end{aligned}$$

This Belyi map is one of about

$$1,900,000,000,000,000,000,000,000,000,000,000,000$$

full Belyi maps with ramification triple

$$(3^{64},\ 2^{84}\,1^{24}, 15^3\,12^5\,9^3\,6^8\,5\;4\;3).$$

Example 3. $h = (G_2(2), (2B, 4D), (3, 1))$ **and** $u = u_4$ **giving degree** $n = 40$ **and** $\mathcal{P} = \{2, 3, 7\}$**.**

$$
\begin{aligned}
f(j,x) = \\
&\Big(64x^{12} - 576x^{11} + 2400x^{10} - 5696x^9 \\
&\qquad +7344x^8 - 3168x^7 - 4080x^6 \\
&\qquad +8640x^5 - 7380x^4 - 1508x^3 \\
&\qquad +8982x^2 - 7644x + 2401\Big)^3 \cdot \\
&\qquad \Big(4x^4 - 20x^3 + 78x^2 - 92x + 49\Big) \\
&-2^8 3^{12} j \left(2x^2 - 4x + 3\right)^8 x^7 (x-2)^3 (x+1)^2,
\end{aligned}
$$

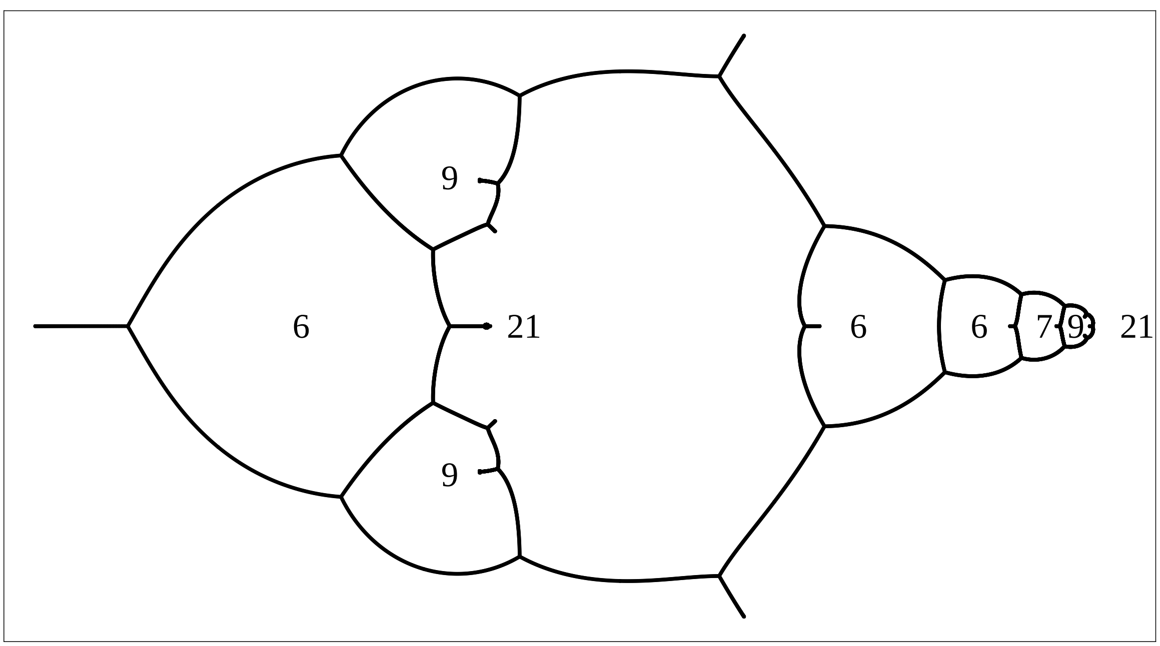
$$
\begin{aligned}
\mathrm{disc}_x(f(j,x)) = \\
&-2^{1148}\, 3^{906}\, 7^{91}\, j^{24}\, (j-1)^{20}.
\end{aligned}
$$

One of $\approx 10{,}000$ full Belyi maps with ramification triple $(3^{12}\, 4,\ 2^{20}, 12\, 8^2\, 7\, 3\, 2)$.

Example 4. $h = (GL_3(2), (22111, 331), (4,1))$ **and** $u = u_5$ **giving degree** $n = 96$ **and** $\mathcal{P} = \{2,3,7\}$**.**

$$f_{96}(j,x) = \left(7411887x^{32} - 316240512x^{31} + 5718682592x^{30} \cdots + 123834728448x - 3869835264\right)^3 - 2^{20}j\left(7x^2-14x+6\right)^{21}\left(2x^3-15x^2+18x-6\right)^9 \cdot x^6\left(x^2+2x-2\right)^6(3x-2)^2.$$

The dessin $\beta^{-1}([0,1])$ in P^1_x:



One of $\approx 3{,}100{,}000{,}000{,}000{,}000$ full Belyi maps with triple $(3^{32},\ 2^{40}\,1^{16}, 21^2\,9^3\,7\,6^3\,2)$.

3G. Towards proving the monodromy conjecture. (Unlike the parallel assertion about number fields) the conjecture may be provable by braid monodromy arguments. Equations, like those of Examples 1-4, are not at all needed.

Verifications in modest degrees are already feasible. E.g., for $h = (S_5, (41, 221), (4, 1))$ the degree of π_h is 1440. A braid monodromy calculation shows that β_{h,u_5} still has full monodromy. It seems feasible and would be interesting to take calculations of this sort into much larger degrees.

What is needed to prove the conjecture is theoretical control over the potential drop in monodromy when one passes from π_h to $\beta_{h,u}$.

Some References. This talk is in the process of becoming *Hurwitz Belyi maps.*

The initial degree 64 example is $U_{8,9}$ from *Chebyshev covers and exceptional number fields* on my homepage.

The full-monodromy theorem with Venkatesh is Theorem 5.1 in *Hurwitz monodromy and full number fields,* to appear in Algebra and Number Theory.

A standard reference on Hurwitz schemes is Bertin and Romagny, *Champs de Hurwitz* Mém. Soc. Math. Fr. 125–126 (2011), 219pp.

Example 4 takes as its starting point Theorem 4.2 from Gunter Malle, *Multi-parameter polynomials with given Galois group*, J. Symbolic Comput. 30 (2000) 717–731.

Braid group computations in higher degree should be feasible using Magaard, Shpectorov, Völklein, *A GAP package for braid orbit computation and applications.* Experiment. Math 12 (2003), no. 4, 385–393.

Braid computations after specialization to Belyi pencils will involve ideas from Jordan Ellenberg. *Galois invariants of dessins d'enfants.* 27-42, Proc. Sympos. Pure Math., 70 (2002).