Hypergeometric Motives I

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Overview of today's talk

1. Definition of a nicely-indexed collection of varieties $X(\vec{p}, \vec{q}, t)$. The "most interesting" part of their cohomology is then a nicely-indexed collection of motives $H(\vec{p}, \vec{q}, t)$ in $\mathcal{M}(\mathbb{Q}, \mathbb{Q})$. Illustrations from the familiar settings $\dim(X(\vec{p}, \vec{q}, t)) \leq 1$.

2. An interlude about Hodge numbers $h^{p,q}$ in general. A combinatorial procedure for computing the Hodge numbers of the $H(\vec{p}, \vec{q}, t)$. This computations show that $H(\vec{p}, \vec{q}, t)$ thoroughly leave the setting of points and curves.

3. Very light sketch of the passage from an $H(\vec{p}, \vec{q}, t)$ to its L-function. Demonstration of how *Magma* is close to calculating complete L-functions of hypergeometric motives automatically, and how they numerically seem to have the expected analytic properties.

Many fundamental topics just briefly mentioned, or omitted!

1. From canonical varieties to hypergeometric motives

Canonical varieties (see [BCM])

Let $\vec{p} = (p_1, \dots, p_r)$ and $\vec{q} = (q_1, \dots, q_s)$ be tuples of positive integers with $\gcd(p_1, \dots, p_r, q_1, \dots, q_s) = 1$, $p_i \neq q_j$ always, and

$$p_1 + \cdots + p_r = q_1 + \cdots + q_s.$$

Let $t \in \mathbb{Q}^\times - \{1\}$. Define $Y(\vec{p}, \vec{q}, t)$ in $\mathbb{P}^{r+s-1}$ by

$$\begin{aligned} x_1 + \cdots + x_r &= y_1 + \cdots + y_s, \\ t x_1^{p_1} \cdots x_r^{p_r} q_1^{q_1} \cdots q_s^{q_s} &= p_1^{p_1} \cdots p_r^{p_r} y_1^{q_1} \cdots y_s^{q_s}. \end{aligned}$$

Notes: • $w := \dim(Y(\vec{p}, \vec{q}, t)) = r + s - 3$.
• $Y(\vec{p}, \vec{q}, t)$ is complicated topologically and can have singularities.
• $t = 1$ makes sense, but then $Y(\vec{p}, \vec{q}, 1)$ has an extra singularity at $(x_1, \dots, x_r, y_1, \dots, y_s) = (p_1, \dots, p_r, q_1, \dots, q_s)$.

Zero-dimensional cases

In the special case $(r, s) = (2, 1)$ the system takes the form

$$Y((a,b),(c),t): \qquad \begin{array}{rcl} x_1 + x_2 & = & y_1, \\ t x_1^a x_2^b c^c & = & a^a b^b y_1^c. \end{array}$$

Dehomogenizing by $y_1 = 1$, eliminating via $x_2 = 1 - x_1$, and abbreviating $x = x_1$ gives a traditional univariate equation

$$tc^c x^a (1-x)^b - a^a b^b = 0.$$

The discriminant of the left side is $a^{a(c-1)} b^{b(c-1)} c^{c^2} (t-1) t^{c-1}$. The generic Galois group is S_c, but there are occasional drops. E.g., $(a, b, c) = (6, 1, 7)$ and $t = 64$ has the 168-element Galois group $PSL_2(7) \cong GL_3(2) \subset S_7$.

A one-dimensional case

Via some new coordinates ([BCM] pages 4-5),

$$Y((6,1),(4,3),t): \quad y^2 = x^3 - \frac{27}{4t}x + \frac{27}{4t}.$$

The right side has discriminant $-2^{-4}3^9(t-1)t^{-3}$. The j-invariant of $Y((6,1),(4,3),t)$ is $1728/(1-t)$.

So for all $t \in \mathbb{Q}^\times - \{1\}$, the motivic Galois group of $Y((6,1),(4,3)),t)$ is GL_2, except for the eleven classical exceptions. These exceptions range in height from

$$t = \frac{189}{125} = \frac{3^3 7}{5^3} \qquad \text{(Potential CM by } \mathbb{Q}(\sqrt{-7}))$$

to

$$t = \frac{3^3 7^2 11^2 19^2 127^2 163}{2^{12}5^3 23^3 29^3} \qquad \text{(Potential CM by } \mathbb{Q}(\sqrt{-163})).$$

Hypergeometric motives

[BCM] elegantly resolves singularities on $Y(\vec{p}, \vec{q}, t)$ to get a family of smooth projective varieties $X(\vec{p}, \vec{q}, t)$, degenerating only at $t \in \{0, 1, \infty\}$. They determine the point count $|X(\vec{p}, \vec{q}, t)(\mathbb{F}_q)|$ with $q = p^f$ and p a good prime, using a **hypergeometric trace formula** of [Greene] and [Katz].

For any $t \in \mathbb{Q}^\times - \{1\}$, all cohomology is represented by algebraic cycles except for the part corresponding to a motive $H(\vec{p}, \vec{q}, t)$ in the middle cohomology.

Example: for $(r, s) = (4, 4)$, the Betti numbers $(b_0, \ldots, b_{10})$ are $(1, 0, 10, 0, 19, b_5, 19, 0, 10, 0, 1)$ with b_5 depending on $(\vec{p}, \vec{q})$. The point count is

$$|X(\vec{p}, \vec{q}, t)(\mathbb{F}_q)| = 1 + 10q + 19q^2 + a_q + 19q^3 + 10q^4 + q^5.$$

Here $a_q = -\mathrm{Trace}(\mathrm{Fr}_q | H(\vec{p}, \vec{q}, t))$ is the interesting quantity.

2. Hodge numbers in general
and
Hodge numbers of hypergeometric motives

Hodge numbers in general, I

The Hodge numbers $h^{p,q} := \dim_{\mathbb{C}} H^q(X, \Omega^p)$ are fundamental invariants of a smooth projective variety X over $\mathbb{C}$.

For a connected w-dimensional variety, one traditionally presents them as a **Hodge diamond**, as in the case of a surface:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,2} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array} \quad = \quad \begin{array}{ccccc} & & 1 & & \\ & c & & c & \\ a & & b & & a \\ & c & & c & \\ & & 1 & & \end{array}$$

One always has left-right symmetry $h^{p,q} = h^{q,p}$ and up-down symmetry $h^{p,q} = h^{w-q,w-p}$.

Hodge numbers in general, II

Genus g curve: $\begin{array}{ccc} & 1 & \\ g & & g \\ & 1 & \end{array}$

Surfaces of low degree in $\mathbb{P}^3$:

1	**2**	**3**	**4**	**5**
$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 1 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 2 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$	$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 4 & & 45 & & 4 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$

Quintic threefold, its mirror, and an abelian threefold:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 1 & & 0 & \\ 1 & & 101 & & 101 & & 1 \\ & 0 & & 1 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 101 & & 0 & \\ 1 & & 1 & & 1 & & 1 \\ & 0 & & 101 & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 3 & & \\ & 3 & & 9 & & 3 & \\ 1 & & 9 & & 9 & & 1 \\ & 3 & & 9 & & 3 & \\ & & 3 & & 3 & & \\ & & & 1 & & & \end{array}$$

Hodge numbers in general, III

The standard way to present a weight w motive is as a summand in the middle cohomology of a w-dimensional variety. One describes the decomposition via **Hodge vectors** $(h^{w,0}, \ldots, h^{0,w})$. E.g., a $K3$ surface with 18 independent algebraic cycles decomposes as

$$(1, 20, 1) = (0, 18, 0) + (\mathbf{1}, \mathbf{2}, \mathbf{1}).$$

Very extreme quintic 3-folds decompose as

$$(1, 101, 101, 1) = (0, 101, 101, 0) + (\mathbf{1}, \mathbf{0}, \mathbf{0}, \mathbf{1})$$

The *Hodge width (= Hodge level= Hodge niveau)* is the largest $|p - q|$ with $h^{p,q}$ nonzero. The generalized Hodge conjecture says that if M is a motive with Hodge width $\underline{w}$, then the Tate twist of $M\left(\frac{w-\underline{w}}{2}\right)$ appears in the cohomology of an $\underline{w}$-dimensional variety.

Hodge numbers of HGMs I (see [Fedorev])

The Hodge vector for $H(\vec{p}, \vec{q}; t)$ is calculated from the roots and poles of the rational function

$$\frac{\prod_{i=1}^{r}(x^{p_i} - 1)}{\prod_{i=1}^{s}(x^{q_i} - 1)}.$$

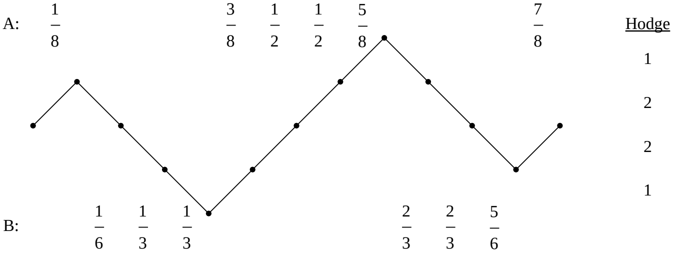
There is typically a lot of cancellation.
Example. For $(\vec{p}, \vec{q}) = ((8, 2, 2, 2), (6, 4, 3, 1))$,

$$\frac{(x^8 - 1)(x^2 - 1)^3}{(x^6 - 1)(x^4 - 1)(x^3 - 1)(x - 1)} = \frac{\Phi_8(x)\Phi_2(x)^2}{\Phi_6(x)\Phi_3(x)^2}.$$

Intertwining. What's essential to the formula is how the roots $\exp(2\pi i\alpha_j)$ and poles $\exp(2\pi i\beta_k)$ intertwine on the unit circle, i.e. how the indices α_j and β_k intertwine on $\mathbb{R}/\mathbb{Z}$.

Hodge numbers of HGMs II

The general procedure is illustrated by how it looks in our example $(\vec{p}, \vec{q}) = ((8, 2, 2, 2), (6, 4, 3, 1))$:



As one goes right, one goes up while passing through an α_j and down when passing through a β_k. From the number of upward steps at a given height, one gets the Hodge numbers. In this weight $r + s - 3 = w = 5$ case, $\vec{h} = (0, 1, 2, 2, 1, 0)$. Tate twisting down to $\underline{w} = 3$, the Hodge numbers become

$$(h^{3,0}, h^{2,1}, h^{1,2}, h^{0,3}) = (1, 2, 2, 1).$$

Hodge numbers of HGMs III

Both extremes are very interesting: **complete intertwining** yields

$$h^{0,0} = (d).$$

This case was intensively studied in [BH]; special motivic Galois groups should be finite, rather than the Sp_d and O_d that occur in essentially all other cases.

Complete separation yields

$$(h^{d-1,0}, \ldots, h^{0,d-1}) = (1, 1 \ldots, 1, 1)$$

which should be families with maximal parameter number 1.

For a given degree, there are $2^{\lfloor d/2 \rfloor}$ intermediate Hodge vectors. Computations shows that in degree ≤ 20 **they all come from HGMs**, except for $(6, 1, 1, 1, 2, 1, 1, 1, 6)$.

3. L-functions of hypergeometric motives: quick demonstration of *Magma* package

Constructing and checking L-functions

For a given $M = H(\vec{p}, \vec{q}, t)$,

1. One has the good Euler factors $L_p(M, s)$ from the hypergeometric trace formula.

2. One has the Gamma factors $L_\infty(M, s)$ from the Hodge number procedure. For $w = 2p$ even, one has the necessary supplemental decomposition $h^{p,p} = h_+^{p,p} + h_-^{p,p}$ which depends on the component of $\mathbb{R}^\times - \{1\}$ containing t.

3. By a variety of techniques, one can often determine at least some of the bad factors $L_p(M, s)$ and some of the contributions p^{c_p} to the conductor N.

4. The expected but almost always unproved analytic properties of $\Lambda(M, s) = N^{s/2} L_\infty(M, s) \prod_p L_p(M, s)$ can be numerically confirmed by *Magma*'s `CheckFunctionalEquation`. In the process, final ambiguities about bad factors are removed.

The family $H((6,1),(7))$ at $t = 2$ and $t = 64$

```
>H := HypergeometricData([*-6,-1,7*]);
>Identify(H,2);
   Artin representation
   S7: (6,4,0,2,3,0,2,0,1,-1,1,0,-1,-1,-1) of
   ext<Q|x^7-6*x^6+15*x^5-20*x^4+15*x^3-6*x^2+x
   -23328/823543>
>Identify(H,64);
    Artin representation
    PSL(2,7): (6,2,0,0,-1,-1) of
    ext<Q|x^7-6*x^6+15*x^5-20*x^4+15*x^3-6*x^2+x
   -729/823543>
```

Magma can compute the associated L-functions, up to bad primes from the general hypergeometric formulas, or completely through its Artin representation package.

The motive $H((6,1),(4,3),189/125)$

```
>H := HypergeometricData([*-6,-1,4,3*]);
>L := LSeries(H,189/125);
>[<p,EulerFactor(L,p)>: p in PrimesInInterval(1,30)];
   [<2, 1>,
    <3, 3*x^2 + 1>,
    <5, 1>,
    <7, 1>,
    <11, 11*x^2 + 4*x + 1>,
    <13, 13*x^2 + 1>,
    <17, 17*x^2 + 1>,
    <19, 19*x^2 + 1>
    <23, 23*x^2 + 8*x + 1>,
    <29, 29*x^2 + 2*x + 1>]
```

The evenness of half the Euler factors reflects the potential CM by $\mathbb{Q}(\sqrt{-7})$ known to be present.

The motives $H((5),(1,1,1,1,1),t)$

This family has Hodge vector $(1,1,1,1)$ and is famous for appearing in the Dwork pencil of threefolds

$$v^5 + w^5 + x^5 + y^5 + z^5 = 5\psi vwxyz.$$

for $t = 1/\psi^5$.

```
>H:=HypergeometricData([*-5,1,1,1,1,1*]);
>L:=LSeries(H,2);
   WARNING: Guessing wild prime information
>Conductor(L); EulerFactor(L,5);
   50000
   1                        (Is this 5-adic information right?)
>CheckFunctionalEquation(L);
   0.000000000000000000000000000000 (Very likely, yes!)
```

The motive $H((3^4),(1^{12}),1)$

This motive has Hodge vector $(1,1,1,0,0,1,1,1)$ and so can first appear in a seven-fold.

```
>H := HypergeometricData([*-3,-3,-3,-3,
                          1,1,1,1,1,1,1,1,1,1,1,1*]);
>L := LSeries(H,1: BadPrimes:=[<3,9,1>]);
```

(overriding *Magma*'s incorrect guess to get a numerically certified L-series)

```
>Evaluate(L,4);
    0.00000....
>Sign(L);
    1.00000....
>LTaylor(L,4,2);
     0.00000... + 0.00000... (s-4) +
     3.247420050405010039020038649853 (s-4)^2
```

Some references

Hypergeometric Motives, with Fernando Rodriguez Villegas and Mark Watkins, in preparation. Several presentations by each of us available online.

Finite hypergeometric functions, by Frits Beukers, Henri Cohen, and Anton Mellit. ArXiv May 12, 2015.

Hypergeometric functions over finite fields, by John Greene, Trans. Amer. Math. Soc. **301** (1987), 77-101.

Exponential Sums and Differential Equations, by Nicholas M. Katz, Annals of Math Studies, **124**, (1990) is an early work emphasizing motivic aspects of hypergeometric functions.

Some references, continued

Variations of Hodge Structure for Hypergeometric Differential operators and parabolic Higgs bundles, by Roman Fedorov, ArXiv May 7, 2015, has the Hodge number formula. Antecedents include works of Terasoma, Corti, Golyshev, Dettweiler, and Sabbah.

Monodromy for the hypergeometric function ${}_nF_{n-1}$, by Frits Beukers and Gert Heckman, Invent. Math. **95** (1989), 325-354, definitively treats the case of complete intertwining.

The HGM package in *Magma* is by Mark Watkins. The L-function package is by Tim Dokchitser.