HURWITZ-BELYI MAPS

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ABSTRACT. The study of the moduli of covers of the projective line leads to the theory of Hurwitz varieties covering configuration varieties. Certain one-dimensional slices of these coverings are particularly interesting Belyi maps. We present systematic examples of such "Hurwitz-Belyi maps." Our examples illustrate a wide variety of theoretical phenomena and computational techniques.

CONTENTS

1. INTRODUCTION

The theory of Belyi maps sits at an attractive intersection in mathematics where group theory, algebraic geometry, and number theory all play fundamental roles. In this paper we first introduce a simply-indexed class of particularly interesting Belyi maps which arise in solutions of Hurwitz moduli problems. Our main focus is then the computation of sample Hurwitz-Belyi maps and the explicit exhibition of their remarkable properties. We expect that our exploratory work here will support future more theoretical studies. We conclude this paper by speculating that as degrees become large, Hurwitz-Belyi maps become extreme outliers among all Belyi maps. The rest of the introduction amplifies on this first paragraph.

1.1. **Belyi maps.** In the classical theory of smooth projective complex algebraic curves, ramified covering maps from a given curve Y to the projective line P^1 play a prominent role. If Y is connected with genus g, then any degree n map $F : \mathsf{Y} \to \mathsf{P}^1$ has $2n + 2g - 2$ critical points in Y, counting multiplicities. For generic F, these critical points y_i are all distinct and moreover the critical values $F(y_i)$ are

also also distinct. A *Belyi map* by definition is a map $\mathsf{Y} \to \mathsf{P}^1$ having all critical values in $\{0, 1, \infty\}$. One should think of Belyi maps as the maps which are as far from generic as possible, with their critical values being moreover normalized to a standard position. The recent paper [24] provides a computationally-focused survey of Belyi maps, with many references.

1.2. **An example.** The main focus of this paper is the explicit construction of Belyi maps with certain extreme properties. A Belyi map from [15] arises from outside the main context of this paper but still exhibits these extremes:

$$\pi : \mathsf{P}^1 \quad \to \quad \mathsf{P}^1,$$

(1.1)
$$x \quad \mapsto \quad \frac{(x+2)^9 x^{18} \left(x^2 - 2\right)^{18} (x-2)}{(x+1)^{16} \left(x^3 - 3x + 1\right)^{16}}.$$

We use this map as an introductory example, because it represents a class of very extreme Belyi maps which provide some context for this paper, as discussed further in §1.3 and §11.1 below.

The degree of π is 64 and the 126 critical points are easily identified as follows. From the numerator $A(x)$, one has the critical points -2, 0, $\sqrt{2}$, $-\sqrt{-2}$, with total multiplicity $8 + 17 + 17 + 17 = 59$ and critical value 0. From the denominator $C(x)$, one has critical points -1, x_2, x_3, x_4 with total multiplicity $15 + 15 + 15 + 15 = 60$ and critical value ∞. Since both $A(x)$ and $C(x)$ are monic, one has $\pi(\infty) = 1$. The exact coefficients in (1.1) are chosen so that the degree of $A(x) - C(x)$ is only 56. This means that ∞ is a critical point of multiplicity $63 - 56 = 7$. As $59 + 60 + 7 = 126$, there can be no critical values outside $\{0, 1, \infty\}$ and so π is indeed a Belyi map.

In general, a degree m Belyi map π has a monodromy group $M_\pi \subseteq S_m$, a number field $F_\pi \subset \mathbb{C}$ of definition, and a finite set $\mathcal{P}_\pi$ of bad primes. We call π *full* if $M_\pi \in \{A_m, S_m\}$. Our example π is full because $M_\pi = S_{64}$. It is defined over $F_\pi = \mathbb{Q}$ because all the coefficients in (1.1) are in $\mathbb{Q}$. It has bad reduction set $\mathcal{P}_\pi = \{2, 3\}$ because numerator and denominator have a common factor in $\mathbb{F}_p[x]$ exactly for $p \in \{2, 3\}$. In the sequel, we almost always drop the subscript π, as it is clear from context.

To orient the reader, we remark that the great bulk of the explicit literature on Belyi maps concerns maps which are not full. Much of this literature, for example [13, Chapter II], focuses on Belyi maps with M a finite simple group different from A_m. On the other hand, seeking Belyi maps defined over $\mathbb{Q}$ is a common focus in the literature. Similarly, preferring maps with small bad reduction sets $\mathcal{P}$ is a common viewpoint.

1.3. **An inverse problem.** To provide a framework for our computations, we pose the following inverse problem: *given a finite set of primes $\mathcal{P}$ and a degree m, find all full degree m Belyi maps π defined over $\mathbb{Q}$ with bad reduction set within $\mathcal{P}$.* The finite set of full Belyi maps in a given degree m is parameterized in an elementary way by group-theoretic data. So, in principle at least, this problem is simply asking to extract those for which $F_\pi = \mathbb{Q}$ and $\mathcal{P}_\pi \subseteq \mathcal{P}$. Our inverse problem is in the spirit of the classical inverse Galois problem [13]; however it focuses on constrained ramification, rather than unusual Galois groups.

While the Belyi map (1.1) may look rather ordinary, it is already unusual for full Belyi maps to be defined over $\mathbb{Q}$. It seems to be extremely rare that their bad

reduction set is so small. In fact, we know of no full Belyi maps defined over $\mathbb{Q}$ with $m \geq 4$ and $|\mathcal{P}_\pi| \leq 1$. For $|\mathcal{P}_\pi| = 2$ we know of only a very sparse collection of such maps [14], [15], as discussed further in our last section here. The largest degree of these with both primes less than seventeen is $m = 64$, coming from (1.1).

1.4. **Hurwitz-Belyi maps.** Suppose now that $\mathcal{P}$ contains the set $\mathcal{P}_T$ of primes dividing the order of a finite nonabelian simple group T. The theoretical setting for this paper is a systematic method of constructing Belyi maps of arbitrarily large degree defined over $\mathbb{Q}$ and ramified within $\mathcal{P}$.

In brief, the method has two steps and goes as follows. First, from T one can build infinitely many natural covers from a Hurwitz variety to a configuration variety. In our notation, these covers are written $\pi_h : \mathsf{Hur}_h^* \to \mathsf{Conf}_\nu$, and the common dimension of both cover and base can be arbitrarily large. Second, there can be many non-trivial maps u from the thrice-punctured projective line $\mathsf{P}^1 - \{0, 1, \infty\}$ into Conf_ν. Let X^0 be the preimage of $u(\mathsf{P}^1 - \{0, 1, \infty\})$ in Hur_h, and let X be its smooth completion. Then the corresponding Hurwitz-Belyi map $\pi_{h,u}$ is the induced map from X to P^1. As we explain in our last section, we expect infinitely many of these $\pi_{h,u}$ to be full, and thus satisfy the remaining condition of our inverse problem.

1.5. **Contents of this paper.** Our viewpoint is that Hurwitz-Belyi maps form a remarkable class of mathematical objects, and are worth studying in all their aspects. This paper focuses on presenting explicit defining equations for systematic collections of Hurwitz-Belyi maps, and exhibits a number of theoretical structures in the process. The defining equations are obtained by two complementary methods. What we call the *standard method* centers on algebraic computations directly with the r-point Hurwitz source. The *braid-triple method* is an alternative method introduced in this paper. It uses the r-point Hurwitz source only to give necessary braid group information; its remaining computations are then the same ones used to compute general Belyi maps.

We focus primarily on the case $r = 4$ which is the easiest case for computations for a given T. This case was studied in some generality by Lando and Zvonkin in [8, §5.5] under the term *megamap*. In the last two sections, we shift the focus to $r \geq 5$, which is necessary to obtain the very large degrees m we are most interested in. The standard method is insensitive to genera of covering curves X, and so we could easily present examples of quite high genus. However, to give a uniform tidiness to our final equations, we present defining equations only in the case of genus zero. Thus the reader will find many explicit rational functions in $\mathbb{Q}(x)$ with properties similar to those of our initial example (1.1). All these rational functions and related information are available in the *Mathematica* file `HBM.m` on the author's homepage.

Section 2 reviews the theory of Belyi maps. Section 3 reviews the theory of Hurwitz maps and explains how carefully chosen one-dimensional slices are Hurwitz-Belyi maps. Of the many Belyi maps appearing in Section 2, two are unexpectedly defined over $\mathbb{Q}$. These maps each appear again in Section 4, with now their rationality obvious from the beginning via the Hurwitz theory.

Section 5 introduces the alternative braid-triple method for finding defining equations. We give general formulas for the preliminary braid computations in the setting $r = 4$. Passing from braid information to defining equations can then be much

more computationally demanding than in our initial examples, and we find equations mainly by p-adic techniques. Section 6 then presents three examples for which both methods work, with these examples having the added interest that lifting invariants force X to be disconnected. In each case, X in fact has two components, each of which is full over the base projective line.

Sections 7, 8, and 9 consider a systematic collection of Hurwitz-Belyi maps, with all final equations computed by the braid-triple method. They focus on the cases where $|\mathcal{P}_T| \leq 3$. By the classification of finite simple groups, the possible $\mathcal{P}_T$ have the form $\{2,3,p\}$ with $p \in \{5,7,13,17\}$. Section 7 sets up our framework and presents one example each for $p = 13$ and $p = 17$. Sections 8 and 9 then give many examples for $p = 5$ and $p = 7$ respectively.

Section 10 takes first computational steps into the setting $r \geq 5$. Working just with $T = A_5$ and $r = 5$, we summarize braid computations which easily prove the existence of full Hurwitz-Belyi maps with bad reduction set $\{2,3,5\}$ and degrees into the thousands. We use the standard method to find equations of two such covers related to $T = A_6$, one in degree 96 and the other in degree 192.

Section 11 concludes by tying the considerations of this paper very tightly to those of [22] and [21]. It conjectures a direct analog for Belyi maps of the main conjecture there for number fields. The Belyi map conjecture responds to the above inverse problem in the case that $\mathcal{P}$ contains the set of primes dividing the order of some finite nonabelian simple group. In particular, it says that there then should be full Belyi maps defined over $\mathbb{Q}$ and ramified within $\mathcal{P}$ of arbitrarily large degree.

1.6. **Notation.** Despite the arithmetic nature of our subject, we work almost exclusively over $\mathbb{C}$. Following [22] and [21], we use a sans serif font for complex spaces as in Y, P^1, Hur_h^*, Conf_ν, or X above.

The phenomenon that allows us to work mainly over $\mathbb{C}$ is that to a great extent geometry determines arithmetic. Thus an effort to find a function $\pi(x) \in \mathbb{C}(x)$ giving a full Belyi map $\pi : \mathsf{P}^1 \to \mathsf{P}^1$ involves choices of normalization. Typically, one can make these choices in a geometrically natural way, and then the coefficients of $\pi(x)$ automatically span the field of definition. When this field is $\mathbb{Q}$, and the normalization is sufficiently canonical, the primes of bad reduction can be similarly read off.

Often there will be several projective lines under consideration at once. When clarifying, we distinguish them by subscripting by the coordinate we are using. We commonly present a Belyi map $\pi : \mathsf{P}_x^1 \to \mathsf{P}_v^1$ not as a rational function $v = A(x)/C(x)$ but rather via the corresponding polynomial equation $A(x) - vC(x) = 0$. This trivial change in perspective has several advantages, one being that it lets one see the three-point property and the primes of bad reduction simultaneously via discriminants. For example, the discriminant of $A(x) - vC(x)$ in our first example (1.1) is $2^{256}3^{126}v^{59}(v-1)^7$.

2. Two Belyi maps unexpectedly defined over $\mathbb{Q}$

This section presents twenty-eight Belyi maps as explicit rational functions in $\mathbb{C}(y)$, two of which are unexpectedly in $\mathbb{Q}(y)$. Via these examples, it provides a quick summary, adapted to this paper's needs, of the general theory of Belyi maps. We will revisit the two rational maps from a different point of view in Section 4. Our three-point computations here are providing models for later r-point computations. Accordingly, we use the letter y as a primary variable.

2.1. **Partition triples.** Let n be a positive integer. Let $\Lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ be a triple of partitions of n, with the λ_τ having all together $n + 2 - 2g$ parts, with $g \in \mathbb{Z}_{\geq 0}$. The two examples pursued in this section are

$$(2.1) \qquad \Lambda' = (322, \, 421, \, 511), \qquad\qquad \Lambda'' = (642, \, 2222211, \, 5322).$$

So the degrees of the examples are $n = 7$ and $n = 12$, and both have $g = 0$.

Consider Belyi maps $F : \mathsf{Y} \to \mathsf{P}^1$ with ramification numbers of the points in $F^{-1}(\tau)$ forming the partition λ_τ, for each $\tau \in \{0, 1, \infty\}$. Up to isomorphism, there are only finitely many such maps. For some of these maps, Y may be disconnected, and we are not interested here in these degenerate cases. Accordingly, let X be the set of isomorphism classes of such Belyi maps with Y connected. One wants to explicitly identify X, and simultaneously get an algebraic expression for each corresponding Belyi map $F_x : \mathsf{Y}_x \to \mathsf{P}^1$. The Riemann-Hurwitz formula says that all these Y_x have genus g.

Note that in the previous paragraph we have finitely many Belyi maps $F_x : \mathsf{Y} \to \mathsf{P}^1$ indexed by the finite set X. In the bulk of this paper, we will have infinitely many maps $F_x : \mathsf{Y}_x \to \mathsf{P}^1$, which are now ramified above more than three points. These less extreme covers will be continuously indexed by the covering curve X in a Belyi map $\pi : \mathsf{X} \to \mathsf{P}^1$. Our notations are chosen so that the computations of this section are in the same notation as the computations of the later sections, even though the position of Belyi maps in these computations is different.

Computations in our current three-point setting can be put into a standard form when $g = 0$ and the partitions λ_0, λ_1, and λ_∞ have in total at least three singletons. Then one can pick an ordered triple of singletons and coordinatize Y by choosing y to take the values 0, 1, and ∞ in order at the three corresponding points. In our two examples, we do this via

$$(2.2) \qquad \Lambda'_* = (3_0 22, \, 4_1 2_x 1, \, 5_\infty 11), \qquad \Lambda''_* = (6_0 4_1 2_x, \, 2222211, \, 5_\infty 322).$$

Also we have chosen a fourth point in each case and subscripted it by x. This choice gives a canonical map from X into $\mathbb{C}$, as will be illustrated in our two examples. When the map corresponding to such a marked triple Λ_* is injective, as it almost always seems to be, we say that Λ_* is a *cleanly marked* genus zero triple.

When $g = 0$ and there is at least one singleton, computations can be done very similarly. All the explicit examples of this paper are in this setting. When $g = 0$ and there are no singletons, one often has to take extra steps, but the essence of the method remains very similar. When $g > 0$, computations are still possible, but they are very much more complicated.

2.2. **The triple Λ' and its associated $4 = 3 + 1$ splitting.** The subscripted triple Λ'_* in (2.2) requires us to consider rational functions

$$F(y) = \frac{1 + c + d}{(1 + a + b)^2} \cdot \frac{y^3(y^2 + ay + b)^2}{y^2 + cy + d}$$

and focus on the equation

(2.3) $5y^4 + 3(a + 2c)y^3 + (4ac + b + 7d)y^2 + (5ad + 2bc)y + 3bd = 5(y - 1)^3(y - x).$

The left side is a factor of the numerator of $F'(y)$ and thus its roots are critical points. The right side gives the required locations and multiplicities of these critical points.

Equating coefficients of y in (2.3) and using also $F(x) = 1$ gives five equations in five unknowns. There are four solutions, indexed by the roots of

(2.4) $$f_{\Lambda'_*}(x) = (x + 2)\left(16x^3 - 248x^2 - 77x - 6\right).$$

In general from a cleanly marked genus zero triple Λ_*, one gets a separable *moduli polynomial* $f_{\Lambda_*}(x)$. The *moduli algebra*

$$K_\Lambda = \mathbb{Q}[x]/f_{\Lambda_*}(x)$$

depends, as indicated by the notation, only on Λ and not on the marking. It is well-defined in the general case when the genus is arbitrary, even though we are not giving a procedure here to find a particular polynomial.

While the computation just presented is typical, the final result is not. We give three independent conceptual explanations for the factorization in (2.4), two in §4.1 and one at the end of §6.3. For context, the splitting of the moduli polynomial is one of just four unexplained splittings on the fourteen-page table of moduli algebras in [11]. While here the degree 7 partition triple yields a moduli algebra splitting as $3 + 1$, in the other examples the degrees are 8, 9, and 9, and the moduli algebras split as $7 + 1$, $8 + 1$, and $8 + 1$.

2.3. **Dessins.** A Belyi map $F : \mathsf{Y} \to \mathsf{P}^1_t$ can be visualized by its *dessin* as follows. Consider the interval $[0, 1]$ in P^1_t as the bipartite graph $\bullet\!\!-\!\!\circ$. Then $\mathsf{Y}_{[0,1]} := F^{-1}([0, 1])$ inherits the structure of a bipartite graph. This bipartite graph, considered always as inside the ambient real surface Y, is the dessin associated to F. A key property is that F is completely determined by the topology of the dessin.

Returning to the example of the previous subsection, the roots indexing the four solutions are

$$\begin{aligned}
x_1 &= -2, & x_2 &\approx 0.153 - 0.018i, \\
x_3 &\approx 15.86, & x_4 &\approx 0.153 + 0.018i.
\end{aligned}$$

The complete first solution is

(2.5) $$F_1(y) = -\frac{y^3 \left(y^2 + 2y - 5\right)^2}{4(2y - 1)(3y - 4)}.$$

The coefficients of the other F_i are cubic irrationalities. The four corresponding dessins in $\mathsf{Y}_i = \mathsf{P}^1_y$ are drawn in Figure 2.1. The scales of the four dessins in terms of the common y-coordinate are quite different. Always the black triple point is at 0 and the white quadruple point is at 1. The white double point is then at x_i.
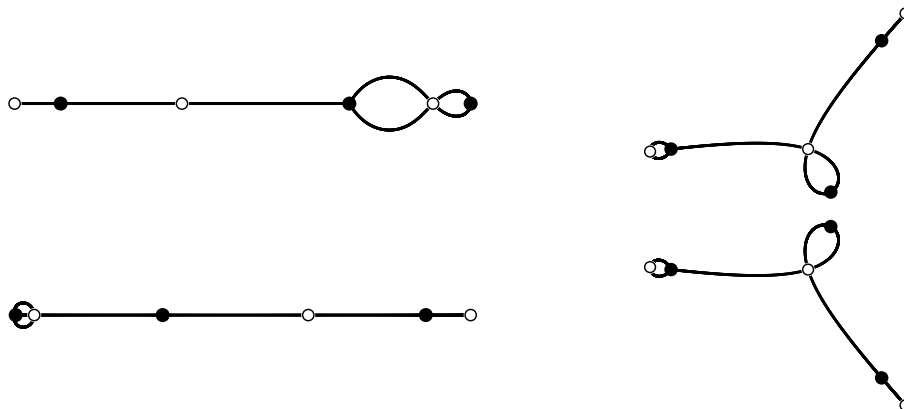
FIGURE 2.1. Dessins $\mathsf{Y}_{x_i,[0,1]} \subset \mathsf{P}^1_y$ corresponding to the points of $\mathsf{X}_{\Lambda'} = \{x_1, x_2, x_3, x_4\}$ with $\Lambda' = (322, 421, 511)$

2.4. **Monodromy.** The dessins visually capture the group theory which is central to the theory of Belyi maps but has not been mentioned so far. Given a degree n Belyi map $F : \mathsf{Y} \to \mathsf{P}^1$, consider the set $\mathsf{Y}_\star$ of the edges of the dessin. Let g_0 and g_1 be the operators on $\mathsf{Y}_\star$ given by rotating minimally counterclockwise about black and white vertices respectively.

The choice of $[0,1]$ as the base graph is asymmetric with respect to the three critical values 0, 1, and ∞. Orbits of g_0 and g_1 correspond to black vertices and white vertices respectively. In our first example, the original partitions $\lambda_0 = 322$ and $\lambda_1 = 421$ can be recovered from each of the four dessins from the valencies of these vertices. On the other hand, the orbits of $g_\infty = g_1^{-1} g_0^{-1}$ correspond to faces. The valence of a face is by definition half the number of edges encountered as one traverses its boundary. Thus $\lambda_\infty = 511$ is recovered from each of the four dessins in Figure 2.1, with the outer face always having valence five and the two bounded faces having valence one.

Let $\mathcal{Y}_*$ be the set of ordered triples (g_0, g_1, g_∞) in S_n such that

- g_0, g_1, and g_∞ respectively have cycle type λ_0, λ_1, and λ_∞,
- $g_0 g_1 g_\infty = 1$,
- $\langle g_0, g_1 \rangle$ is a transitive subgroup of S_n.

Then S_n acts on $\mathcal{Y}_*$ by simultaneous conjugation, and the quotient is canonically identified with $\mathsf{Y}_\star$.

For each of the thirty-one dessins of this section, the monodromy group $\langle g_0, g_1 \rangle$ is all of S_n. Indeed the only transitive subgroup of S_7 having the three cycle types of Λ' is S_7, and the only transitive subgroup of S_{12} having the three cycle types of Λ'' is S_{12}.

2.5. **Galois action.** Let $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be absolute Galois group of $\mathbb{Q}$. The "profound identity" mentioned in the introduction centers on the fact that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts naturally on the set X of Belyi maps belonging to any given Λ. In the favorable cleanly marked situation set up in §2.1, one has $X \subset \overline{\mathbb{Q}}$ and the action on X is the restriction of the standard action on $\overline{\mathbb{Q}}$.

A broad problem is to describe various ways in which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ may be forced to have more than one orbit. Suppose $x, x' \in \mathsf{X}$ respectively give rise to monodromy groups $\langle g_0, g_1 \rangle$ and $\langle g_0', g_1' \rangle$. If these monodromy groups are not conjugate in S_n then certainly x and x' are in different Galois orbits. Malle's paper [11] repeatedly illustrates the next most common source of decompositions, namely symmetries with respect to certain base-change operators $\mathsf{P}_t^1 \to \mathsf{P}_t^1$. The two splittings in this section do not come from either of these simple sources.

2.6. **The triple Λ'' and its associated $24 = 23+1$ splitting.** Here we summarize the situation for Λ''. Again the computation is completely typical, but the result is atypical. The clean marking on Λ'' identifies $\mathsf{X}_{\Lambda''}$ with the roots of

$(5x + 4)\cdot$

$(48828125x^{23} + 283203125x^{22} - 4345703125x^{21} - 21400390625x^{20} + 134842187500x^{19}$

$+ 461968375000x^{18} - 1670830050000x^{17} - 2095451850000x^{16} + 7249113240000x^{15}$

$+ 6576215456000x^{14} - 23053309281280x^{13} - 10284915779584x^{12} + 50191042453504x^{11}$

$+ 9449308979200x^{10} - 74715419574272x^9 + 5031544553472x^8 + 71884253429760x^7$

$- 35243151065088x^6 - 41613745192960x^5 + 29347637362688x^4 + 14541349978112x^3$
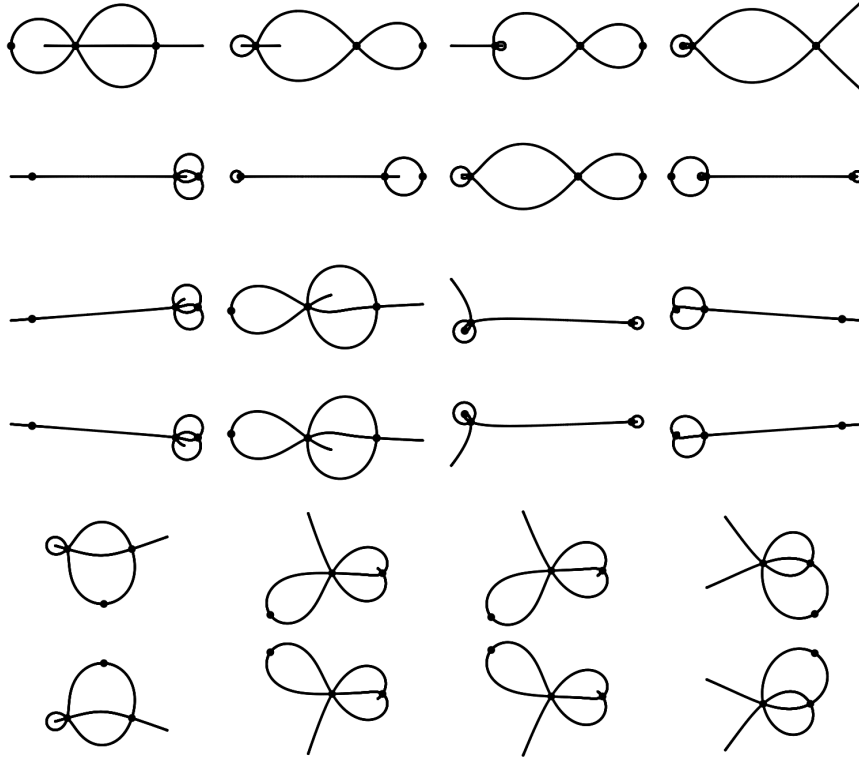
$+ 1765701320704x^2 + 100126425088x + 2684354560).$

FIGURE 2.2.   Dessins in $\mathsf{Y}_{x,[0,1]} \subset \mathsf{P}_y^1$ corresponding to the twenty-four points $x \in \mathsf{X}_{\Lambda''}$ with $\Lambda'' = (642, 2222211, 5322)$

The twenty-four associated dessins are drawn in Figure 2.2. The cover $F_{-4/5}$: $\mathsf{P}^1_y \to \mathsf{P}^1_t$ is given by

$$(2.6) \qquad t = \frac{5^5 y^6 (y-1)^4 (5y+4)^2}{2^4 3^3 (2y+1)^3 \left(5y^2 - 6y + 2\right)^2}.$$

This splitting of one cover away from the other twenty-three covers is explained in §4.3.

In choosing conventions for using dessins to represent covers, one often has to choose between competing virtues, such as symmetry versus simplicity. Figure 2.2 represents the standard choice when λ_1 has the form $2^a 1^b$: one draws the white vertices just as regular points, because they are not necessary for recovering the cover. With this convention there are just three highlighted points in each of the dessins in Figure 2.2: black dots of valence 6, 4, 2 at $y = 0$, 1, x. The rational cover, with $x = -4/5$, appears in the upper left.

2.7. **Bounds on bad reduction.** Let n be a positive integer and let $\Lambda = (\lambda_0, \lambda_1, \lambda_\infty)$ be a triple of partitions of n as above. Let $\mathcal{P}^{\mathrm{loc}}$ be the set of primes dividing a part of one of the λ_i. Let $\mathcal{P}^{\mathrm{glob}}$ be the set of primes less than or equal to n. In our two examples $\mathcal{P}^{\mathrm{loc}} = \{2, 3, 5\}$ and $\mathcal{P}^{\mathrm{glob}}$ is larger, by $\{7\}$ and $\{7, 11\}$ respectively.

Let K_Λ be the moduli algebra associated to Λ. Let D_Λ be its discriminant, i.e. the product of the discriminants of the factor fields. In our two examples, $D_{\Lambda'} = -2^3 \, 3 \, 5^3 \, 7$ and $D_{\Lambda''} = 2^{38} \, 3^{25} \, 5^{18} \, 7^6$. Let $\mathcal{P}_\Lambda$ be the set of primes dividing D_Λ. Then one always has $\mathcal{P}_\Lambda \subseteq \mathcal{P}^{\mathrm{glob}}$. Of course if $K_\Lambda = \mathbb{Q}$, then one has $\mathcal{P}_\Lambda = \emptyset$. Our experience is that once $[K_\Lambda : \mathbb{Q}]$ has moderately large degree, $\mathcal{P}_\Lambda$ is quite likely to be all or almost all of $\mathcal{P}^{\mathrm{glob}}$, as in the two examples.

Suppose now that $\pi : \mathsf{Y} \to \mathsf{P}^1$ is a Belyi map defined over $\mathbb{Q}$. Then its set $\mathcal{P}$ of bad primes satisfies

$$(2.7) \qquad \mathcal{P}^{\mathrm{loc}} \subseteq \mathcal{P} \subseteq \mathcal{P}^{\mathrm{glob}}.$$

For our two examples, $\mathcal{P}$ coincides with its lower bound $\{2, 3, 5\}$. The conceptual explanations of the splitting given in Section 3 also explain why the remaining one or two primes in $\mathcal{P}^{\mathrm{glob}}$ are primes of good reduction.

3. Hurwitz maps, Belyi pencils, and Hurwitz-Belyi maps

In §3.1 we very briefly review the formalism of dealing with moduli of maps $\mathsf{Y} \to \mathsf{P}^1_t$ with r critical values. A key role is played by Hurwitz covering maps $\pi_h : \mathsf{Hur}^*_h \to \mathsf{Conf}_\nu$. In §3.2 we introduce the concept of a Belyi pencil $u : \mathsf{P}^1 - \{0, 1, \infty\} \to \mathsf{Conf}_\nu$ and in §3.3 we give three important examples in $r = 4$. Finally §3.4 combines the notion of Hurwitz map and Belyi pencil in a straightforward way to obtain the general notion of a Hurwitz-Belyi map $\pi_{h,u}$.

3.1. **Hurwitz maps.** Consider a general degree n map $F : \mathsf{Y} \to \mathsf{P}^1_t$ as in §1.1. Three fundamental invariants are

- Its global monodromy group $G \subseteq S_n$.
- The list $C = (C_1, \ldots, C_k)$ of distinct conjugacy classes of G arising as non-identity local monodromy transformations.
- The corresponding list $(D_1, \ldots, D_k)$ of disjoint finite subsets $D_i \subset \mathsf{P}^1$ over which these classes arise.

To obtain a single discrete invariant, we write $\nu = (\nu_1, \ldots, \nu_k)$ with $\nu_i = |D_i|$. The triple $h = (G, C, \nu)$ is then a *Hurwitz parameter* in the sense of [22, §2] or [21, §3].

A Hurwitz parameter h determines a *Hurwitz moduli space* Hur_h whose points x index maps $F_x : \mathsf{Y}_x \to \mathsf{P}_t^1$ of type h. The Hurwitz space covers the *configuration space* Conf_ν of all possible divisor tuples $(D_1, \ldots, D_k)$ of type ν. The common dimension of Hur_h and Conf_ν is $r = \sum_i \nu_i$, the number of critical values of any F_x.

Sections 2-4 of [22] and Section 3 of [21] provide background on Hurwitz maps, some main points being as follows. There is a group-theoretic formula for a mass $\overline{m}$ which is an upper bound and often agrees with the degree m of $\pi_h : \mathsf{Hur}_h \to \mathsf{Conf}_\nu$. If all the C_i are rational classes, and under weaker hypotheses as well, then the covering of complex varieties descends canonically to a covering of varieties defined over the rationals, $\pi_h : \mathrm{HUR}_h \to \mathrm{CONF}_\nu$. The set $\mathcal{P}_h$ of primes at which this map has bad reduction is contained in the set $\mathcal{P}_G$ of primes dividing $|G|$.

On the computational side, [21] provides many examples of explicit computations of Hurwitz covers. Because the map π_h is equivariant with respect to PGL_2 actions, we normalize to take representatives of orbits and thereby replace $\mathsf{Hur}_h \to \mathsf{Conf}_\nu$ by a similar cover with three fewer dimensions. Our computations in the previous section for $h = (S_7, (322, 421, 511), (1, 1, 1))$ and $h = (S_{12}, (642, 2222211, 5322), (1, 1, 1))$ illustrate the case $r = 3$. Computations in the cases $r \geq 4$ proceed quite similarly. The next section gives some simple examples and a collection of more complicated examples is given in the companion paper [16].

Let $\mathrm{Out}(G, C)$ be the subgroup of $\mathrm{Out}(G)$ which fixes all classes C_i in C. Then $\mathrm{Out}(G, C)$ acts freely on Hur_h. For any subgroup $Q \subseteq \mathrm{Out}(G, C)$, we let Hur_h^Q be the quotient Hur_h/Q and let $\pi_h^Q : \mathsf{Hur}_h^Q \to \mathsf{Conf}_\nu$ be the corresponding covering map. One can expect that Q will usually play a very elementary role. For example, it can be somewhat subtle to get the exact degree m of a map π_h, but then the degree of π_h^Q is just $m/|Q|$.

We already have the simpler notation Hur_h for $\mathsf{Hur}_h^{\{e\}}$. Similarly, following [22], we use $*$ as a superscript to represent the entire group $\mathrm{Out}(G, C)$. In the literature, Hur_h is often called an inner Hurwitz space while Hur_h^* is an outer Hurwitz space. In the entire sequel of this paper, the only Q that we will consider are these two extreme cases. It is important for us to descend to the $*$-level to obtain fullness.

3.2. **Belyi pencils.** For any ν as above, the variety CONF_ν naturally comes from a scheme over $\mathbb{Z}$. Thus for any commutative ring R, we can consider the set $\mathrm{CONF}_\nu(R)$. Section 8 of [22] and then the sequel paper [21] considered $R = \mathbb{Q}$ and its subrings $\mathbb{Z}[1/\mathcal{P}] = \mathbb{Z}[\{1/p\}_{p \in \mathcal{P}}]$. From fibers $\mathsf{Hur}_{h,u} \in \mathrm{HUR}_h(\overline{\mathbb{Q}})$ above points in $\mathrm{CONF}_\nu(\mathbb{Z}[1/\mathcal{P}])$ one gets interesting number fields, the Hurwitz number fields of the title of [21].

Our focus here is similar, but more geometric. A Belyi pencil u is an algebraic map

$$(3.1) \qquad\qquad u : \mathsf{P}_v^1 - \{0, 1, \infty\} \to \mathsf{Conf}_\nu,$$

with image not contained in a single PGL_2 orbit. One can think of v as a time-like variable here. The Belyi pencil u then can be understood as giving r points in P_t^1, typically moving with v. There are ν_i points of color i; points are indistinguishable except for color, and they always stay distinct except for collisions at $v \in \{0, 1, \infty\}$.

To make the similarity clear, for R a ring let

$$R\langle v\rangle = R[v, \frac{1}{v(v-1)}].$$

Then a Belyi pencil can be understood as a point in $\mathrm{CONF}_\nu(\mathbb{C}\langle v\rangle)$. We say that the Belyi pencil u is *rational* if it is in $\mathrm{CONF}_\nu(\mathbb{Q}(\langle v\rangle))$. For rational pencils, one has a natural bad reduction set $\mathcal{P}_u$. It is the smallest set $\mathcal{P}$ with $u \in \mathrm{CONF}_\nu(\mathbb{Z}[1/\mathcal{P}]\langle v\rangle)$.

Remark. A complicated Belyi pencil. In this paper we will actually use only the three very simple Belyi pencils of the next subsection, and also the simple Belyi pencil (10.1). However general Belyi pencils can be much more complicated. For example, consider the eight-tuple

$$\begin{aligned}
&\left((t^6 - 8vt^3 + 9vt^2 - 2v^2\right), \left(t^6 - 3t^5 + 10vt^3 - 15vt^2 + 9vt - 2v^2\right),\\
(3.2) \quad &\left(t^6 - 6vt^5 + 15vt^4 - 20vt^3 + 6v^2t^2 + 9vt^2 - 6v^2t + v^2\right),\\
&\left(t^4 - 2t^3 + 2vt - v\right), \left(t^4 - 4vt + 3v\right), \left(2t^3 - 3t^2 + v\right), (t), \{\infty\}).
\end{aligned}$$

The product of the seven irreducible polynomials presented has leading coefficient 2 and discriminant $2^{161}3^{266}v^{125}(v-1)^{125}$. Thus $u : \mathsf{P}_v^1 - \{0, 1, \infty\} \to \mathsf{Conf}_{6,6,6,4,4,3,1,1}$ is a Belyi pencil with bad reduction set $\{2, 3\}$.

3.3. Belyi pencils for $r = 4$. Three Belyi pencils play a special role in the case $r = 4$, and we denote them by $u_{1,1,1,1}$, $u_{2,1,1}$, and $u_{3,1}$. For $u_{1,1,1,1}$, we keep our standard variable v. To make $u_{2,1,1}$ and $u_{3,1}$ stand out when they appear in the sequel, we switch the time-like variable v to respectively w and j for them. These special Belyi pencils are then given by

$$(3.3) \quad (\{v\}, \{0\}, \{1\}, \{\infty\}), \quad (D_w, \{0\}, \{\infty\}), \quad (D_j, \{\infty\}).$$

Here the divisors D_w and D_j are the root-sets of $t^2 + t + \frac{1}{4(1-w)}$ and $4(1-j)t^3 + 27jt + 27j$ respectively. So the three Belyi pencils are all rational, and their bad reduction sets are respectively $\{\}$, $\{2\}$, and $\{2, 3\}$.

The images of these Belyi pencils are curves

$$\mathsf{U}_{1,1,1,1} \subset \mathsf{Conf}_{1,1,1,1}, \quad \mathsf{U}_{2,1,1} \subset \mathsf{Conf}_{2,1,1}, \quad \mathsf{U}_{3,1} \subset \mathsf{Conf}_{3,1}.$$

The three curves are familiar as coarse moduli spaces of elliptic curves. Here $\mathsf{U}_{1,1,1,1} = \mathsf{Y}(2)$ parametrizes elliptic curves with a basis of 2-torsion, $\mathsf{U}_{2,1,1} = \mathsf{Y}_0(2)$ parametrizes elliptic curves with a 2-torsion point, and $\mathsf{U}_{3,1} = \mathsf{Y}(1)$ is the j-line parametrizing elliptic curves. The formulas

$$(3.4) \quad w = \frac{(2v-1)^2}{9}, \quad j = \frac{(3w+1)^3}{(9w-1)^2} = \frac{2^2(v^2-v+1)^3}{3^3v^2(v-1)^2}$$

give natural maps between these three bases: $\mathsf{P}_v^1 \to \mathsf{P}_w^1 \to \mathsf{P}_j^1$.

Remark. The two other 4-point cases. The cases $\nu = (2, 2)$ and $\nu = (4)$ are complicated by the presence of extra automorphisms. Any configuration $(D_1, D_2) \in \mathsf{Conf}_{2,2}$ is in the PGL_2 orbit of a configuration of the special form $(\{0, \infty\}, \{a, 1/a\})$. This latter configuration is stabilized by the automorphism $t \mapsto 1/t$. Similarly, a configuration $(D_1) \in \mathsf{Conf}_4$ has a Klein four-group of automorphisms. To treat these two cases, the best approach seems to be modify the last two pencils in (3.3) to $(D_w, \{0, \infty\})$ and $(D_j \cup \{\infty\})$. Outside of a quick example for $\nu = (4)$ in the remark containing (8.10), we do not pursue any explicit examples with such ν in this paper.

3.4. **Hurwitz-Belyi maps.** We can now define the objects in our title.

Definition 3.1. *Let $h = (G, C, \nu)$ be a Hurwitz parameter, let Q be a subgroup of $\mathrm{Out}(G, C)$, and let $\pi_h^Q : \mathsf{Hur}_h^Q \to \mathsf{Conf}_\nu$ be the corresponding Hurwitz map. Let $u : \mathsf{P}_v^1 - \{0, 1, \infty\} \to \mathsf{Conf}_\nu$ be a Belyi pencil. Let*

$$(3.5) \qquad\qquad \pi_{h,u}^Q : \mathsf{X}_{h,u}^Q \to \mathsf{P}_v^1$$

be the Belyi map obtained by pulling back the Hurwitz map via the Belyi pencil and canonically completing. A Belyi map obtainable by this construction is a Hurwitz-Belyi map.

Recall from the end of §3.1 that Q plays a passive role. We usually take Q to be all of $\mathrm{Out}(G, C)$, in which case we replace the superscript simply by $*$. In the common case when $\mathrm{Out}(G, C)$ is trivial, we can abbreviate further by omitting the superscript.

When $r = 4$ and u is one of the three maps (3.3), then we are essentially not specializing, as we are taking a set of representatives for the PGL_2 orbits on Conf_ν. We allow ourselves to drop the u in this situation, writing e.g. $\pi_h : \mathsf{X}_h \to \mathsf{P}_j^1$. On the other hand, when $r \geq 5$ we are truly specializing a cover of $(r - 3)$-dimensional varieties to a Belyi map.

Rationality and bad reduction are both essential to this paper. If h and u are both defined over $\mathbb{Q}$, then so is $\pi_{h,u}$. If h has bad reduction set $\mathcal{P}_h$ and u has bad reduction set $\mathcal{P}_u$ then the bad reduction set of $\pi_{h,u}$ is within $\mathcal{P}_h \cup \mathcal{P}_u$. All the examples we pursue in this paper satisfy $\mathcal{P}_u \subseteq \mathcal{P}_h$.

4. THE TWO RATIONAL BELYI MAPS AS HURWITZ-BELYI MAPS

This section presents some first examples in the setting $r = 4$, and in particular interprets the two rational Belyi maps of Section 2 as Hurwitz-Belyi maps.

4.1. **A degree 7 Hurwitz-Belyi map: computation and dessins.** To realize the Belyi map (2.5) as a Hurwitz-Belyi map, we start from the Hurwitz parameter

$$(4.1) \qquad h = (S_6, (2_x 1111, 3_0 3, 3_1 111, 4_\infty 11), (1, 1, 1, 1)).$$

Here and in the sequel we often present Hurwitz parameters with subscripts which indicate our normalization, without being as formal about markings as we were in Section 2. Thus the subscript 0 in $3_0 3$ causes us to write $y^2 - a$ in the next equation, rather than say $y^2 + zy - a$; this type of normalization on the second-highest order term does not introduce irrationalities.

The marked Hurwitz parameter (4.1) tells us to consider rational functions of the form

$$(4.2) \qquad\qquad F(y) = \frac{(y^2 - a)^3 (b + c + 1)}{(1 - a)^3 (y^2 + by + c)}$$

and the equation

$$(4.3) \qquad\qquad 4y^3 + 5by^2 + 2ay + ab = 4(y - 1)^2 (y - x).$$

The left side of (4.3) is a factor of the numerator of $F'(y)$ and thus its roots are critical points. The right side gives the required locations and multiplicities of these critical points.
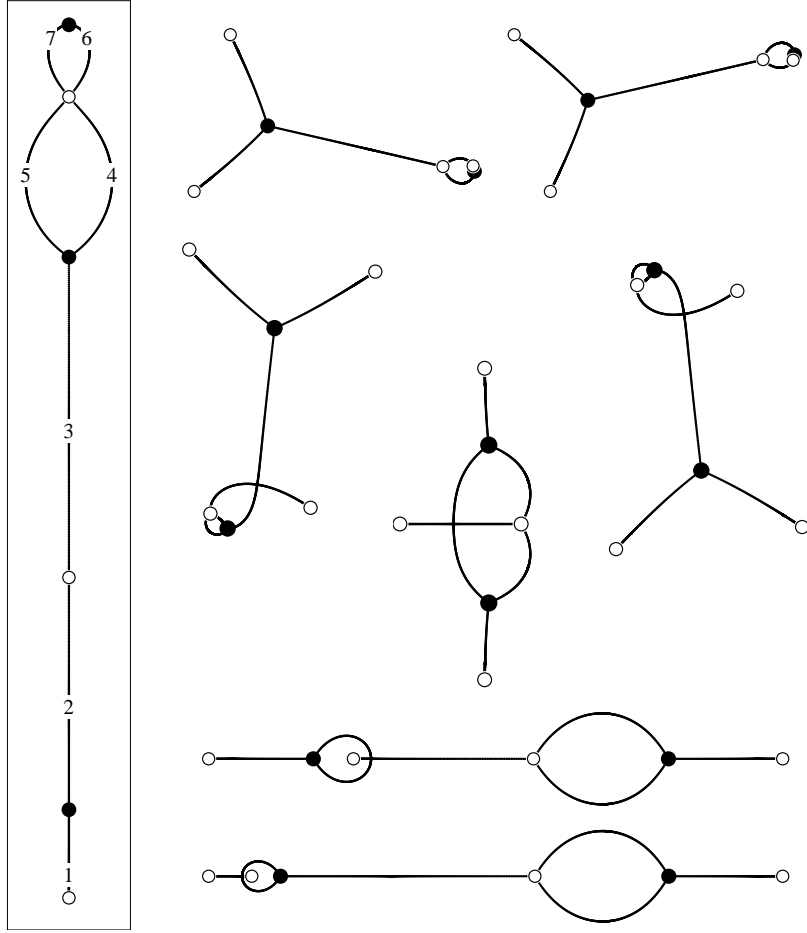
FIGURE 4.1. Left: The dessin $\mathsf{X}_{h,[0,1]} \subset \mathsf{P}^1_x$ belonging to $h = (S_6, (21111, 33, 3111, 411), (1,1,1,1))$, with real axis pointing up; the integers i are at preimages $x_i \in \mathsf{X}_{h,[0,1]}$ of $1/2 \in \mathsf{P}^1_v$. Right: the dessins $\mathsf{Y}_{h,x_i,[0,1]} \subset \mathsf{P}^1_y$ for the seven i.

Equating coefficients of powers of y in (4.3) and solving, we get

$$(4.4) \qquad a = \frac{5x}{x+2}, \qquad b = -\frac{4}{5}(x+2), \qquad c = \frac{4x^2 + 5x + 4}{3(x+2)}.$$

Summarizing, we have realized X_h as the complex projective x-line and identified each $\mathsf{Y}_{h,x}$ with the complex projective y-line P^1_y so that the covering maps $F_{h,x} : \mathsf{P}^1_y \to \mathsf{P}^1_t$ become

$$(4.5) \qquad F_{h,x}(y) = \frac{\left((x+2)y^2 - 5x\right)^3}{4(2x-1)\left(-15(x+2)y^2 + 12(x+2)^2 y - 5(4x^2 + 5x + 4)\right)}.$$

Since the fourth critical value of $F_{h,x}$ is just $F_{h,x}(x)$, we have also coordinatized the Hurwitz-Belyi map $\pi_h : \mathsf{X}_h \to \mathsf{P}^1_v$ to

$$(4.6) \qquad \pi_h(x) = -\frac{x^3 \left(x^2 + 2x - 5\right)^2}{4(2x-1)(3x-4)}.$$

Said more explicitly, to pass from the right side of (4.5) to the right side (4.6), one substitutes x for y and cancels $x^2 + 2x - 5$ from top and bottom.

The rational function (4.6) appeared already as (2.5), with its dessin printed in the upper left of Figure 2.1. This connection is our first explanation of why (2.4) splits. It also explains why the bad reduction set of the rational Belyi map is in $\{2, 3, 5\}$.

Figure 4.1 presents the current situation pictorially, with $v = 1/2$ chosen as a base point. The elements of $\mathsf{X}_{h,1/2} = \pi_h^{-1}(1/2)$ are labeled in the box at the left, where the real axis runs from bottom to top for a better overall picture. For each $x \in \mathsf{X}_{h,1/2}$, a corresponding dessin $\mathsf{Y}_{x,[0,1]}$ is drawn to its right. Like the standard dessins of §2.3, these dessins have black vertices, white vertices, and faces. However they each also have five vertices of a fourth type which we are not marking, corresponding to the five parts of the partition 21111. The valence of this type of vertex with ramification number e is $2e$, so only the extra vertex coming from the critical point with $e = 2$ is visible on Figure 4.1. The action of the braid group to be discussed in §5.1 can be calculated geometrically from these dessins.

4.2. **Cross-parameter agreement.** An interesting phenomenon that we will see repeatedly in later sections is *cross-parameter agreement*. By definition, cross-parameter agreement occurs when two different Hurwitz parameters give rise to isomorphic Hurwitz covers. At present, as mentioned in [21, §3.6], some of the these agreements are explained by the Katz middle convolution operator. However there are unexplained agreements as well that do not seem accidental. Already the phenomenon of cross-parameter agreement occurs for our septic Belyi map, which we realize in a second way as a Hurwitz-Belyi map as follows.

For the normalized Hurwitz parameter

$$\hat{h} = (S_5, (2_z 111, 221, 3_1 11, 3_\infty 2_0), (1, 1, 1, 1)),$$

the computation is easier than it was for the Hurwitz parameter h of (4.1). An initial form of $\hat{F}(y)$, analogous to (4.2), is

$$(4.7) \qquad \hat{F}(y) = \frac{(y - c) \left(y^2 + ay + b\right)^2}{(1-c)y^2(a+b+1)^2}.$$

Analogously to (4.5), the covering maps $\mathsf{P}^1_y \to \mathsf{P}^1_t$ are
$$(4.8)$$
$$\hat{F}_z(y) = \frac{(4yz + 2y - z)\left(-2y^2 z - y^2 + 6yz^2 + 14yz + 6y + 12z^2 + 12z + 3\right)^2}{4y^2(3z+2)^5}.$$

Analogously to (2.2) the Hurwitz-Belyi map $\mathsf{P}^1_z \to \mathsf{P}^1_v$ is

$$(4.9) \qquad \hat{\pi}(z) = \hat{F}_z(z) = \frac{\left(4z^2 + z\right)\left(4z^3 + 25z^2 + 18z + 3\right)^2}{4z^2(3z+2)^5}.$$

The map (4.9) agrees with the map (2.2) via the substitution $z = (1 - 2x)/(3x - 4)$. In terms of Figure 4.1, the dessin at the left remains exactly the same, up to

change of coordinates. In contrast, the seven sextic dessins at the right would be each replaced by a corresponding quintic dessin.

4.3. **Two degree** 12 **Hurwitz-Belyi maps.** In our labeling of conjugacy classes mentioned in §1.6, we are accounting for the fact that the five cycles in A_5 fall into two classes, with representatives $(1,2,3,4,5) \in 5a$ and $(1,2,3,4,5)^2 = (1,3,5,2,4) \in 5b$. Consider two Hurwitz parameters, as in the left column:

$$h_{aa} = (A_5, (5a, 311, 221), (2,1,1)), \qquad (\beta_0, \beta_1, \beta_\infty) = (5331, 222222, 5322),$$

$$h_{ab} = (A_5, (5a, 5b, 311, 221), (1,1,1,1)), \quad (\beta_0, \beta_1, \beta_\infty) = (642, 2222211, 5322).$$

Applying the outer involution of A_5 turns h_{aa} and h_{ab} respectively into similar Hurwitz parameters h_{bb} and h_{ba}, and so it would be redundant to explicitly consider these latter two.

It is hard to computationally distinguish $5a$ from $5b$. We will deal with this problem by treating h_{aa} and h_{bb} simultaneously. Thus we working formally with

$$h = (S_5, (5, 311, 221), (2,1,1)),$$

ignoring that the classes do not generate S_5. A second problem is that there are only eight parts altogether in the partitions 5, 5, 311, and 221, so the covering curves Y have genus two.

To circumvent the genus two problem, we use the braid-triple method, as described later in Section 5. The mass formula [21, §3.5] applies to h, with only the two abelian characters of S_5 contributing. It says that the corresponding cover $\mathsf{X}_h \to \mathsf{P}^1_w$ has degree

$$\frac{|C_5|^2|C_{311}||C_{221}|}{|A_5|^2} = \frac{24^2 \cdot 20 \cdot 15}{120 \cdot 60} = 24.$$

A braid group computation of the type described in §5.2 says X_h has two components, each of degree 12. The braid partition triples are given in the right column above. The β_τ then enter the formalism of Section 2 as the λ_τ there. Conveniently. each cover sought has genus zero, and so the covers are easily computed. The resulting polynomials are

$$f_{12aa}(w,x) = x^5 \left(9x^2 - 21x + 16\right)^3 (x+3) - 2^8 w(x-1)^3 \left(9x^2 - 12x + 8\right)^2,$$

$$f_{12ab}(w,x) = 5^5(x-1)^4 x^6 (5x+4)^2 - 2^4 3^3 w(2x+1)^3 \left(5x^2 - 6x + 2\right)^2.$$

Up to the simultaneous letter change $y \leftrightarrow x$, $t \leftrightarrow w$, the equation $f_{12ab}(w,x) = 0$ defines the exact same map as (2.6). The current context explains why this map is defined over $\mathbb{Q}$ and has bad reduction at exactly $\{2,3,5\}$.

Dessins corresponding to f_{12aa} and f_{12ab} are drawn in Figure 4.2. The two dessins present an interesting contrast: the dessin on the left of Figure 4.2 is the unique dessin with its partition triple, while the dessin on the right is one of the 24 locally equivalent dessins drawn in Figure 2.2.

Remark. An M_{12} specialization. Specializing Hurwitz-Belyi maps yields interesting number fields. Except for this remark, we are saying nothing about this application, because we discuss specialization of Hurwitz covers quite thoroughly in [19] and [21]. However the polynomial $f_{12ab}(1/4, x)$, or equivalently

$$x^{12} - 24x^{10} + 180x^8 - 60x^6 - 2520x^5 + 4320x^4 - 2520x^3 + 864x^2 - 216,$$
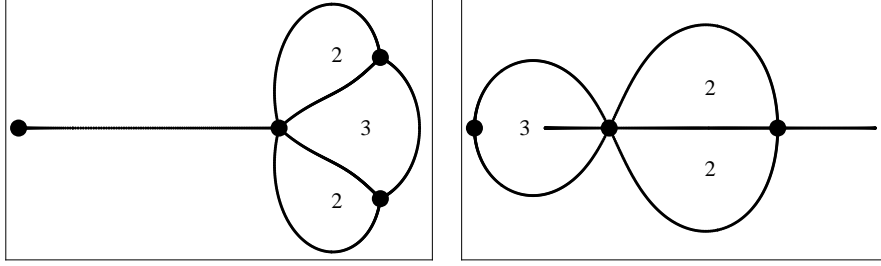
FIGURE 4.2. Rational degree 12 dessins coming from Hurwitz parameters involving irrational classes. h_{12aa} yielding partition triple $(5331, 222222, 5322)$ on the left and h_{12ab} yielding $(642, 2222211, 5322)$ on the right

unexpectedly has the Mathieu group M_{12} as its Galois group. The splitting field has root discriminant $2^{3/2}3^{3/2}5^{23/20} \approx 93.55$. Comparing with the discussion in [20, §6.2], one sees that this field is currently the second least ramified of known M_{12} Galois fields, being just slightly greater than the current minimum $2^{25/12}3^{10/11}5^{13/10} \approx 93.23$.

5. THE BRAID-TRIPLE METHOD

Our first Hurwitz-Belyi map (4.6) was computed with the standard method. After §5.1 gives background on braids, §5.2 and §5.3 describe the alternative braid-triple method, already used twice in §4.3. The two methods are complementary, as we explain in the short §5.4.

A key step in the braid-triple method is to pass from a Belyi pencil u to a corresponding *braid triple* $B = (B_0, B_1, B_\infty)$. In this paper, we use the braid-triple method only for four u, and the corresponding B are given by the simple formulas (5.3), (5.4), (5.5), and (10.2). We do not pursue the general case here; our policy in this paper is to be very brief with respect to braid groups, saying just enough to allow the reader to replicate our computations of individual covers.

5.1. **Algebraic background on braid groups.** The *Artin braid group* on r strands is the most widely known of all braid groups, and our summary here follows [22, §3]. The group is defined via $r - 1$ generators and $\binom{r-1}{2}$ relations:

$$(5.1) \qquad \mathrm{Br}_r = \left\langle \sigma_1, \ldots, \sigma_{r-1} : \begin{array}{ll} \sigma_i\sigma_j = \sigma_j\sigma_i, & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, & \text{if } |i-j| = 1 \end{array} \right\rangle.$$

The assignment $\sigma_i \mapsto (i, i+1)$ extends to a surjection $\mathrm{Br}_r \twoheadrightarrow S_r$. For every subgroup of S_r one gets a subgroup of Br_r by pullback. Thus, in particular, one has surjections $\mathrm{Br}_\nu \twoheadrightarrow S_\nu$ for $S_\nu = S_{\nu_1} \times \cdots \times S_{\nu_r}$.

Given a finite group G, let $\mathcal{G}_r \subset G^r$ be the set of tuples $(g_1, \ldots, g_r)$ with the g_i generating G and satisfying $g_1 \cdots g_r = 1$. The braid group Br_r acts on the right of $\mathcal{G}_r$ by the braiding rule

$$(5.2) \qquad (\ldots, g_{i-1}, g_i, g_{i+1}, g_{i+2}, \ldots)^{\sigma_i} = (\ldots, g_{i-1}, g_{i+1}, g_i^{g_{i+1}}, g_{i+2}, \ldots).$$

The group G acts diagonally on $\mathcal{G}_r$ by simultaneous conjugation. The actions of Br_ν and G commute with one another.

Let $h = (G, C, \nu)$ be an r-point Hurwitz parameter, with $C = (C_1, \ldots, C_k)$, $\nu = (\nu_1, \ldots, \nu_k)$, and $\sum_{i=1}^{k} \nu_i = r$, all as usual. Consider the subset $\mathcal{G}_h$ of $\mathcal{G}_r$, consisting of tuples with $g_j \in C_i$ for $\sum_{a=1}^{i-1} \nu_a < j \leq \sum_{a=1}^{i} \nu_a$. This subset is stable under the action of $\mathrm{Br}_\nu \times G$. The group Br_ν therefore acts on the right of the quotient set $\mathcal{F}_h = \mathcal{G}_h/G$.

The actions of Br_ν on $\mathcal{F}_h$ all factor through a certain quotient HBr_ν. Terminology is not important for us here, but for comparison with the literature we remark that HBr_ν is the quotient of the standard Hurwitz braid group by its two-element center. Choosing a base point $\star$ and certain identifications appropriately, the group HBr_ν is identified with the fundamental group $\pi_1(\mathsf{Conf}_\nu, \star)$, in a way which makes the action of HBr_ν on $\mathcal{F}_h$ agree with the action of $\pi_1(\mathsf{Conf}_\nu, \star)$ on the base fiber $\pi_h^{-1}(\star) \subset \mathsf{Hur}_h$.

Let Z be the center of G, this center being trivial for most of our examples. Let $\mathrm{Aut}(G, C)$ be the subgroup of $\mathrm{Aut}(G)$ which stabilizes each conjugacy class C_i in C. Then not only does G/Z act diagonally on $\mathcal{G}_h$, but so does the entire overgroup $\mathrm{Aut}(G, C)$. The group $\mathrm{Out}(G, C)$ introduced in §3.1 is the quotient of $\mathrm{Aut}(G, C)$ by G/Z. Quotienting by the natural action of a subgroup $Q \subseteq \mathrm{Out}(G, C)$, gives the base fiber $\mathcal{F}_h^Q$ corresponding to the cover $\mathsf{Hur}_h^Q \to \mathsf{Conf}_\nu$.

5.2. Step one: computation of braid triples.

A Belyi pencil $u : \mathsf{P}_v^1 \to \mathsf{Conf}_\nu$ determines, up to conjugacy depending on choices of base points and a path between them, an *abstract braid triple* (B_0, B_1, B_∞) of elements of HBr_ν. These elements have the property that in any Hurwitz-Belyi map $\pi_{h,u} : \mathsf{X}_{h,u} \to \mathsf{P}^1$, the images of the B_τ in their action on $\mathcal{F}_h$ give the global monodromy of the cover. When $\mathcal{F}_h$ is identified with $\{1, \ldots, m\}$, we denote the image of B_τ by $b_\tau \in S_m$ and its cycle partition by β_τ. We call (b_0, b_1, b_∞) a *braid permutation triple*. As we have already done before, we call $(\beta_0, \beta_1, \beta_\infty)$ a *braid partition triple*.

For the three 4-point Belyi pencils introduced in (3.3), the abstract braid triples are

(5.3) $u_{1,1,1,1}$: $(B_0, B_1, B_\infty) = (\sigma_1^2, \sigma_2^2, \sigma_2^{-2}\sigma_1^{-2})$,

(5.4) $u_{2,1,1}$: $(B_0, B_1, B_\infty) = (\sigma_1, \sigma_1^{-1}\sigma_2^{-2}, \sigma_2^2)$,

(5.5) $u_{3,1}$: $(B_0, B_1, B_\infty) = (\sigma_1\sigma_2, \sigma_2^{-1}\sigma_1^{-2}, \sigma_1)$.

The triple for $u_{1,1,1,1}$ is given in [8, §5.5.2]. The other two can be deduced by quadratic and then cubic base change, using (3.4). An important point is that, in the quotient group HBr_ν, the B_1 for $u_{2,1,1}$ and $u_{3,1}$ have order 2 and the B_0 for $u_{3,1}$ has order 3.

It is worth emphasizing the conceptual simplicity of our braid computations. They repeatedly use the generators σ_i of (5.1) and their actions on $\mathcal{F}_h$ from (5.2). However they do not explicitly use the relations in (5.1). Likewise they do not explicitly use the extra relations involved in passing from Br_ν to HBr_ν. Our actual computations are at the level of the permutations $b_\tau \in S_m$ rather than the level of the braid words B_τ. At the permutation level, all these relations automatically hold.

Computationally, we realize $\mathcal{F}_h$ via a set of representatives in $\mathcal{G}_h$ for the conjugation action. A difficulty is that the set $\mathcal{G}_h$ in which computations take place is large. Relatively naive use of (5.1) and (5.2) suffices for the braid computations

presented in the next five sections. To work as easily with larger groups and/or larger degrees, a more sophisticated implementation as in [9] would be essential.

5.3. **Step 2: passing from a braid triple to an equation.** Having computed a braid partition triple $(\beta_0, \beta_1, \beta_\infty)$ belonging to $\pi_{h,u}$, one can then try to pass from the triple to an equation for $\pi_{h,u}$ by algebraic methods. We did this in §4.3 for two partition triples with degree $m = 12$. For one, as described in §2.6, the desired $\pi_{h,u}$ is just one of $\mu = 24$ locally equivalent covers, the one defined over $\mathbb{Q}$.

The braid partition triples $(\beta_0, \beta_1, \beta_\infty)$ arising in the next five sections have degrees m into the low thousands. The number μ of Belyi maps with a given such braid partition triple is likely to be more than 10^{100} in some cases. The numbers 12 and 24 are therefore being replaced by very much larger m and μ respectively. It is completely impractical to follow the purely algebraic approach of getting all the maps belonging to the given $(\beta_0, \beta_1, \beta_\infty)$ and extracting the desired rational one. Instead, there are three more feasible techniques for computing only the desired cover.

For almost all the covers in this paper, Step 2 was carried out by a p-adic technique for finding covers defined over $\mathbb{Q}$ explained in detail in [10]. Here one picks a good prime p for the cover sought, and first searches for a tame cover with the correct $(\beta_0, \beta_1, \beta_\infty)$ defined over $\mathbb{F}_p$. Commonly, one finds several covers, and one cannot yet tell which is the reduction of the cover sought. One then uniquely lifts all these candidates iteratively to $\mathbb{Z}/p^c$ for some large c. This step requires solving linear equations and is easy. We commonly took $c = 50$. Then one recognizes the coefficients of the lifted covers as p-adically near rational numbers. In practice this is easy too, and only one of the initial solutions over $\mathbb{F}_p$ gives small height rational numbers. One concludes by checking that the monodromy of the cover constructed really does agree with the braid permutation triple (b_0, b_1, b_∞). The efficiency of this p-adic technique decreases rapidly with p. Since the covers we pursue all have a small prime of good reduction, typically 5 or 7, the technique is well adapted to our situation.

The second and third technique have been recently introduced, and both are undergoing further development. They take the permutation triple (b_0, b_1, b_∞) rather than the partition triple $(\beta_0, \beta_1, \beta_\infty)$ as a starting point. Thus they isolate the cover sought immediately, and there is no issue of a large local equivalence class. The technique of [6] centers on power series while the technique of [7] centers on numerically solving partial differential equations. Schiavone used the programs described in [6] to compute (7.5) here. Our tables in Sections 8-10 present braid information going well beyond where one can currently compute equations, in part to provide targets for these developing computational methods.

5.4. **Comparison of the two methods.** We used the standard method many times in [21] in the context of constructing covers of surfaces. In the present context of curves, the braid-triple method complements the standard method as follows. In the standard method, one can expect the difficulty of the computation to increase rapidly with the genus g_Y of Y_x and the degree n of the cover $\mathsf{Y}_x \to \mathsf{P}^1_t$. In the braid-triple method, these measures of difficulty are replaced by the genus g_X of the curve $\mathsf{X}_{h,u}$ and the degree m of the cover $\mathsf{X}_{h,u} \to \mathsf{P}^1_v$. The quantities (g_Y, n) and (g_X, m) are not tightly correlated with each other, and in practice each method

has a large range of parameters for which it works well while the other method does
not.

6. HURWITZ-BELYI MAPS EXHIBITING SPIN SEPARATION

This section presents three Hurwitz-Belyi maps for which we were able to find
a defining equation by both the standard and the braid-triple method. Each map
has the added interesting feature that the covering curve X_h has two components.
We explain this splitting by means of lifting invariants. Many of the covers in the
next four sections are similarly forced to split via lifting invariants.

6.1. **Lifting in general.** Decomposition of Hurwitz varieties was studied by Fried
and Serre. Here we give a very brief summary of the longer summary given in
[21, §4]. The decompositions come from central extensions $\tilde{G}$ of the given group
G. The term *spin separation* is used because many double covers are induced from
the double cover Spin_n of the special orthogonal group SO_n via an orthogonal
representation.

Let $h = (G, C, \nu)$ be a Hurwitz parameter. First, one has the Schur multiplier
$H_2(G, \mathbb{Z})$, always abbreviated in this paper as $H_2(G)$. Any universal central ex-
tension $\tilde{G}$ of G has the form $H_2(G).G$. Second, one has a quotient $H_2(G, C)$ of
the Schur multiplier, with $H_2(G, C).G$ being the largest quotient in which each
C_i splits completely into $|H_2(G, C)|$ different conjugacy classes. Third, one has a
torsor $H_h = H_2(G, C, \nu)$ over $H_2(G, C)$. So $|H_h| = |H_2(G, C)|$, but the set H_h does
not necessarily have a distinguished point like the group $H_2(G, C)$ does.

The group $\text{Out}(G, C)$ defined in §3.1 acts on the set H_h. For any subgroup
$Q \subseteq \text{Out}(G, C)$, one has a natural map from the component set $\pi_0(\text{Hur}_h^Q)$ of Hur_h^Q
to H_h^Q. The most common behavior is that these maps $\pi_0(\text{Hur}_h^Q) \to H_h^Q$ are
bijective. As said already in §3.1, our main interest is in $Q = \text{Out}(G, C)$, in which
case we replace $\text{Out}(G, C)$ by $*$ as a superscript.

In practice, the key groups $H_2(G, C)$ and $\text{Out}(G, C)$ are extremely small. In the
next three subsections $H_2(G, C)$ has order 2, 3, and 3 respectively, while $\text{Out}(G, C)$
has order 1, 2, and 1. We explain lifting in some detail in these subsections and
also in §8.2 where $H_2(G, C)$ and $\text{Out}(G, C)$ can be slightly larger.

6.2. **A degree** $25 = 15 + 10$ **family.** Applying the mass formula [21, (3.6)] to
a Hurwitz parameter $h = (G, C, \nu)$ requires the use of the character table of G.
Common choices for G in this paper are A_5 and S_5. Table 6.1 gives the character
table for these two groups, as well as their Schur double covers $\tilde{A}_5$, and $\tilde{S}_5$. In
this subsection, we use this table to illustrate how mass formula computations for
a given Hurwitz parameter h appear in practice, including refinements involving
covering groups $\tilde{G}$.

The character table for A_5 is given simply by the upper left 5-by-5 block. The
remaining character tables require the use of Atlas conventions. The double cover
$\tilde{A}_5$ has the listed nine characters. The nine conjugacy classes arise because all but
the class 221 splits into two classes. We label these classes of $\tilde{A}_5$ according to
whether the order of a representing element is even $(+)$ or odd $(-)$. Thus $5a$ splits
into $5a+$ and $5a-$. The printed character values refer to the class with odd order
elements. Thus, e.g., $\chi_8(311+) = 1$ but $\chi_8(311-) = -1$.

Only the groups A_5 and $\tilde{A}_5$ are used in the example of this subsection, but S_5
and $\tilde{S}_5$ are equally common in the sequel and we explain them here. The group S_5

	1^5	221	311	$5a$	$5b$	2111	41	32
$|C_i|$	1	15	20	12	12	10	30	20
C_i	1^5	221	311	$5a$	$5b$	2111	41	32
χ_1	1	1	1	1	1	1	1	1
χ_2	3	-1	0	$-b$	$-b'$	0	0	0
χ_3	3	-1	0	$-b'$	$-b'$			
χ_4	4	0	1	-1	-1	2	0	-1
χ_5	5	1	-1	0	0	1	-1	1
Orders of	1	4	3	5	5	2	8	6
classes	2		6	10	10		8	6
χ_6	2	0	-1	b	b'	0	0	0
χ_7	2	0	-1	b'	b			
χ_8	4	0	1	-1	-1	0	0	$\sqrt{-3}$
χ_9	6	0	0	1	1	0	$\sqrt{-2}$	0

TABLE 6.1. Character tables for A_5, S_5, $\tilde{A}_5$, and $\tilde{S}_5$, with the abbreviations $b = (-1 + \sqrt{5})/2$ and $b' = (-1 - \sqrt{5})/2$.

has seven conjugacy classes, the classes $5a$ and $5b$ having merged to a single class 5. The corresponding seven characters are the printed χ_1, χ_4, χ_5, the sum $\chi_2 + \chi_3$ extended by zero, and the twists $\chi_1\epsilon$, $\chi_4\epsilon$, and $\chi_5\epsilon$. Here ϵ is the sign character, taking value 1 on A_5 and -1 on $S_5 - A_5$. The cover $\tilde{S}_5$ has twelve characters, the seven from before and the five new ones $\chi_6 + \chi_7$, χ_8, $\chi_8\epsilon$, χ_9, and $\chi_9\epsilon$.

For the example of this subsection, let $h = (A_5, (311, 5a), (3, 1))$. Because of 0's appearing as character values, only the characters χ_1 and χ_4 appear when evaluating the mass formula:

$$\overline{m}_h = \frac{|C_1|^3 |C_2|}{|G|^2} \sum_{i=1}^{5} \frac{\chi_i(C_1)^3 \chi_i(C_2)}{\chi_i(1)^2} = \frac{20^3 12}{60^2} \left(\frac{1^3(1)}{1^2} + \frac{1^3(-1)}{4^2} \right) = \frac{20}{3} \frac{15}{16} = 25.$$

Because A_5 does not have a proper subgroup meeting the both the classes 311 and $5a$, the desired degree is just the mass, $m_h = \overline{m}_h = 25$.

The joint paper [22] was originally planned to include this h as an example. The curves Y_x parameterized have genus one but Venkatesh nonetheless computed the Belyi map $\pi_h : X \to P_j^1$ by the standard method, seeing directly that X breaks into two components, each of genus zero, of degree 15 and 10 over P_j^1. The present author simultaneously used the braid-triple method, using (5.5) to get the braid partition triples $(3331, 22222, 541)$ and $(33333, 22222221, 5433)$. Both methods end at the explicit equations (8.4) and (8.5).

To explain the splitting, consider the Hurwitz parameters

$$(6.1) \qquad h^+ = (\tilde{A}_5, (311+, 5a+), (3, 1)), \qquad h^- = (\tilde{A}_5, (311+, 5a-), (3, 1)).$$

Let $(g_1, g_2, g_3, g_4) \in \mathcal{G}_h$. For $i = 1, 2, 3$, let $\tilde{g}_i$ be the unique preimage of g_i in $311+$. Then there is a unique lift $\tilde{g}_4$ of g_4 which satisfies $\tilde{g}_1 \tilde{g}_2 \tilde{g}_3 \tilde{g}_4 = 1$. This lift can be in either $5a+$ or $5a-$. In this way one gets a map from $\mathcal{G}_h$ to $H_h = \mathbb{Z}/2$. This invariant does not change under either the braid or conjugation action.

The mass formula [21, (3.6)], applied to $\tilde{G}$ now, lets one find the degrees of the factors. In this simple case, where proper subgroups of $\tilde{G}$ are not involved, one has

$$
\begin{aligned}
m_h^{\pm} &= \frac{1}{2} \frac{|C_1|^3 |C_2|}{|G|^2} \sum_{i=1}^{9} \frac{\chi_i(C_1)^3 \chi_i(C_2)}{\chi_i(1)^2} \\
&= \frac{1}{2} \frac{20^3 12}{60^2} \left(\frac{1^3(1)}{1^2} + \frac{1^3(-1)}{4^2} \pm \frac{(-1)^3(b+b')}{2^2} \pm \frac{1^3(-1)}{4^2} \right) \\
&= \frac{40}{3} \left(1 - \frac{1}{16} \pm \frac{1}{4} \pm \frac{-1}{16} \right) = \frac{40}{3} \left(\frac{15}{16} \pm \frac{3}{16} \right) = \frac{5}{2}(5 \pm 1) = 15, 10.
\end{aligned}
$$

Similar mass computations let one properly identify components with lifting invariants in general. We make these identifications, typically with no further comments, in the next two subsections and many times in §8-10.

6.3. A degree $70 = 30 + 40$ family: rational cubic splitting. Let

$$
h = (A_7, \ (22111, \ 511, \ 322), \ (2, 1, 1)).
$$

The large singletons 5 and 3 help keep the standard method within computational feasibility. By a direct application of this method, one sees at the end that the degree m is 70 and there is a splitting into two components of degrees 30 and 40.

In the braid-triple method the order of events is reversed. Mass formula computations says that the desired $\mathsf{X}_h^* \to \mathsf{P}_w^1$ has degree 70. A braid group computation using (5.4) says that X_h^* has two components of degrees 30 and 40. The monodromy groups are A_{30} and S_{40} respectively, with braid partition triples

$$
\begin{aligned}
(\beta_0, \beta_1, \beta_\infty) &= (7^2\, 5\, 3\, 2^4, \ 2^{14}\, 1^2, \ 6\, 5^2\, 4\, 3^3\, 1), \\
(\beta_0, \beta_1, \beta_\infty) &= (7^2\, 5^2\, 4^2\, 2^3\, 1^2, \ 2^{20}, \ 5^2\, 4^3\, 3^6).
\end{aligned}
$$

As the total number of parts is 32 and 42 respectively, the genus is zero in each case.

The second step in the braid-triple method is challenging, since the smallest prime not in $\mathcal{P}_{A_7}$ is 11. This step is only within feasibility because of the splitting $70 = 30 + 40$, and the fact that one can compute the two components independently. Explicit equations are

$$
\begin{aligned}
f_{30}(w, x) &= 2^2 3^3 \left(7x^2 + 14x + 4\right)^7 x^5 (2x+1)^3 \left(x^2 + 3x + 1\right)^2 \left(2x^2 + x + 2\right)^2 \\
&\quad + w \left(7x^2 + 6x + 2\right)^5 (5x+2)^4 \left(14x^3 + 39x^2 + 18x + 2\right)^3 (x+2),
\end{aligned}
$$

$$
\begin{aligned}
f_{40}(w, x) &= 2^2 3^4 \left(5x^2 - 12x + 3\right)^7 \left(5x^2 - 15x + 12\right)^5 \left(x^2 - 3x + 6\right)^4 \\
&\quad \left(4x^2 - 15x + 15\right)^2 x(5x - 9) \\
&\quad + w \left(x^2 - 3\right)^5 \left(5x^3 - 45x^2 + 120x - 108\right)^4 \\
&\quad \left(400x^6 - 2700x^5 + 7425x^4 - 10530x^3 + 7830x^2 - 2430x - 27\right)^3.
\end{aligned}
$$

The polynomial discriminants are

$$
\begin{aligned}
D_{30}(w) &= -2^{450} 3^{285} 5^{95} 7^{105} w^{22} (w-1)^{14}, \\
D_{40}(w) &= 2^{930} 3^{1254} 5^{230} 7^{105} w^{29} (w-1)^{20}.
\end{aligned}
$$

Modulo squares these quantities are -105 and $7w$, reflecting the fact that monodromy groups and generic Galois groups are (A_{30}, S_{30}) in the first case and (S_{40}, S_{40}) in the second.

The group A_7 has the unusually large maximal non-split central extension $6.A_7$. For both this subsection and the next, only the subextension $3.A_7$ is relevant because all the classes C_i split in it, while the class 22111 is inert in $2.A_7$. In the notation reviewed in §6.1, this reduction is expressed by an identification $H_2(A_7, C) = \mathbb{Z}/3$. The group $\mathrm{Out}(A_7, C)$ is all of $\mathrm{Out}(A_7) = \{\pm 1\}$ because the classes 22111, 511, and 322 are all stabilized by the outer involution. The action of $\{\pm 1\}$ on $\mathbb{Z}/3$ is the nontrivial one where -1 acts by negation. The degree 30 component corresponds to the identity class $0 \in \mathbb{Z}/3$ while the degree 40 component corresponds to the orbit of the two nonidentity classes in $\mathbb{Z}/3$.

The promised third conceptual explanation of the degree splitting $4 = 3 + 1$ for $h = (A_7, (322, 421, 511), (1, 1, 1))$ from (2.4) is in our present context. All three classes split in $3.A_7$ while only the last two split in $2.A_7$. All three classes are stable under outer involution. So here again $\mathrm{Out}(A_7, C) = \{\pm 1\}$ acts nontrivially on $H_2(A_7, \mathbb{Z}) = \mathbb{Z}/3$. In this case the degree one factor corresponds to $0 \in \mathbb{Z}/3$ while the degree three factor corresponds to $\{-1, 1\} \subset \mathbb{Z}/3$.

6.4. A degree $42 = 21 + 21$ family: irrational cubic splitting.

Lando and Zvorkin [8, §5.4] investigated splitting of Hurwitz covers in some generality. The unique splitting in their context that they could not conceptually explain comes from the Hurwitz parameter

$$h = (A_7, \ (22111, 7a), \ (3, 1)).$$

Here one has splitting of the form $42 = 21 + 21$. In this subsection we complement their study of this example by both giving an equation and explaining the splitting.

Computing using (5.5), one gets that the two components have the same braid partition triple, namely

$$(6.2) \qquad\qquad (\beta_0, \ \beta_1, \ \beta_\infty) = (3^7, \ 2^{10} \, 1, \ 6 \, 5 \, 4 \, 3^2).$$

This agreement is in contrast to the previous subsection, where the two components even had different degrees. Lando and Zvonkin speculated (p. 333) that the agreement is explained by the two components being Galois conjugate.

Indexing the two maps arbitrarily by $\epsilon \in \{+, -\}$, each map we seek fits as the right vertical map in a Cartesian square:

$$(6.3) \qquad\qquad
\begin{array}{ccc}
\tilde{\mathsf{X}}^\epsilon & \to & \mathsf{X}^\epsilon \\
\downarrow & & \downarrow \\
\mathsf{P}^1_v & \to & \mathsf{P}^1_j
\end{array} \ .$$

Here the bottom map is the degree six S_3 cover given in (3.4), and so the top map is a degree six S_3 cover as well.

There are $7 + 11 + 5 = 23$ parts in all in (6.2), so that the genus of each X_ϵ is 0 by the Riemann-Hurwitz formula. Lando and Zvonkin worked first with the base-changed cover. Here the braid partition is $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_\infty)$, with each $\tilde{\beta}_\tau = 5333322$. As $7 + 7 + 7 = 21$, the genus is 1. Jones and Zvonkin [5] carried out the S_3 descent as we are doing here.

We find via explicit equations that the two components are indeed conjugate with respect to the two choices $s = \pm\sqrt{21}$:

$$f_{21\pm}(j, x) =$$

$$2 \cdot 5 \cdot \left(2700000x^7 + x^6(630000s - 3780000) + x^5(724500 - 829500s)\right.$$

$$+x^4(2228100 - 474600s) + x^3(1404725s - 5328225)$$

$$+x^2(7020216 - 1485456s) + x(856800s - 4384800) - 252000s + 972000\big)^3$$

$$\pm 3 \cdot 7^7 j(5239s - 21429)x^5(10x - 9)^4 \left(150x^2 + x(40s - 15) - 8s + 88\right)^3 .$$

Figure 6.1 draws the dessins in P_x^1 corresponding to Cover 21+ on the left and its
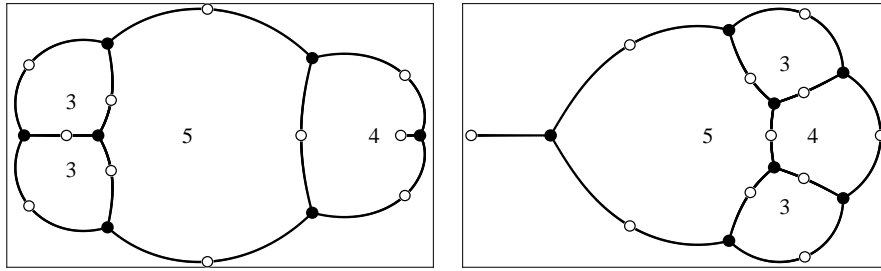
FIGURE 6.1.   Conjugate dessins, with 21+ on left and 21− on right

conjugate Cover 21− on the right. The preimages in P_x^1 of $\infty \in \mathsf{P}_j^1$ are indicated in the picture by their ramification numbers, with the undrawn $\infty \in \mathsf{P}_x^1$ also being a preimage with ramification number 6. Figure 6.1 gives the correct analytic shape of Figure 3 of [5], and, after base change, the correct shape of Figure 5.15 of [8].

The splitting is induced by the existence of $3.A_7$ as in the previous subsection. Again one has an identification $H_2(A_7, C) = \mathbb{Z}/3$. Here however, because $7a$ is not stabilized by the outer involution of A_7, the group $\mathrm{Out}(A_7, C)$ is trivial. Accordingly one has a natural function from components of X to $\mathbb{Z}/3$. One would generally expect all three preimages to have one component each. In this case, the preimages of 0, 1, and −1 are respectively empty, X^+ and X^-.

7. Hurwitz-Belyi maps with $|G| = 2^a 3^b p$ and $\nu = (3, 1)$

In this section, we set up a framework for studying the Hurwitz-Belyi maps coming from a systematic collection of 4-point Hurwitz parameters h. Here and in the next two sections, we carry out the first part of the braid-triple method for all these h, obtaining braid permutation triples (b_0, b_1, b_∞) and thus braid partition triples $(\beta_0, \beta_1, \beta_\infty)$. In many low degree cases, we carry out the second part as well, obtaining a defining equation for the cover.

7.1. **Restricting to $|\mathcal{P}| = 3$ and $\nu = (3, 1)$.** To respond to the inverse problem of §1.3, we consider only $h = (G, C, \nu)$ giving covers defined over $\mathbb{Q}$. To keep our computational study of manageable size we impose two severe restrictions. First, we require that G be almost simple with exactly three primes dividing its order. Second, we restrict attention to the case $\nu = (3, 1)$. There are many more cases within computational reach which are excluded because of these two restrictions. The rest of this subsection elaborates on the two restrictions.

Almost simple groups divisible by at most three primes. There are exactly eight nonabelian simple groups T for which the set $\mathcal{P}_T$ of primes dividing $|T|$ has size at most 3. In all cases, the order has the form $2^a 3^b p$ and the classical list is as in Table 7.1. References into this classification literature and the complete list in the much more complicated case $|\mathcal{P}_T| = 4$ are in [25].

p	T	$|T|$			H_2	A	p	T	$|T|$			H_2	A
5	A_5	60	$=$	$2^2 3^1 5$	2	2	7	$SL_3(2)$	168	$=$	$2^3 3^1 7$	2	2
5	A_6	360	$=$	$2^3 3^2 5$	6	2^2	7	$SL_2(8)$	504	$=$	$2^3 3^2 7$	1	3
5	$W(E_6)^+$	25920	$=$	$2^5 3^4 5$	2	2	7	$SU_3(3)$	6048	$=$	$2^5 3^3 7$	1	2
13	$SL_3(3)$	5616	$=$	$2^4 3^3 13$	1	2	17	$PSL_2(17)$	2448	$=$	$2^4 3^2 17$	2	2

TABLE 7.1.   The eight simple groups of order $2^a 3^b p$ and related information

The column H_2 gives the order of the Schur multiplier $H_2(T)$. Non-trivial entries here are the source of spin separation as explained in the previous section. The column A in Table 7.1 gives the structure of the outer automorphism group of T. So in every case except $T = A_6$ there are two groups G to consider, T itself and $\mathrm{Aut}(T) = T.A$. For $T = A_6$ one has, in Atlas order, the extensions S_6, $PGL_2(9)$, and M_{10}, as well as the full extension $\mathrm{Aut}(A_6) = A_6.2^2$.

Attractive features of the case $\nu = (3,1)$. The restriction $\nu = (3,1)$ is chosen for several reasons. First, it makes tables much shorter, and in fact Tables 8.1, 8.3, 9.1, 9.2 are complete. The case $\nu = (2,2)$ would have similar length and the case $\nu = (4)$ would be even shorter. However we stay away from both these alternatives as the involutions discussed at the end of §3.3 complicate the situation. Second, covers in a given degree m tend to have considerably smaller genus for $\nu = (3,1)$ than they do for $\nu = (2,1,1)$ or $(1,1,1,1)$. In fact our tables show that for $\nu = (3,1)$, covers can have genus zero into quite high degree. Third, in the case $\nu = (3,1)$ and $(\beta_0, \beta_1) = (3^{m/3},\ 2^{m/2})$, Beukers and Montanus [1] described a method which allows one to solve the given system with m unknowns by first solving an auxiliary system with approximately $m/3$ unknowns. This method generalizes to the full $(3,1)$ case of $(\beta_0, \beta_1) = (3^a 1^{m-3a}, 2^b 1^{m-2b})$; we used it simultaneously with the p-adic technique sketched in §5.3 to extend the reach of our calculations. Finally, as discussed in §3.3, the base P^1_j is the familiar j-line. Transitive degree-m covers $\mathsf{X}_h \to \mathsf{P}^1_j$ correspond to index-m subgroups of $PSL_2(\mathbb{Z})$ and we are in a very classical setting.

7.2. **Agreement and indexing.** As discussed in §4.2, the interesting phenomenon of cross-parameter agreement says that different Hurwitz parameters can give rise to isomorphic coverings. When the two groups involve different nonabelian simple groups T, as in the initial example of §4.2, we use the term *cross-group* agreement. We note cross-group agreement in our tables mainly by referencing a common equation. Two covers appearing even for T involving different primes are

$$\begin{aligned}
(7.1) \qquad f_{3,1}(j,x) &= (x-4)x^3 + 4j(2x+1), \\
(7.2) \qquad f_{4,3,2}(j,x) &= (4x^3 - 3x + 2)^3 - 27jx^3(3x-2)^2.
\end{aligned}$$

These covers are capable of arising for $|T|$ of the form 2^*3^*p for various p because their discriminants are respectively $-2^{12}3^3j^2(j-1)^2$ and $2^{54}3^{39}j^6(j-1)^4$.

The covers (7.1), (7.2) illustrate our convention of indexing by the braid partition β_∞. This partition and also $\beta_0 = 3^a1^{m-3a}$ can be read off from the presented polynomial. The remaining partition $\beta_1 = 2^b1^{m-2b}$ governing the factorization of $f_{\beta_\infty}(1,x)$ is then determined by the fact that we give polynomials only in genus zero cases.

7.3. A degree 46 map with bad reduction set $\{2,3,13\}$.

The next two sections focus on Hurwitz-Belyi maps coming from groups of order 2^a3^bp for $p \in \{5,7\}$. Here and in the next subsection, to give a sense of completeness, we give one map each for $p \in \{13,17\}$. From $h = (PGL_3(3), (2B,4B), (3,1))$ we get the braid partition triple $(\beta_0, \beta_1, \beta_\infty) = (3^{14}1^4,\ 2^{23},\ 13^2\,8\,6\,3\,2\,1)$. Our final polynomial is

$$f_{13^2,8,6,3,2,1}(j,x) =$$
$$\left(16x^4 + 40x^3 - 3x^2 - 116x - 8\right)\left(4096x^{14} + 20480x^{13} - 25856x^{12} - \right.$$
$$196736x^{11} + 47189x^{10} + 680764x^9 - 69384x^8 - 1135104x^7 + 7638144x^6$$
$$\left. -16337408x^5 + 9620480x^4 - 2785280x^3 + 741376x^2 - 16384x - 32768\right)^3$$
$$-2^{13}3^{12}j(x-2)^3x^6(x+4)^2(2x-1)\left(3x^2 + 2x - 4\right)^{13}.$$

The discriminant of this polynomial is $-2^{2260}3^{1371}13^{351}(j-1)^{23}j^{28}$. Modulo squares this discriminant is $-39(j-1)$. The factor of $j-1$ is known from the outset by the oddness of β_1 and β_∞.

7.4. A degree 54 map with bad reduction set $\{2,3,17\}$.

The Hurwitz parameters

$$(7.3) \quad h_1 = (SL_2(17), (17a+, 3a-), (3,1)) \text{ and } h_2 = (PGL_2(17), (2B,6A), (3,1))$$

each give conjugate braid permutation triples, with common braid partition triple

$$(7.4) \qquad\qquad (\beta_0, \beta_1, \beta_\infty) = (3^{17}\,1^3,\ 2^{27},\ 9^3\,8^2\,4^2\,2\,1).$$

An equation was determined by Schiavone using improvements of the techniques described in [6]:

$$f_{9^3,8^2,4^2,2,1}(j,x) =$$
$$\left(x^3 + 12x^2 + 12x - 8\right) \cdot \left(x^{17} - 52x^{16} + 42136x^{15} - 593008x^{14} + 10147846x^{13}\right.$$
$$+225862160x^{12} + 1467000268x^{11} + 6342760760x^{10} + 593082769x^9$$
$$(7.5) \qquad -1815237116x^8 - 5586407260x^7 - 258348008x^6 + 8975722736x^5$$
$$\left. -8292246656x^4 + 3424464320x^3 - 664160384x^2 + 44883968x - 131072\right)^3$$
$$-2^43^9jx\left(x^2 - 71x + 32\right)^4\left(x^2 + 2x - 1\right)^8\left(x^3 + 18x^2 - 48x - 8\right)^9.$$

It seems that this cross-parameter agreement is one of an infinite family indexed by odd primes as follows. Generalize h_1 to $(SL_2(p), (pa+, 3a-), (3,1))$. Generalize h_2 to $(PGL_2(p), (2B,6A), (3,1))$ when $p \equiv \pm 5$ (12) and to $(PSL_2(p), (2b,6a), (3,1))$ when $p \equiv \pm 1$ (12). Then mass computations confirm that both covers have degree

$$m = \frac{p^2 - 5}{4} + \begin{cases} p & \text{if } p \equiv 1\ (3) \\ -p & \text{if } p \equiv 2\ (3) \end{cases}.$$

Braid computations say that indeed the covers are isomorphic at least for $p \le 19$. For $m = 5$ and 7 the degrees are 0 and 18 respectively, these cases arising in §8.1 and §9.1.

8. HURWITZ-BELYI MAPS WITH $|G| = 2^a 3^b 5$ AND $\nu = (3, 1)$

In this section we work in the framework set up in §7.1 and present a systematic collection of Hurwitz-Belyi maps having bad reduction at exactly $\{2, 3, 5\}$.

Cross-group agreement. Before getting to the individual groups, we present equations for covers involved in cross-group agreement. The covers

$$(8.1) \qquad f_{5,3,1}(j, x) = 5^2 \left(5x^3 - 45x^2 + 39x + 25\right)^3 - 2^{14} 3^3 j x^3 (3x - 25),$$

$$(8.2) \qquad f_{5,3,2}(j, x) = \left(9x^3 + 3x^2 - 53x + 81\right)^3 (x + 9) - 2^{14} 3^2 j x^3 (3x - 5)^2,$$

$$(8.3) \qquad f_{5,4,3}(j, x) = \left(4x^4 - 24x^3 + 24x^2 - 48x + 27\right)^3 - 2^2 3^3 j x^3 (3x - 4)^5$$

appear for all three groups. The covers

$$(8.4) \qquad f_{5,4,1}(j, x) = \left(16x^3 - 87x^2 + 48x + 16\right)^3 (16x + 1) - 2^2 3^{12} j x^4 (x - 5),$$

$$(8.5) \quad f_{5,4,3^2}(j, x) = 4 \left(256x^5 + 640x^4 - 440x^3 - 3325x^2 - 6400x - 4096\right)^3$$
$$- 3^{12} j x^4 \left(32x^2 + 95x + 80\right)^3$$

appear for the simple groups A_5 and A_6. The covers

$$(8.6) \qquad f_{5,1}(j, x) = \left(x^2 - 5\right)^3 - 3^3 j (2x - 5),$$

$$(8.7) \qquad f_{5^2,4,2}(j, x) = 2^7 x \left(18x^5 - 144x^4 + 336x^3 - 224x^2 + 801x - 162\right)^3$$
$$- j(2x - 9)^2 \left(36x^2 - 52x - 9\right)^5,$$

appear for A_6 and $W(E_6)^+$. Several larger degree covers also appear for both A_6 and $W(E_6)^+$. A polynomial for the smallest of these is

$$f_{10,8^2,6,5,4^2,3}(j, x) =$$
$$(8.8) \qquad \left(184528125x^{16} - 984150000x^{15} + 2263545000x^{14} - 2768742000x^{13}\right.$$
$$+ 1616849100x^{12} + 181316880x^{11} - 1023304104x^{10} + 721510416x^9$$
$$- 166620402x^8 - 72763728x^7 + 59318552x^6 - 4952016x^5 - 12051828x^4$$
$$+ 7406640x^3 - 2117016x^2 + 314928x - 19683\right)^3$$
$$+ 2^{20} 3^8 j \left(9x^2 - 10x + 3\right)^8 x^6 (5x - 3)^5 \left(3x^2 - 1\right)^4 (3x - 1)^3.$$

8.1. **The simple group $A_5 \cong SL_2(4) \cong PSL_2(5)$.** Tables 8.1, 8.3, 9.1, and 9.2 have a similar structure, which we explain now drawing on Table 8.1 where $T = A_5$ for examples. The top left subtable gives degrees of components of Hurwitz-Belyi maps $\mathsf{X}_h^* \to \mathsf{P}_j^1$ for $h = (T, (C_1, C_2), (3, 1))$. Here C_1 and C_2 are distinct conjugacy classes in T. When the lifting invariant set H_h^* from §6.1 is trivial, a single number is typically printed. For $T = A_5$, this triviality occurs exactly if 221 is one of the C_i, as from Table 6.1 for A_5, only 221 is inert in the double cover $\tilde{A}_5$. When H_h^* is canonically $\mathbb{Z}/2$, typically two numbers are printed; the top and bottom numbers respectively give the degrees of X_h^{*+} and X_h^{*-} over P_j^1.

The remaining subtables on the left sides of Tables 8.1, 8.3, 9.1, and 9.2 similarly give degrees of components of Hurwitz-Belyi maps, bit now for $h =$

$(T.2, (C_1, C_2), (3, 1))$. When $H_h = H_h^*$ has 2 elements but is not canonically $\mathbb{Z}/2$, typically again two numbers are printed in a column. These numbers are necessarily the same. In general, a number is put in italics when the corresponding component is not defined over $\mathbb{Q}$. For $G = S_5$, irrationality occurs exactly if $\{C_1, C_2\} = \{41, 32\}$, a key point being that 41 and 32 each split into two irrational classes in $\tilde{S}_5$. Other possibilities for H_h^* occur only for $T = A_6$ and will be discussed in §8.2. In general, if there is splitting beyond that forced by lifting invariants then the corresponding degree is written as a list of the component degrees separated by commas. This extra splitting does not occur on Table 8.1 and we expect it to be rare in general. Indeed for $G = A_5$ and any (C, ν), it never occurs on the level of the entire r-dimensional Hurwitz cover $\mathsf{Hur}_h^* \to \mathsf{Conf}_\nu$ [4].

$C_1 \backslash C_2$	221	311	$5b$
221	•	0	$10a$
311	12	•	15
			$10b$
$5a$	4	9	4
		0	0

$C_1 \backslash C_2$	2111	41	32
2111	•	0	0
41	32	•	*36*
			36
32	$10b$	*16*	•
		16	

#	M	g	β_0	β_1	β_∞	Eqn.
2+	A_4	0	3 1	2^2	3 1	(7.1)
1+	A_9	0	3^3	$2^4 1$	5 3 1	(8.1)
1+	S_{10a}	0	$3^3 1$	2^5	5 3 2	(8.2)
2+	S_{10b}	0	$3^3 1$	2^5	5 4 1	(8.4)
1+	S_{12}	0	3^4	$2^5 1^2$	5 4 3	(8.3)
1+	S_{15}	0	3^5	$3^7 1$	5 4 3^2	(8.5)
1	A_{32}	0	$3^{10} 1^2$	2^{16}	10 6 5 4^2 3	(8.9)

Two pairs defined over $\mathbb{Q}(\sqrt{6})$					
1	A_{16}	0	$3^5 1$	2^8	6 5 4 1	(8.11)
1	A_{36}	0	3^{12}	2^{18}	10 6 5 4^2 3^2 1	

TABLE 8.1. Left: Degrees of components of Hurwitz-Belyi covers with parameters $(G, (C_1, C_2), (3, 1))$ with $G = A_5$ or S_5. Right: further information on these covers.

We are interested primarily in rational covers and we distinguish non-isomorphic rational covers of the same degree by identifying labels. This convention highlights cross-parameter agreement. Thus on the left half of Table 8.1 the two 4's and the two $10b$'s each represent isomorphic covers.

The left half of Tables 8.1 8.3, 9.1, or 9.2, as just described, is well thought of as the Hurwitz half. The right half can then be considered the Belyi half, as it makes no reference to its Hurwitz sources beyond the column #. Here a number printed under # just repeats the number of Hurwitz sources from the left half; a + sign represents cross-group agreement, as it indicates that the cover also arises elsewhere in this paper for a different T. While our focus is on Hurwitz-Belyi covers defined over $\mathbb{Q}$, when there is space we include extra lines for Hurwitz-Belyi covers not defined over $\mathbb{Q}$.

Equations for the first six lines of the top right subtable of Table 8.1 have already been presented in the context of cross-group agreement. An equation for the seventh line is

$$f_{10,6,5,4^2,3}(j, x) = \left(x^{10} - 38x^9 + 591x^8 - 4920x^7 + 24050x^6 - 71236x^5 + 125638x^4\right.$$

(8.9)
$$\left. -124536x^3 + 40365x^2 + 85050x - 91125\right)^3 \left(x^2 - 14x - 5\right)$$

$$+ 2^{20} 3^3 j x^6 (x-5)^5 \left(x^2 - 4x + 5\right)^4 (x-9)^3.$$

Note that the four-point covers $\mathsf{Y}_x \to \mathsf{P}_t^1$ corresponding to the seventh line have genus one, and so (8.9) would be hard to compute by the standard method. The tables of this and the next section give many examples where g_Y is large but $g_X = 0$. As we are systematically using the braid-triple method, g_Y is irrelevant and the tables present $g = g_X$.

Remark. The excluded case $\nu = (4)$. Tables 8.1, 8.3, 9.1, and 9.2 exclude the case $C_1 = C_2$ to stay in the context of $\nu = (3, 1)$. The excluded cases $(G, C_1, (4))$ are interesting too and we mention one of them. For $h = (A_5, (311), (4))$, the cover X_h^{*+} is given by $f_{5,3,1}(j, x)$ from (8.1) while X_h^{*-} is empty. This h is our first of three illustrations of a general theorem of Serre [23] as follows. Consider Hurwitz parameters

$$h = (A_n, (e_1 1^{n-e_1}, \ldots, e_k 1^{n-e_k}), (\nu_1, \ldots, \nu_k))$$

with all e_i odd, so that one has a lifting invariant and thus an equation $\mathsf{X}_h^* = \mathsf{X}_h^{*+} \coprod \mathsf{X}_h^{*-}$. Suppose $\sum \nu_i(e_i - 1) = 2n - 2$ so that the genus g_Y is 0. Then the general theorem says,

$$(8.10) \qquad\qquad \textit{If } \prod e_i^{\nu_i} \overset{8}{\equiv} \left\{ \begin{array}{c} \pm 1 \\ \pm 3 \end{array} \right. \textit{ then } \mathsf{X}_h^* = \left\{ \begin{array}{c} \mathsf{X}_h^{*+} \\ \mathsf{X}_h^{*-} \end{array} \right. .$$

Table 8.1 shows that X_h^{*-} is empty for $(5a, 311)$ and $(5a, 5b)$ as well, even though $g_Y > 0$ and so Serre's theorem does not apply in these cases.

Remark. A conjugate pair of irrational covers. Covers not defined over $\mathbb{Q}$ arise naturally in our situation, and the left half of Table 8.3 refers to two pairs of irrational covers. Letting $s = \pm\sqrt{6}$, equations for the smaller degree pair are

$$f_{6,5,4,1}(j, x) =$$
$$(3x + s - 3) \left(-225x^5 + 1305x^4 s + 4005x^4 - 8932x^3 s - 22662x^3 + 6594x^2 s\right.$$
$$(8.11) \qquad \left.+ 16254x^2 - 28476xs - 69741x + 11673s + 28593\right)^3$$
$$-12288jx^5(5x-9)^4(53236s + 130401)(-15x + 76s + 186).$$

8.2. The simple group $A_6 \cong Sp_4(2)' \cong PSL_2(9)$. In terms of both its Schur multiplier $H_2 \cong \mathbb{Z}/6$ and its outer automorphism group $A \cong (\mathbb{Z}/2)^2$, the group $T = A_6$ is the most complicated group on Table 7.1. Table 8.2 gives information on conjugacy classes.

Conventions about the (C_1, C_2) entry in the left half of Table 8.3 have been given in §8.1 whenever $|H_2(G, C)| \in \{1, 2\}$. The remaining possibilities are as follows. Three entries in a single row separated by semicolons means $|H_h| = 3$ and $\mathrm{Out}(G, C)$ acts trivially on H_h, so that $|H_h^*| = 3$ as well. This possibility arises three times, always in the form $(a; b; b)$. By the typeface convention of §8.1, this means that the degree a component is rational and the degree b components are conjugate. Two entries in a single row separated by semicolons means $|H_h| = 3$ but $\mathrm{Out}(G, C)$ acts nontrivially on H_h, so that $|H_h^*| = 2$. This possibility also arises three times, always in the form $(c; d)$. Here both components are rational, as indeed in these three cases $c \neq d$. Instances of these two situations were described already in §6.4 and §6.3, where degrees were $(a; b; b) = (0; 21; 21)$ and $(c, d) = (30; 40)$ respectively. In these situations, one generally expects $a \approx b$ and $c \approx d/2$.

$H_2(A_6)$	$=$	6				$H_2(S_6)$	$=$	2		
$\mathrm{Out}(A_6)$	$=$	2^2				$\mathrm{Out}(S_6)$	$=$	2		
2211	3111	33	42	$5a$	$5b$	21111	222	411	6	321
3	2	2	6	6	6	1	1	1	2	2

$H_2(PGL_2(9))$	$=$	3			$H_2(M_{10})$	$=$	3
$\mathrm{Out}(PGL_2(9))$	$=$	2			$\mathrm{Out}(M_{10})$	$=$	2
2222	811A	811B	10A	10B	4411	82C	82D
1	2	2	2	2	3	3	3

TABLE 8.2. Information on conjugacy classes in A_6 and conjugacy classes not in A_6 of its three overgroups $S_6 = A_6.2_1$, $PGL_2(9) = A_6.2_2$, and $M_{10} = A_6.2_3$. The last row gives the number of classes in $\tilde{G}$ mapping to the given class in G.

In the case $(5a, 5b)$, one has $H_h \cong \mathbb{Z}/6$ and $\mathrm{Out}(G, C)$ has order two. The non-trivial element of $\mathrm{Out}(G, C)$ acts by negation, so that H_h^* has order four. The natural action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on H_h is trivial, and one would generally expect four rational components. In this case, the natural map $\pi_0(\mathsf{X}_h^*) \to H_h^*$ is injective but not surjective, and X_h^* has only three components. The cases $(42, 51a)$ and $(51b, 42)$ are similar to $(5a, 5b)$ but now all components are defined over $\mathbb{Q}(\sqrt{10})$. The two degree 24 components have their dessin drawn in the website associated to [1].

A blank in the (C_1, C_2) slot means that covers belonging to this slot are isomorphic to those of (C_1^α, C_2^α) for some α in $\mathrm{Out}(G) - \mathrm{Out}(G, C)$. For example the $(411, 6)$ slot is left blank because the cover is the same as that represented by the $(411, 321)$ slot. It is this non-triviality of $\mathrm{Out}(G) - \mathrm{Out}(G, C)$ that makes some of the covers involving $51a$ and/or $51b$ rational, even though the classes $51a$ and $51b$ are conjugate to each other. Among the further things to note on Table 8.3 are two isomorphic unforced decompositions of the form $46 = 42 + 4$. Also the cover $96b$ is unexpectedly nonfull. Finally, a second instance of Serre's theorem (8.10) is $(C_1, C_2) = (3111, 51a)$, so that X_h^{*-} is forced to be empty.

8.3. **The simple group $W(E_6)^+ \cong PSp_4(3) \cong PSU_4(2)$.** The group $W(E_6) = W(E_6)^+.2$ has twenty-five conjugacy classes. As for all Weyl groups, all the classes are rational. Ten classes are in $W(E_6) - W(E_6)^+$ and ten classes stay rational classes in $W(E_6)^+$. The remaining five conjugacy classes of $W(E_6)$, namely $3ab$, $6ab$, $6cd$, $9ab$, and $12ab$, split into two classes in $W(E_6)^+$. If we were presenting complete tables for $\nu = (1, 1, 1, 1)$, there would thousands of lines. Even complete tables for $(3, 1)$ would have hundreds of lines. Accordingly, Table 8.4 presents just some of Hurwitz-Belyi covers in a self-explanatory format.

One of the new covers has the remarkably small degree nine:

$$(8.12) \qquad f_{5,4}(j, x) = 5^2 \left(10x^3 + 15x^2 + 48x - 100\right)^3 + 3^{15} j x^4.$$

The other three new covers are

$$f_{9,6,5,4}(j, x)$$
$$= \left(9x^8 - 72x^7 + 180x^6 - 104x^5 - 26x^4 - 568x^3 + 1620x^2 - 1944x + 729\right)^3$$

$C_1\backslash C_2$	2211	33	42	51b
2211	•	0	12	10a;15
3111	12	0 6	16	10a 0
42	9; 48	108	•	60;40 60;40
51a	60;0	45 54	24;40 24;40	9;45 0;42

$C_1\backslash C_2$	21111	411	321	6
21111	•	0	0	0
411	10a	•	252	
321	84	544	•	396 396
6	88			•

$C_1\backslash C_2$	22222	811B	10B
22222	•	16	10b
811A	96a	96b 108	100 100
10A	4, 42	64 64	4, 42 54

$C_1\backslash C_2$	4411	82D
4411	•	656;672;672
82C	164;168;168	66; 90;90

#	M	g	β_0	β_1	β_∞	Eqn.
2+	A_4	0	$3\,1$	2^2	$3\,1$	(7.1)
1+	A_6	0	3^2	$2^2 1^2$	$5\,1$	(8.6)
2+	A_9	0	3^3	$2^4\,1$	$5\,3\,1$	(8.1)
3+	S_{10a}	0	$3^3 1$	2^5	$5\,3\,2$	(8.2)
1+	S_{10b}	0	$3^3\,1$	2^5	$5\,4\,1$	(8.4)
2+	S_{12}	0	3^4	$2^5\,1^2$	$5\,4\,3$	(8.3)
1+	S_{15}	0	3^5	$2^7 1$	$5\,4\,3^2$	(8.5)
2+	A_{16}	0	$3^5 1$	2^8	$5^2\,4\,2$	(8.7)
3	S_{42}	0	3^{14}	2^{21}	$10\,8^2\,4^2\,3^2\,1^2$	
2	A_{45}	0	3^{15}	$2^{22}\,1$	$8^2\,5^3\,4^2\,3^2$	
1+	A_{48}	0	3^{16}	$2^{22}\,1^4$	$10\,8^2\,6\,5\,4^2\,3$	(8.8)
2	S_{54}	0	3^{18}	2^{27}	$10\,8^2\,5^2\,4^2\,3^3\,1$	
1	S_{60}	0	3^{20}	$2^{29}\,1^2$	$10\,8^2\,5^4\,4^2\,3^2$	
1	S_{66}	0	$3^{21}\,1^3$	2^{33}	$10^2\,8^3\,6^3\,2^1\,1^2$	
1+	A_{84}	0	$3^{27}\,1^3$	2^{42}	$10^3\,8^2\,6^3\,5\,4^3\,2\,1$	
1	A_{88}	0	$3^{28}\,1^4$	2^{44}	$10^3\,8^4\,6\,5^2\,3^3\,1$	
1	A_{96a}	1	3^{32}	2^{48}	$10^2\,8^3\,6^3\,5^4\,4^2\,3^2$	
1	G_{96b}	1	3^{32}	2^{48}	$10^2\,8^3\,6^3\,5^4\,4^2\,3^2$	
2+	A_{108}	0	3^{36}	2^{54}	$10^2\,8^4\,6^2\,5^2\,4^4\,3^2$	
1	A_{164}	0	$3^{53}\,1$	2^{82}	$10^6\,8^5\,6^5\,4^8\,1^2$	
1	A_{252}	0	3^{84}	2^{126}	$10^4\,8^9\,6^6\,5^{10}\,4^8\,3^5\,1$	
1	A_{544}	3	3^{181}	2^{272}	$10^{18}\,8^{18}\,6^{15}\,5^{10}\,4^{13}\,3^8\,1^4$	
1	A_{656}	7	$3^{217}\,1^5$	2^{328}	$10^{26}\,8^{20}\,6^{25}\,4^{21}\,1^2$	

A pair defined over $\mathbb{Q}(\sqrt{10})$					
1 A_{24}	0	3^8	2^{12}	$8\,5\,4\,3^2\,1$	

TABLE 8.3. Left: Degrees of components of Hurwitz-Belyi covers with parameters $(G,(C_1,C_2),(3,1))$ with $G = A_6$, S_6, $PGL_2(9)$, or M_{10}. Right: further information on these covers.

$$(8.13) \qquad +2^{16}j(x-3)^4 x^6 (2x-3)^5,$$

$$
\begin{aligned}
f_{5^4,4,3}(j,x) &= \big(1024x^9 - 13824x^8 + 81360x^7 - 272928x^6 + 585144x^5 - 879336x^4 \\
&\qquad +1012365x^3 - 896832x^2 + 516096x - 131072\big)^3
\end{aligned}
$$

$$(8.14) \qquad -54j(72x^4 - 508x^3 + 1350x^2 - 1629x + 768)^5 x^3,$$

$$
\begin{aligned}
f_{10,9,5,2^2}(j,x) &= \big(3125x^9 - 9375x^8 + 7500x^7 - 6500x^6 + 9150x^5 - 4410x^4 - 2484x^3 \\
&\qquad -2916x^2 - 2187x + 6561\big)^3 (x-3)
\end{aligned}
$$

$$(8.15) \qquad +2^{22}3^3 jx^9(5x-6)^5 \big(3x^2+2x+3\big)^2.$$

C_1	C_2	M	g	β_0	β_1	β_∞	Eqn.
			Full low degree covers				
$3d$	$4a$	A_6	0	3^2	$2^2\,1^2$	$5\,1$	(8.6)
$3C$	$9A$	A_9	0	3^3	$2^4\,1$	$5\,3\,1$	(8.1)
		S_9	0	3^3	$2^3\,1^3$	$5\,4$	(8.12)
		S_{10}	0	$3^3\,1$	2^5	$5\,3\,2$	(8.2)
		S_{12}	0	3^4	$2^5\,1^2$	$5\,4\,3$	(8.3)
$6a$	$4a$	A_{16}	0	$3^5\,1$	2^8	$5^2\,4\,2$	(8.7)
$6a$	$2b$	A_{24}	0	3^8	$2^{10}\,1^4$	$9\,6\,5\,4$	(8.13)
$3c$	$9a$	S_{27}	0	3^9	$2^{13}\,1$	$5^4\,4\,3$	(8.14)
$4a$	$2b$	S_{28}	0	$3^9\,1$	$2^{13}\,1^2$	$10\,9\,5\,2^2$	(8.15)

C_1	C_2	M	g	β_0	β_1	β_∞	Eqn.
			Agreement with covers coming from A_6				
$4a$	$3c1$	A_{48}	0	3^{16}	$2^{22}\,1^4$	$10\,8^2\,6\,5\,4^2\,3$	(8.8)
$6e$	$6a$	A_{84}	0	$3^{27}\,1^3$	2^{42}	$10^3\,8^2\,6^3\,5\,4^3\,2\,1$	
$4a$	$6a$	A_{108}	0	3^{36}	2^{54}	$10^2\,8^3\,6^3\,5^4\,4^4\,3^3\,1$	

C_1	C_2	M	g	β_0	β_1	β_∞	Eqn.
			Large degree genus zero examples of spin separation				
$3d$	$5a$	A_{165}	0	3^{55}	$2^{80}1^5$	$12^4\,9^4\,6^7\,5^2\,4^2\,3^6\,2^1\,1$	
$3d$	$5a$	S_{225}	0	3^{75}	$2^{109}\,1^7$	$12^4\,9^7\,6^7\,5^7\,4^6\,3^3\,2^2$	
$3d$	$9b$	S_{189}	0	3^{63}	$2^{93}1^3$	$12^4\,9^6\,6^7\,5^4\,4^4\,3^9$	
$3d$	$9b$	S_{234}	0	3^{78}	2^{117}	$12^4\,9^6\,6^7\,5^7\,4^7\,3^8\,2\,1$	

C_1	C_2	M	g	β_0	β_1	β_∞	Eqn.
			An example where all allowed bases appear in β_∞				
$4D$	$6G$	S_{3186}	82	3^{1062}	2^{1593}	$24^6\,18^{27}\,12^{43}\,10^{78}\,9^{27}\,8^{54}$	
						$6^{24}\,5^{48}\,4^{28}\,3^{23}\,2^9\,1^2$	

TABLE 8.4.   Invariants of some covers with $G = W(E_6)^+$ or $W(E_6)$

In the entire table for $T = W(E_6)^+$, there are only twelve integers which can appear as parts for β_∞. The last line of Table 8.4 gives the smallest degree cover where all these integers actually appear.

9. HURWITZ-BELYI MAPS WITH $|G| = 2^a 3^b 7$ AND $\nu = (3,1)$

This section is very parallel in structure to the previous one, and presents a systematic collection of Hurwitz-Belyi maps having bad reduction at exactly $\{2, 3, 7\}$.

Cross-group agreement. Again we present equations for covers involved in cross-group agreement before getting to the individual groups. Now we have only two:

$$(9.1) \qquad f_{4,3}(j,x) \;=\; 4(x-12)\left(9x^2 - 20x - 27\right)^3 + 3 \cdot 7^7 j x^3,$$

$$(9.2) \qquad f_{7,4,3^2,1}(j,x) \;=\; \left(9x^6 - 126x^4 + 252x^3 - 63x^2 - 252x + 196\right)^3$$
$$+\,2^6 j (3x-2)^4 \left(3x^2 - 9x + 7\right)^3 (3x+14).$$

The cover $f_{7,4,3^2,1}(j,x)$ was first found by Malle [12] in connection with the group $PGL_2(7)$.

9.1. The simple group $PSL_2(7) \cong SL_3(2)$.

Equations for the first three Hurwitz-Belyi maps have been given already. For the fourth, an equation is

$C_1\backslash C_2$	2a	3a	4a	7b
2a	•	0	0	7
3a	60	•	64 / 64	70 / 63
4a	18b	36 / 36	•	28 / 28
7a	4	0 / 18a	16 / 16	0 / 4

#	M	g	β_0	β_1	β_∞	Eqn.
1+	A_4	0	3 1	2^2	3 1	(7.1)
1+	S_7	0	$3^2 1$	$2^3 1$	4 3	(9.1)
2+	S_{18a}	0	3^6	2^9	7 4 3^2 1	(9.2)
1	S_{18b}	0	3^6	2^9	7 6 3 1^2	(9.3)
1	S_{60}	1	3^{20}	$2^{29}\, 1^2$	14 8^2 7 6^2 $4^2$3	
1	A_{63}	1	3^{21}	$2^{32}1$	14 8^2 7 6^2 4^2 3^2	
2	S_{70}	0	$3^{23}\, 1$	2^{35}	14 8^2 7 6^2 4^2 3^4 1	

$C_1\backslash C_2$	2B	6A
2B	•	18a
6A	70	•

	Two pairs defined over $\mathbb{Q}(\sqrt{2})$					
1	A_{16}	0	$3^5 1$	2^8	7 4 3 2	(9.4)
1	A_{28}	0	$3^9 1$	2^{14}	8 7 6 3^2 1	

TABLE 9.1. Left: Degrees of components of Hurwitz-Belyi covers with parameters $(G, (C_1, C_2), (3, 1))$ and $G \in \{PSL_2(7), PGL_2(7)\}$. Right: further information on the covers.

$$f_{7,6,3,1^2}(j,x) = \left(9x^6 - 102x^5 + 295x^4 - 212x^3 + 39x^2 + 90x + 9\right)^3$$

$$(9.3) \qquad\qquad -2^{14}jx^6(2x-3)^3\left(9x^2 - 66x - 7\right).$$

Remark. A conjugate pair of irrational covers. The left half of Table 9.1 refers to four pairs of irrational covers. Letting $s = \pm\sqrt{2}$, equations for the smallest degree pair are

$$f_{7,4,3,2}(j,x) =$$
$$\left(-7x + 19s + 27\right)\left(-49x^5 + x^4(217s - 63) + x^3(332s - 478)+\right.$$
$$\left. x^2(154s - 658) + x(196s + 147) + 441s + 637\right)^3$$
$$(9.4) \qquad -216jx^4(35123s + 49688)(-2x + s - 4)^2(7s - 4x)^3.$$

9.2. The simple group $SL_2(8)$.

The group $T = SL_2(8)$ has outer automorphism group A of order 3. All the corresponding Hurwitz parameters h satisfying the conditions of §7.1 have $G = T$, as those of the form $(T.3, (C_1, C_2), (3, 1))$ have at least $\mathbb{Q}(\sqrt{-3})$ in their field of definition and hence break the rationality restriction in §7.1.

Since the Schur multiplier of $SL_2(8)$ is trivial, there is no spin separation. However Table 9.2 exhibits so many Galois degeneracies that it seems likely that at least some of them are forced by deeper reasons. We describe some of these degeneracies here. Our conventions follow the Atlas: if $g \in 7a$, then $g^2 \in 7b$ and $g^4 \in 7c$; similarly, if $g \in 9a$, then $g^2 \in 9b$ and $g^4 \in 9c$.

For the case $h = (SL_2(8), (7b, 7c), (3, 1))$, the mass and degree from the mass formula [22, (3.6)] are $\overline{m} = m = 97$. A braid calculation gives two degeneracies; first there are two orbits, of size 7 and 90 respectively. Second, the monodromy group for the degree 90 orbit is imprimitive, with image inside the wreath produce

$C_1\backslash C_2$	$2a$	$3a$	$7c$	$9c$
$2a$	$\bullet$	$18a$	70	54
$3a$	16	$\bullet$	49	33
$7a$	84	$18b, 63$	$4, 84$	*81*
$7b$			$7, 90$	
$9a$	$4, 36$	$4, 36$	*49*	33
$9b$				33

#	M	g	β_0	β_1	β_∞	Eqn.
3+	A_4	0	$3\ 1$	2^2	$3\ 1$	(7.1)
1+	S_7	0	$3^2 1$	$2^3 1$	$4\ 3$	(9.1)
1	A_{16}	0	$3^4 1^4$	2^8	$9\ 7$	(9.5)
1	S_{18a}	0	$3^5 1^3$	2^9	$9\ 7\ 2$	(9.6)
1+	S_{18b}	0	3^6	2^9	$7\ 4\ 3^2\ 1$	(9.2)
3	A_{33}	0	3^{11}	$2^{16}\ 1$	$9\ 7^3\ 1^3$	(9.7)
2	A_{36}	1	3^{12}	2^{18}	$9\ 7^3\ 3^2$	
1	A_{49}	1	$3^{16}\ 1$	$2^{24}\ 1$	$9^3\ 7^3\ 1$	
1	S_{54}	0	3^{18}	2^{27}	$9^2\ 7^3\ 3^3\ 2^3$	
1	S_{63}	0	$3^{20}\ 1^3$	$2^{31}\ 1$	$9^3\ 7^3\ 4^3\ 3$	
1	S_{70}	0	$3^{23}1$	2^{35}	$9^3\ 7^4\ 3^3\ 2^3$	
2	A_{84}	1	3^{28}	2^{42}	$9^3\ 7^5\ 4^4\ 3^2$	
1	G_{90}	0	3^{30}	2^{45}	$9^3\ 7^6\ 4^3\ 3^2\ 1^3$	(9.8)

TABLE 9.2. Left: Degrees of components of Hurwitz-Belyi covers with parameters $(SL_2(8), (C_1, C_2), (3, 1))$. Right: further information on these covers.

m	$\lvert\langle g_i\rangle\rvert$	M	$g_1 \in 7b$	$g_2 \in 7b$	$g_3 \in 7b$	$g_4 \in 7c$
7	504	S_7	(359467182)	(287516439)	(236478159)	(127698453)
90	504	G_{90}	(318954762)	(978436512)	(978436512)	(127698453)

m	$\lvert\langle g_i\rangle\rvert$	M	$g_1 \in 7a$	$g_2 \in 7a$	$g_3 \in 7a$	$g_4 \in 7c$
4	504	A_4	(132674598)	(863972514)	(832465917)	(124835697)
84	504	A_{84}	(396482715)	(685427319)	(853716942)	(127698453)
9	56	$SL_2(8)$	(329518746)	(124835697)	(827153964)	(127698453)
9	56	$SL_2(8)$	(136927485)	(124835697)	(185294736)	(127698453)
1/7	7	S_1	(136927485)	(136927485)	(136927485)	(149375286)

TABLE 9.3. Top: Representatives for braid orbits of $\mathrm{HBr}_{3,1}$ on $\mathcal{F}_h$, for $h = (SL_2(8), (7a, 7c), (3, 1))$. Bottom: Representatives for braid orbits of $\mathrm{HBr}_{3,1}$ on $\mathcal{F}_h$ for $h = (SL_2(8), (7b, 7c), (3, 1))$, followed representatives of three degenerate orbits

$S_3 \wr S_{30}$. Representatives in $\mathcal{G}_h$ of the two braid orbits on $\mathcal{F}_h$ are given in the top part of Table 9.3.

The the case $h = (SL_2(8), (7a, 7c), (3, 1))$, the mass is $\overline{m} = 106\frac{1}{7}$ and the degree is $m = 88$. The degree decomposes, $m = 4 + 84$, and the degenerate piece decomposes as well, $\overline{m} - m = 9 + 9 + \frac{1}{7}$. The two components with $\langle g_i\rangle = 56$ have the same monodromy group $SL_2(8)$, with the rigid braid partition triple $(3^3,\ 2^4 1,\ 7\ 1^2)$. Representatives in $\mathcal{G}_4$ are given in the bottom part of Table 9.3 for all five orbits. Note that the representative of the orbit with mass $1/7$ has the very simple form (g, g, g, g^4). All the braid computations in this paper involve r-tuples of permutations like the ones exhibited in Table 9.3.

Given how differently behaved the last two parameters are, one might expect that the parameters $(SL_2(8), (9a, 9c), (3, 1))$ and $(SL_2(8), (9b, 9c), (3, 1))$ would be differently behaved as well. However here the Galois degeneracy is in the other direction: not only is $\overline{m} = m = 33$ in each case, but the two degree 33 covers are isomorphic. Moreover, these covers are also isomorphic to the cover arising from $(SL_2(8), (3a, 9c), (3, 1))$.

It is because of the 3-element outer automorphism group that the covers considered above are all rational, despite the fact that $7a$, $7b$, $7c$ and $9a$, $9b$, $9c$ are defined only over the cyclic cubic fields $\mathbb{Q}(\cos(2\pi/7))$ and $\mathbb{Q}(\cos(2\pi/9))$ respectively. In contrast, the three-element group $\text{Out}(SL_2(8))$ is not large enough to make the covers indexed by $(SL_2(8), (9a, 7c), (3, 1))$ and $(SL_2(8), (7a, 9c), (3, 1))$ rational. They are each defined over a cyclic cubic field ramified at both 7 and 9. As reported by Table 9.2, their degrees are 49 and 81 respectively. Like most of the covers in the upper right of Table 9.2, they are full of genus zero.

Equations for three covers coming only from $SL_2(8)$ are

$$(9.5) \qquad f_{9,7}(j, x) = \left(441x^4 + 1764x^3 + 702x^2 - 140x + 49\right)^3$$
$$\left(343x^4 + 2940x^3 + 6594x^2 - 468x + 63\right) - 2^{42}jx^7,$$

$$(9.6) \qquad f_{9,7,2}(j, x) = \left(7^4 x^5 - 441x^4 - 3366x^3 + 2430x^2 - 3^7 x + 3^7\right)^3$$
$$\left(49x^2 + 6x + 9\right)(x + 3) - 2^{30} 3^9 j x^9 (x - 1)^2,$$

$$(9.7) \quad f_{9,7^3,1^3}(j, x) = \left(16x^{11} + 256x^{10} + 1312x^9 + 2208x^8\right.$$
$$-1248x^7 - 6720x^6 - 1512x^5 + 5652x^4$$
$$\left. -6147x^3 - 3912x^2 + 11712x - 1536\right)^3$$
$$+ 108j(x - 1)(x + 2)(x + 8)\left(8x^3 + 15x^2 - 9x - 8\right)^7.$$

The cover $f_{9,7,2}(j, x)$ was found by Hallouin [3]. For $h = (SL_2(8), (7a, 7b), (3, 1))$, an equation for the degree thirty intermediate cover is

$$f_{9,7^2,4,3,1^2}(j, x) =$$
$$\left(11664x^{10} + 31104x^9 - 38880x^8 - 276960x^7 - 458528x^6 - 245952x^5\right.$$
$$(9.8) \qquad \left. +244440x^4 + 549396x^3 + 475389x^2 + 225504x + 46656\right)^3$$
$$-2^2 3^2 7^7 j\left(8x^2 + 15x + 9\right)^7 x^4(x - 3)\left(3x^2 + 6x + 4\right).$$

9.3. The simple group $G_2(2)' \cong PSU_3(3)$.

In parallel with the §8.3, the third simple group of order $2^a 3^b 7$ is substantially larger than the first and second group. Again we present only some sample Hurwitz-Belyi maps, following the format used in §8.3.

The first block on Table 9.4 represents cases where the Hurwitz-Belyi map has degree 1 and hence is uninteresting in the present context. These three rigid cases are closely related and are studied in detail in [18], starting from Proposition 3.1 there. These three cases serve as a reminder that non-trivial Hurwitz-Belyi maps measure a failure of rigidity.

The last two genus zero covers on Table 9.4 come only from $T = PSU_3(3)$. Equations are

$$f_{8,7,6,3}(j, x) =$$

C_1	C_2	M	g	β_0	β_1	β_∞	Eqn.
				Degree one covers corresponding to rigid Hurwitz parameters			
$3a$	$4a$	S_1	0	1	1	1	$x - j$
$4a$	$4b$	S_1	0	1	1	1	$x - j$
$4a$	$2a$	S_1	0	1	1	1	$x - j$
				Genus zero covers of small degree			
$4a$	$6a$	G_4	0	3^2	2^3	$4\,1^2$	(7.1)
$4c$	$2a$	S_7	0	$3^2\,1$	$2^3\,1$	$4\,3$	(9.1)
$4a$	$3b$	G_9	0	3^3	$2^4\,1$	$4\,3\,2$	(7.2)
$4c$	$3a$	S_{18}	0	3^6	2^9	$7\,4\,3^2\,1$	(9.2)
$4D$	$2B$	A_{24}	0	$3^7\,1^3$	2^{12}	$8\,7\,6\,3$	(9.9)
$2B$	$4D$	A_{40}	0	$3^{12}\,1^4$	2^{20}	$12\,8^2\,7\,3\,2$	(9.10)
				An unforced splitting to two full covers			
$6a$	$4b$	S_{135}	1	$3^{43}\,1^6$	$2^{65}\,1^5$	$14^2\,12^4\,8^2\,6^6\,4\,3$	
$6a$	$4b$	S_{180}	3	3^{60}	$2^{87}\,1^6$	$14^2\,12^4\,8^7\,7^2\,6^4\,3^2\,2^2$	
				An example where all allowed bases appear in β_∞			
$8b$	$2a$	S_{750}	25	$3^{248}\,1^6$	2^{375}	$24^5\,16^{10}\,14^{12}\,12^9\,8^8\,7^2$ $6^{13}\,4^5\,3^2\,2^5\,1^2$	

TABLE 9.4. Invariants of some covers with $G = G_2'(2)$ or $G_2(2)$

$$(9.9) \qquad \frac{4\left(4x^7 + 22x^6 - 60x^5 - 166x^4 + 236x^3 + 858x^2 - 3626x + 2401\right)^3}{(2x - 1)\left(2x^2 + 16x - 49\right)}$$
$$+3^{18} j x^7 (x-2)^6 (x+4)^3,$$

$$f_{12,8^2,7,3,2}(j,x) =$$
$$\left(64x^{12} - 576x^{11} + 2400x^{10} - 5696x^9 + 7344x^8 - 3168x^7 - 4080x^6\right.$$
$$\left. +8640x^5 - 7380x^4 - 1508x^3 + 8982x^2 - 7644x + 2401\right)^3$$
$$(9.10) \qquad \left(4x^4 - 20x^3 + 78x^2 - 92x + 49\right)$$
$$-2^8 3^{12} j \left(2x^2 - 4x + 3\right)^8 x^7 (x-2)^3 (x+1)^2.$$

10. SOME 5-POINT HURWITZ-BELYI MAPS

All the explicit Hurwitz-Belyi maps presented in the paper so far have had ramification number $r = 4$. This section presents some examples with $r = 5$, as a first indication of how things look when r increases.

10.1. **A Belyi pencil for $\nu = (4, 1)$ yielding 3-2-∞ maps.** Sections 7-9 built many Hurwitz-Belyi maps from the single Belyi pencil $u_{3,1}$ into $\mathsf{Conf}_{3,1}$. This pencil has the remarkable property that it produces braid permutation triples (b_0, b_1, b_∞)

in S_m with b_0 and b_1 of order 3 and 2 respectively. This property kept genera very low in §7-9.

Abbreviating $k = j - 1$, let

(10.1) $$s(j, t) = k^2 t^4 - 6jkt^2 - 8jkt - 3j^2.$$

Define $u : \mathsf{P}_j^1 - \{0, 1, \infty\} \to \mathsf{Conf}_{4,1}$ by $j \mapsto (D_1(j), \{\infty\})$, with $D_1(j) \subset \mathsf{P}_t^1$ the roots of $s(j, t)$. Let

(10.2) $$B_0 = \sigma_1 \sigma_2 \sigma_3^2, \qquad\qquad B_1 = (\sigma_1 \sigma_2 \sigma_3)^2.$$

A braid calculation says that the abstract braid triple of the Belyi pencil u is $(B_0, B_1, B_1^{-1} B_0^{-1})$, and that B_0 and B_1 likewise have orders 3 and 2 in $\mathrm{HBr}_{4,1}$ respectively.

Two Hurwitz-Belyi maps built from u are considered in [21]. First, for $h = (S_5, (2111, 5), (4, 1))$ the Hurwitz-Belyi map $\pi_{h,u}$ is full and an equation is given in §4.1 there. This Hurwitz-Belyi map reappears in Table 10.1 here. For $h = (SL_3(2), (22111, 421), (4, 1))$ the degree is 192. After quotienting by the natural action of $\mathrm{Out}(SL_3(2))$, one gets a full degree 96 map with equation given in [21, §8.2].

10.2. **A table of 3-2-∞ maps from $T = A_5$.** We begin with the smallest nonabelian simple group $T = A_5$ and build our Hurwitz parameters from $G \in \{A_5, S_5\}$. Table 10.1 gives all Hurwitz-Belyi maps $\pi_{h,u}^* : \mathsf{X}_{h,u}^* \to \mathsf{P}_j^1$ with $h = (G, (C_1, C_2), (4, 1))$ and u the Belyi pencil (10.1). The complications described at the end of §3.3 arising in the passage from $(3, 1)$ to (4) do not arise when one passes from $(4, 1)$ to (5). Accordingly, Table 10.1 also includes cases of the form $h = (G, (C_1), (5))$, written on the table as $h = (G, (C_1, C_1), (4, 1))$. Otherwise, Table 10.1 has a format very similar to the first two tables in each of Sections 8 and 9.

There is one instance of cross-parameter agreement: the Belyi map for $(A_5, (5a), (5))$ and $(A_5, (221), (5))$ are isomorphic; this Belyi maps occurs for a third time in the next section, where we get an equation for it. Spin separation is near generic as follows. If (C_1, C_2) contains either 221 or 2111, then the Belyi cover $\mathsf{X}_{h,u}^*$ is always connected. Otherwise both C_1 and C_2 split in the Schur double cover and one has the spin separation $\mathsf{X}_{h,u} = \mathsf{X}_{h,u}^{*+} \coprod \mathsf{X}_{h,u}^{*-}$. In all cases $\mathsf{X}_{h,u}^{*\epsilon}$ has one component except that $\mathsf{X}_{h,u}^{*+}$ is empty for $(C_1, C_2) = (5a, 5a)$ and $\mathsf{X}_{h,u}^{*-}$ has two components for $(C_1, C_2) = (5a, 5b)$.

Several patterns in Table 10.1 merit comments. First, the first two monodromy groups under the header M are odd, being S_3 and $H_{9a} = 9T13$. However, from the left half of Table 10.1, they arise together as an even intransitive dodecic group. With this packaging, all monodromy groups are even, including $H_{9b} = 9T11$. Second, just like in all the tables in the previous two sections, the exponent on 1 in β_0 is always very small; however, in contrast to these previous tables, the exponent on 1 in β_1 is not always small. Finally, a phenomenon present in the tables of the previous two sections is more visible here because of the different organization: the general nature of β_∞ depends on whether G is A_5 or S_5.

10.3. **Two unexpectedly similar 3-2-∞ maps built from $T = A_6$.** Consider the two Hurwitz parameters on the left:

  $h_{96} = (A_6, (3111), (5))$, $(\beta_0, \beta_1, \beta_\infty) = (3^{36}, \, 2^{44} \, 1^8, \, 15^3 \, 9^3 \, 5 \, 3^6 \, 1)$,

$C_1\backslash C_2$	221	311	5b
221	96	135	150
311	288	126 216	225 300
5a	64	9b 108	45 3b, 9a
5b			0 96

M	g_X	β_0	β_1	β_∞
S_{3b}	0	3	2 1	2 1
H_{9a}	0	3^3	$2^3\,1^3$	6 3
H_{9b}	0	3^3	$2^4\,1$	6 2 1
A_{45}	0	3^{15}	$2^{22}\,1$	$9^3\,6\,3^3\,2\,1^2$
A_{64}	1	$3^{21}1$	2^{32}	$15\,9^3\,6^2\,5\,2^2\,1$
A_{96}	0	3^{32}	$2^{44}\,1^8$	$15^3\,9^3\,5\,3^6\,1$
A_{108}	1	3^{36}	$2^{48}\,1^{12}$	$15^4\,9^3\,6\,5^2\,3\,2$
A_{126}	0	3^{42}	$2^{58}\,1^{10}$	$15^3\,9^2\,6^8\,5\,3^3\,1$
A_{135}	0	3^{45}	$2^{60}\,1^{15}$	$15^3\,9^6\,6^4\,5\,3^2\,1$
A_{150}	1	3^{50}	$2^{70}\,1^{10}$	$15^3\,9^6\,6^6\,5\,3^2\,2^2$
A_{216}	3	3^{72}	$2^{102}\,1^{12}$	$15^5\,9^9\,6^8\,3^4$
A_{225}	7	3^{75}	$2^{112}\,1^1$	$15^8\,9^4\,6^{10}\,5\,2^2$
A_{288}	7	3^{96}	$2^{140}\,1^8$	$15^8\,9^{11}\,6^{10}\,5\,2^2$
A_{300}	4	3^{100}	$2^{140}\,1^{20}$	$15^8\,9^{12}\,6^{10}\,5\,3\,2^2$
A_{25}	0	3^81	$2^{10}\,1^5$	12 9 4
A_{192}	1	3^{64}	$2^{88}\,1^{16}$	$18^2\,12^{10}\,4^2\,3^9\,1$
A_{225}	7	3^{75}	$2^{112}\,1^1$	$18^2 12^{10} 6^{10} 4^2 1^1$
A_{255}	7	3^{85}	$2^{122}\,1^{11}$	$18^6\,12^{10}\,4^2\,3^6\,1$
A_{300}	6	3^{100}	$2^{140}\,1^{20}$	$18^6\,12^{10}\,6^{10}\,4^2\,3\,1$
A_{336}	9	3^{112}	$2^{160}\,1^{16}$	$18^8\,12^{10}\,6^{10}\,4^2\,3\,1$
A_{1089}	34	3^{363}	$2^{528}\,1^{33}$	$18^{24}\,12^{37}\,6^{34}\,4\,2^2\,1$
A_{1107}	36	3^{369}	$2^{540}\,1^{27}$	$18^{24}\,12^{38}\,6^{34}\,4^2\,3\,2^2$
A_{1225}	46	$3^{408}1$	$2^{612}\,1$	$18^{24}\,12^{47}\,6^{35}\,4^3\,2^3\,1$
A_{1275}	40	3^{425}	$2^{620}\,1^{35}$	$18^{24}\,12^{51}\,6^{35}\,4^3\,3\,2^3$
A_{1440}	40	3^{480}	$2^{704}\,1^{32}$	$18^{24}\,12^{60}\,6^{34}\,4^4\,3^{21}\,2^2\,1$

$C_1\backslash C_2$	221	311	5
21111	0	0	25
41	1440	1107 1089	1275 1225
32	336	255 192	225 300

TABLE 10.1. Invariants of $\pi_{h,u}$ for $h = (G, (C_1, C_2), (4,1))$ and u the five-point pencil (10.1). Top: $G = A_5$ Bottom: $G = S_5$

$$h_{192} = (A_6, (3111,\ 2211), (4,1)), \quad (\beta_0, \beta_1, \beta_\infty) = (3^{64},\ 2^{84}\,1^{24},\ 15^3\,12^5\,9^3\,6^8\,5\,4\,3).$$

These cases are amenable to a standard calculation because the five-point covers $\mathsf{Y}_x \to \mathsf{P}^1$ all have genus zero. A mass formula calculation says that the two parameters have their indicated degrees.

Since the Y_x have genus zero, Serre's theorem (8.10) applies and the degree 96 cover $\mathsf{X}_{96}^* := \mathsf{X}_{h_{96},u}^*$ does not exhibit spin separation, as $\mathsf{X}_{96}^* = \mathsf{X}_{96}^{*-}$. The braid monodromy computation using (10.2) shows that in fact the monodromy group is A_{96}. The braid partition triple is as indicated above, and so X_{96}^* also has genus zero. The standard computation eventually yields $f_{15^3,9^3,5,3^6,1}(j,x) =$

$$\left(3x^8 - 6x^7 - 60x^6 + 202x^5 - 110x^4 - 74x^3 - 52x^2 - 10x - 1\right)^3$$
$$\left(729x^{24} - 10206x^{23} + 15552x^{22} - 2045790x^{21} + 52397442x^{20} - 543319218x^{19}\right.$$
$$+3209261832x^{18} - 12210163074x^{17} + 31525143435x^{16} - 55955395164x^{15}$$
$$+66094935696x^{14} - 43882703964x^{13} - 2654708692x^{12} + 42096515820x^{11}$$
$$-51857004992x^{10} + 37353393228x^9 - 17942013057x^8 + 5711207034x^7$$
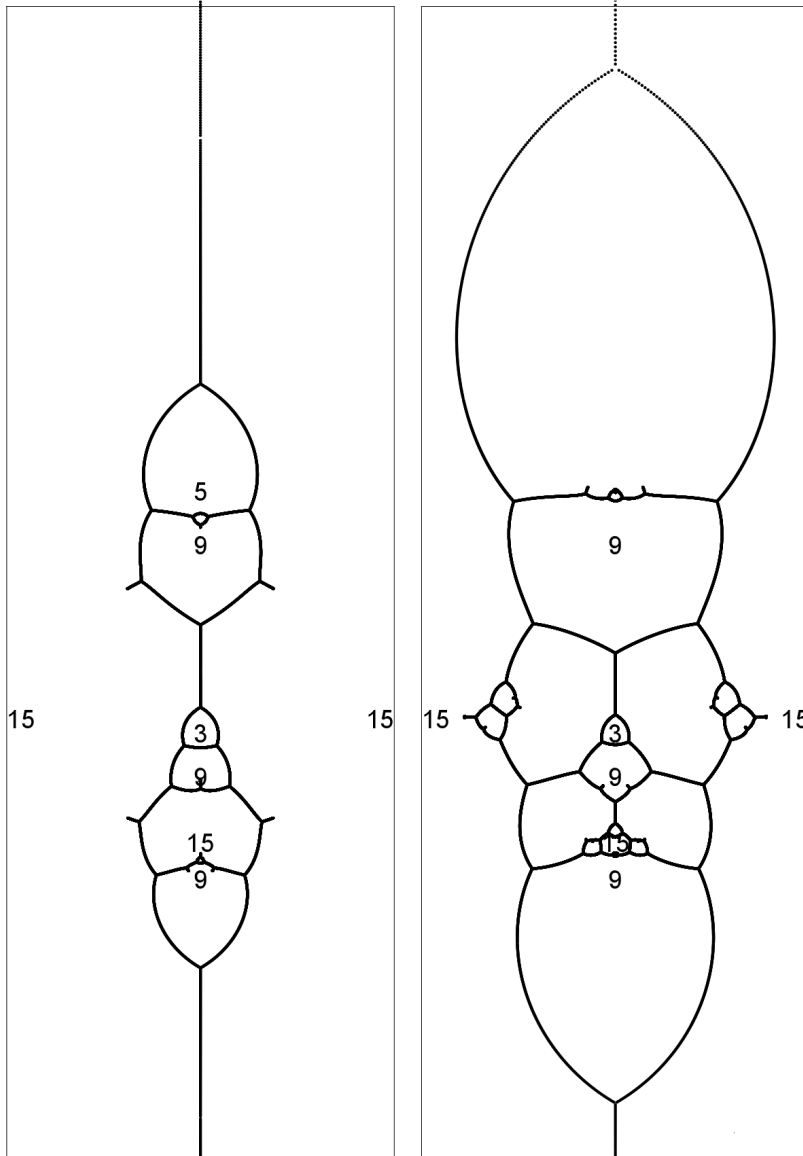
FIGURE 10.1. Dessins in $\mathsf{X}^*_{h,u}$ for $h = (A_6, (3111), (5))$ on the left and $h = (A_6, (3111, 2211), (4, 1))$ on the right, illustrating the common locations of the three 9's and the three 15's. The real axis runs vertically through the center of both pictures.

$$-1071984720x^6 + 65222394x^5 + 12734514x^4 - 1277306x^3 - 182088x^2$$
$$-3850x - 3)^3$$
$$+ 2^{10}j \left[3x^3 - 7x^2 + 11x - 1\right]^{15} \left[3x^3 - 9x^2 + 3x + 1\right]^9 [x - 3]^5$$

$$\left(x^4 + 8x^3 - 36x^2 + 17x + 1\right)^3 \left[x - 1\right]^3 \left[x\right].$$

The case $h = h_{192}$ has monodromy group A_{192}, braid partition triple as above, and genus zero. There is a remarkable and unexpected similarity between the coefficients of j in the two defining equations, symbolized by

$$15_A^3 \, 9_B^3 \, 5_3 \, 3^4 \, 3 \, 3_1 \, 1_0 \sim 15_A^3 \, 12^5 \, 9_B^3 \, 6^8 \, 5_3 \, 4_0 \, 3_1.$$

Here the subscripts 3, 1, and 0 indicate that we are normalizing so that the coordinates induced on X_{96}^* and X_{192}^* have some similarity. The unexpected similarity is that the cubic polynomials corresponding to the two A's coincide and likewise the cubic polynomials corresponding to the two B's coincide. All these agreeing factors are bracketed in the two displayed polynomials. The second polynomial is too large to print, but an excerpt containing the part relevant for the current discussion is $f_{15^3, 12^5, 9^3, 6^8, 5, 4, 3}(j, x) =$

$$\left(14659268544 x^{64} - 1012884030720 x^{63} + 33879848424192 x^{62} + \cdots\right.$$

$$\left. + 40857490944 x^5 - 1245316608 x^4 + 28200960 x^3 - 569088 x^2 + 11008 x - 64\right)^3$$

$$- 2^4 3^6 j \left[3x^3 - 7x^2 + 11x - 1\right]^{15} \left(6x^5 - 36x^4 + 72x^3 - 64x^2 + 23x - 4\right)^{12}$$

$$\left[3x^3 - 9x^2 + 3x + 1\right]^9$$

$$\left(9x^8 - 72x^7 + 240x^6 - 444x^5 + 474x^4 - 280x^3 + 72x^2 - 12x + 1\right)^6$$

$$\left[x - 3\right]^5 \left[x\right]^4 \left[x - 1\right]^3.$$

Our situation presents many challenges. For example, we have not worked out equations for the four covers of largest degree on Table 10.1 with $g_X = 0$. From the degrees given in the table, 45, 96, 126, and 135, the last three are certainly beyond current implementations of the braid-triple method. However, if some part of these equations could be determined ahead of time, perhaps by understanding better how parts of $f_{96}(j, x)$ repeat in $f_{192}(j, x)$ as just discussed, these computations might be brought into the range of feasibility.

As a second example of a challenge, it would be interesting to build analogs of Table 10.1 both for other simple groups T and other Belyi pencils u. The braid monodromy programs described in [9] would allow one to go quite far. For example, consider the Hurwitz parameter $h = (S_6, (6, 51), (4, 1))$. Both classes split in the double cover $\tilde{S}_6$, so one has a decomposition $\mathsf{X}_{h,u} = \mathsf{X}_{h,u}^+ \coprod \mathsf{X}_{h,u}^-$. The mass formula applied to the group $\tilde{S}_6$ says that the degrees are 49275 and 65400 respectively. Magaard has verified that indeed both $\mathsf{X}_{h,u}^\epsilon$ are full over P_j^1, with monodromy groups A_{49275} and A_{65400}.

11. Expectations in large degree

In [22] with Venkatesh and then in the sequel [21], we formulated and supported an unboundedness conjecture for number fields. This final section transposes these considerations from number fields to Belyi maps, with emphasis on phenomena particular to the Belyi map setting.

11.1. **Full Belyi maps with at most two bad primes.** Consider Belyi maps defined over $\mathbb{Q}$ with bad reduction within a given set of primes $\mathcal{P}$. For any prime p and any exponent k, it is elementary to get 3^k different degree p^k such Belyi maps $\mathsf{P}^1 \to \mathsf{P}^1$ with monodromy group a p-group and bad reduction set $\{p\}$ [17].

For any two distinct primes p, ℓ and certain k, mod ℓ-reductions of hypergeometric monodromy representations give degree $(\ell^{2k}-1)/(\ell-1)$ Belyi maps with primitive monodromy group $PSp_{2k}(\ell)$ and bad reduction set $\{p, \ell\}$. Fixing $\{p, \ell\}$, the number of such Belyi maps for a given k can be arbitrarily large.

In contrast, it seems very difficult to construct full Belyi maps defined over $\mathbb{Q}$ with bad reduction within a two-element set $\mathcal{P}$. Returning to the inverse problem of §1.3, write $B_{\mathcal{P}}(m)$ for the number of isomorphism classes of full Belyi maps defined over $\mathbb{Q}$ with bad reduction within $\mathcal{P}$. If π contributes to $B_{\mathcal{P}}(m)$, then typically the compositions $\sigma \circ \pi$ for $\sigma \in \langle t \mapsto 1 - t, t \mapsto 1/t \rangle = \mathrm{Sym}(\{0, 1, \infty\})$ all contribute separately, so in a sense the numbers $B_{\mathcal{P}}(m)$ are inflated by a factor of six. However the $B_{\mathcal{P}}(m)$ enter the unboundedness conjecture below only in a qualitative way, and so this duplication is not important to us.

To provide context for the unboundedness conjecture and support the discussion afterwards, we summarize here what we know about the numbers $B_{\mathcal{P}}(m)$ for $|\mathcal{P}| \le 2$. The trinomial equation $y^k - kty + (k-1)t = 0$ gives a Belyi map ramified exactly at the set $\mathcal{P}_k$ of prime divisors of $k(k-1)$. Thus, as an interesting example, $\mathcal{P}_9 = \{2, 3\}$. Otherwise one has only the possibilities involving Mersenne primes $M_r = 2^r - 1$ and the Fermat primes $F_r = 2^{2^r} + 1$, namely $(k-1, k) = (M_r, 2^r)$ and $(k-1, k) = (2^{2^r}, F_r)$. In [15], we are giving two more sequences of covers $T_{k-1,k}$ and $U_{k-1,k}$, also ramified exactly at $\mathcal{P}_k$. Degrees are now larger, being $k(k-1)/2$ and $(k-1)^2$ respectively. Our initial degree 64 example (1.1) is $U_{8,9}$.

From [14] we know also that $B_{\{2,3\}}(m)$ is positive for $m \in \{28, 33\}$. Otherwise we do not currently know of any instances with $|\mathcal{P}| \le 2$ and $m \ge 20$ with $B_{\mathcal{P}}(m)$ positive beyond the three sequences just described.

11.2. **An unboundedness conjecture.** The following conjecture is a direct analog of Conjecture 1.1 of [21]:

Conjecture 11.1. *Let $B_{\mathcal{P}}(m)$ be the number of full degree m Belyi maps defined over $\mathbb{Q}$ with bad reduction within $\mathcal{P}$. Suppose that $\mathcal{P}$ contains the set of primes dividing the order of a finite nonabelian simple group. Then the numbers $B_{\mathcal{P}}(m)$ can be arbitrarily large.*

Our heuristic argument for Conjecture (11.1) is essentially the same as the argument made in [22] and [21] for its number field analog. Namely we expect that Hurwitz-Belyi maps $\pi_{h,u}$ already give enough maps to make $B_{\mathcal{P}}(m)$ arbitrarily large.

In more detail, given $\mathcal{P}$ as in the conjecture, there is at least one nonabelian finite simple group T with $\mathcal{P}_T \subseteq \mathcal{P}$. From Hurwitz parameters $h = (G, C, \nu)$, with G of the form $T^k.A$ as in [22, §5.1], supplemented if necessary by rational lifting invariants ℓ, there are infinitely many full covers $\mathsf{Hur}_h^{*\ell} \to \mathsf{Conf}_\nu$ defined over $\mathbb{Q}$ with bad reduction within $\mathcal{P}$. From [19, §8] or [17], there are infinitely many appropriately matching rational Belyi pencils, even with bad reduction set consisting of a single prime. For Conjecture 11.1 to be false, there would be have to be a systematic drop from fullness when one specializes from the full family to the Belyi pencil. We have seen occasional drops from fullness in [21, §6] and on some of the tables in §8-10 here. However these seem to represent a low degree phenomenon, and there is no evidence of systematic drops in asymptotically large degrees.

We have already noted an important difference between Hurwitz number fields and Hurwitz-Belyi maps in §3.2. Namely for the former, the specialization step is arithmetic, as the ground field becomes $\mathbb{Q}$, but for the latter, the specialization step stays within geometry, as the ground field becomes only $\mathbb{C}(v)$. In particular, it seems to us that Conjecture 11.1 is more within reach than its analog, as it may be possible to prove it using braid groups.

11.3. **Complements.** To conclude very speculatively, say that $\mathcal{P}$ is *anabelian* if it contains the set of primes dividing the order of a finite nonabelian simple group, and *abelian* otherwise. This terminology seems appropriate to us because we suspect that there are connections between the material in this paper and investigations into anabelian geometry as defined in [2].

Conjecture 11.1 gives a partial qualitative response to the inverse problem set up in §1.3. One could ask for a more complete qualitative response. A guess we find attractive is

- If $\mathcal{P}$ is abelian, then $B_{\mathcal{P}}(m)$ is eventually zero.
- If $\mathcal{P}$ is anabelian, then $B_{\mathcal{P}}(m)$ is unbounded because of Hurwitz-Belyi maps, but still zero for m in a set of density one.

We put forward the analogous guess for number fields in [21, §4.6].

The first bulleted statement is supported by the extreme paucity of known Belyi maps contributing to $B_{\{p,\ell\}}(m)$, as reported in §11.1. The second part of the second bulleted statement is motivated by the exponential dependence of the asymptotic mass formula [22, (3-7)] on the multiplicities ν_i. Evidence either supporting or opposing this vision would be most welcome.

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