A DATABASE OF NUMBER FIELDS

JOHN W. JONES AND DAVID P. ROBERTS

Abstract. We describe an online database of number fields which accompanies this paper. The database centers on complete lists of number fields with prescribed invariants. Our description here focuses on summarizing tables and connections to theoretical issues of current interest.

1. Introduction

A natural computational problem is to completely determine the set $\mathcal{K}(G, D)$ of all degree n number fields K with a given Galois group $G \subseteq S_n$ and a given discriminant D. Many papers have solved instances of this problem, some relatively early contributions being [Hun57, Poh82, BMO90, SPDyD94].

This paper describes our online database of number fields at

`http://hobbes.la.asu.edu/NFDB/` .

This database gives many complete determinations of $\mathcal{K}(G, D)$ in small degrees n, collecting previous results and going well beyond them. Our database complements the Klüners-Malle online database [KM01], which covers more groups and signatures, but is not as focused on completeness results and the behavior of primes. Like the Klüners-Malle database, our database is searchable and intralinked.

Section 2 explains in practical terms how one can use the database. Section 3 explains some of the internal workings of the database, including how it keeps track of completeness. Section 4 presents tables summarizing the contents of the database in degrees $n \leq 11$, which is the setting of most of our completeness results. The section also briefly indicates how fields are chosen for inclusion on the database and describes connections with previous work.

The remaining sections each summarize an aspect of the database, and explain how the tabulated fields shed some light on theoretical issues of current interest. As a matter of terminology, we incorporate the signature of a field into our notion of discriminant, considering the formal product $D = -^s|D|$ to be the discriminant of a field with s complex places and absolute discriminant $|D|$.

Section 5 focuses on the complete list of all 11279 quintic fields with Galois group $G = S_5$ and discriminant of the form $-^s2^a3^b5^c7^d$. The summarizing table here shows that the distribution of discriminants conforms moderately well to the mass heuristic of [Bha07]. Section 6 summarizes lists of fields for more nonsolvable groups, but now with attention restricted to discriminants of the form $-^sp^aq^b$ with $p < q$ primes.

2010 *Mathematics Subject Classification.* 11R21, 11R32.

This work was partially supported by a grant from the Simons Foundation (#209472 to David Roberts).

Sections 7 and 8 continue to pursue cases with $D = -^{s}p^{a}q^{b}$, but now for octic groups G of 2-power order. Section 7 treats the cases $p > 2$ and discusses connections to tame maximal nilpotent extensions as studied in [BE11, BP00]. Section 8 treats the case $p = 2$ and takes a first step towards understanding wild ramification in some of the nilpotent extensions studied in [Koc02].

Sections 9 and 10 illustrate progress in the database on a large project initiated in [JR07]. The project is to completely classify Galois number fields with root discriminant $|D|^{1/n}$ at most the Serre-Odlyzko constant $\Omega := 8\pi e^{\gamma} \approx 44.76$. Upper bounds on degrees coming from analysis of Dedekind zeta functions [Mar82, Odl90] play a prominent role. The database gives many solvable fields satisfying the root discriminant bound. In this paper for brevity we restrict attention to nonsolvable fields, where, among other interesting things, modular forms [Bos07], [Sch12] sometimes point the way to explicit polynomials.

The database we are presenting here has its origin in posted versions of the complete tables of our earlier work [JR99]. Other complete lists of fields were posted sporadically in the next ten years, while most fields and the new interface are recent additions. Results from the predecessors of the present database have occasionally been used as ingredients of formal arguments, as in e.g. [HKS06, OT05, Dah08]. The more common use of our computational results has been to guide investigations into number fields in a more general way. With our recent enhancements and this accompanying paper, we aim to increase the usefulness of our work to the mathematical community.

2. Using the database

A simple way to use the database is to request $\mathcal{K}(G, D)$, for a particular (G, D). A related but more common way is to request the union of these sets for varying G and/or D. Implicit throughout this paper and the database is that fields are always considered up to isomorphism. As a very simple example, asking for quartic fields with any Galois group G and discriminant D satisfying $|D| \leq 250$ returns Table 2.1.

Results below are proven complete					
rd(K)	grd(K)	D	h	G	Polynomial
3.29	6.24	$-^{2}3^{2}13^{1}$	1	D_4	$x^4 - x^3 - x^2 + x + 1$
3.34	3.34	$-^{2}5^{3}$	1	C_4	$x^4 - x^3 + x^2 - x + 1$
3.46	3.46	$-^{2}2^{4}3^{2}$	1	V_4	$x^4 - x^2 + 1$
3.71	6.03	$-^{2}3^{3}7^{1}$	1	D_4	$x^4 - x^3 + 2x + 1$
3.87	3.87	$-^{2}3^{2}5^{2}$	1	V_4	$x^4 - x^3 + 2x^2 + x + 1$
3.89	15.13	$-^{2}229^{1}$	1	S_4	$x^4 - x + 1$

Table 2.1. Results of a query for quartic fields with absolute discriminant ≤ 250, sorted by root discriminant

In general, the monic polynomial $f(x) \in \mathbb{Z}[x]$ in the last column defines the field of its line, via $K = \mathbb{Q}[x]/f(x)$. It is standardized by requiring the sum of the absolute squares of its complex roots to be minimal, with ties broken according to the conventions of Pari's *polredabs*. Note that the database, like its local analog

[JR06], is organized around non-Galois fields. However, on a given line, some of the information refers to a Galois closure K^g.

The Galois group $G = \mathrm{Gal}(K^g/\mathbb{Q})$ is given by its common name, like in Table 2.1, or its T-name as in [BM83, PAR13, GAP06] if it does not have a very widely accepted common name. Information about the group—essential for intelligibility in higher degrees—is obtainable by clicking on the group. For example, the database reports $10T42$ as having structure $A_5^2.4$, hence order $60^2 4 = 14400$; moreover, it is isomorphic to $12T278$, $20T457$, and $20T461$.

Continuing to explain Table 2.1, the column D prints $-^s|D|$, where s is the number of complex places and $|D|$ is given in factored form. This format treats the infinite completion $\mathbb{Q}_\infty = \mathbb{R}$ on a parallel footing with the p-adic completions $\mathbb{Q}_p$. If $n \leq 11$, then clicking on any appearing prime p links into the local database of [JR06], thereby giving a detailed description of the p-adic algebra $K_p = \mathbb{Q}_p[x]/f(x)$. This automatic p-adic analysis also often works in degrees $n > 11$.

The root discriminant $\mathrm{rd}(K) = |D|^{1/n}$ is placed in the first column, since one commonly wants to sort by root discriminant. Here and later we often round real numbers to the nearest hundredth without further comment. When it is implemented, our complete analysis at all ramifying primes p automatically determines the Galois root discriminant of K, meaning the root discriminant of a Galois closure K^g. The second column gives this more subtle invariant $\mathrm{grd}(K)$. Clicking on the entry gives the exact form and its source. Often it is better to sort by this column, as fields with the same Galois closure are then put next to each other. As an example, quartic fields with $G = D_4$ come in twin pairs with the same Galois closure. The twin K^t of the first listed field K on Table 2.1 is off the table because $|D(K^t)| = 3^1 13^2 = 504$; however $\mathrm{grd}(K^t) = \mathrm{grd}(K) = 3^{1/2}13^{1/2} \approx 6.24$.

Class numbers are given in the column h, factored as $h_1 \cdots h_d$ where the class group is a product of cyclic groups of size h_i. There is a toggle button, so that one can alternatively receive narrow class numbers in the same format. To speed up the construction of the table, class numbers were computing assuming the generalized Riemann hypothesis; they constitute the only part of the database that is conditional. Theoretical facts about class groups can be seen repeatedly in various parts of the database. For example, let n be an odd positive integer and consider a degree n field K with dihedral Galois group D_n. Let L be its Galois closure with unique quadratic subfield F. Let p be a prime not dividing $2n$ and consider the p-parts of all class groups in question. Then, decomposing via the natural D_n action on $\mathrm{Cl}_p(L)$ and using the triviality of $\mathrm{Cl}_p(\mathbb{Q})$, one gets

$$\mathrm{Cl}_p(L) \cong \mathrm{Cl}_p(K)^2 \times \mathrm{Cl}_p(F). \tag{2.1}$$

One explicit example comes from the unique D_7 field K with Galois root discriminant $1987^{1/2} \cong 44.58$. Illustrating (2.1), the database reports $\mathrm{Cl}(L) = 13 \cdot 13$, $\mathrm{Cl}(K) = 13$, and $\mathrm{Cl}(F) = 7$.

When the response to a query is known to be complete, then the table is headed by the completeness statement shown in Table 2.1. As emphasized in the introduction, keeping track of completeness is one of the most important features of the database. The completeness statement often reflects a very long computational proof, even if the table returned is very short.

There are many other ways to search the database, mostly connected to the behavior of primes. For example, one can restrict the search to find fields with restrictions on $\mathrm{ord}_p(D)$, or one can search directly for fields with Galois root discriminant in a given range. On the other hand, there are some standard invariants of fields that the database does not return, such as Frobenius partitions and regulators. The database does allow users to download the list of polynomials returned, so that it can be used as a starting point for further investigation.

3. Internal structure

The website needs to be able to search and access a large amount of information. It uses a fairly standard architecture: data is stored in a MySQL database and web pages are generated by programs written in Perl.

A MySQL database consists of a collection of tables where each table is analogous to a single spreadsheet with columns representing the types of data being stored. We use data types for integers, floating point numbers, and strings, all of which come in various sizes, i.e., amount of memory devoted to a single entry. When searching, one can use equalities and inequalities where strings are ordered lexicographically.

When a user requests number fields, the Perl program takes the following steps:

1. Construct and execute a MySQL query to pull fields from the database.
2. Filter out fields which satisfy all of the user's requirements when needed (see below).
3. Check completeness results known to the database.
4. Generate the output web page.

The main MySQL table has one row for each field. There are columns for each piece of information indicated by the input boxes in the top portion of the search screen, plus columns for the defining polynomial (as a string), and an internal identifier for the field. The only unusual aspect of this portion of the database is how discriminants are stored and searched. The difficulty stems from the fact that many number fields in the database have discriminants which are too large to store in MySQL as integers. An option would be to store the discriminants as strings, but then it would be difficult to search for ranges: string comparisons in MySQL are lexicographic, so '11' comes before '4'. Our solution is to store absolute discriminants $|D|$ as strings, but prepend the string with four digits which give $\lfloor \log_{10}(|D|) \rfloor$, padded on the left with zeros as needed. So, 4 is stored as '00004', 11 is '000111', etc. This way we can use strings to store each discriminant in its entirety, but searches for ranges work correctly.

The MySQL table of number fields also has a column for the list of all primes which ramify in the field, stored as a string with a separator between primes. This is used to accelerate searches when it is clear from the search criteria that only a small finite list of possibilities can occur, for example, when the user has checked the box that "Only listed primes can ramify".

Information on ramification of specific primes can be input in the bottom half of the search inputs. To aid in searches involving these inputs, we have a second MySQL table, the ramification table, which stores a list of triples. A triple $(\text{field identifier}, p, e)$ indicates that p^e exactly divides the discriminant of the corresponding field. The most common inputs to the bottom half of the search page work well with this table, namely those which list specific primes and allowable discriminant exponents. However, the search boxes allow much more general inputs, i.e., where a range of values is allowed for the prime and the discriminant exponent allows both 0 and positive values. It is possible to construct MySQL queries for inputs of this sort, but they are complicated, involve subqueries, and are relatively slow. Moreover, a search condition of this type typically rules out relatively few number fields. If a user does make such a query, we do not use the information at this stage. Instead, we invoke Step (2) above to select fields from the MySQL query which satisfy these additional requirements.

The database supports a variety of different types of completeness results. Complicating matters is that these results can be interrelated. We use four MySQL tables for storing ways in which the data is complete. In describing them, G denotes the Galois group of a field, n is the degree, s is the number of complex places, and $|D|$ is the absolute discriminant, as above. The tables are

A. store (n, s, B) to indicate that the database is complete for fields with the given n and s such that $|D| \leq B$;
B. store (n, s, G, B) to indicate that the database is complete for fields with the given n, s, and G such that $|D| \leq B$;
C. store (n, S, L) where S is a list of primes and L is a list of Galois groups to indicate that the database is complete for degree n fields unramified outside S for each Galois group in L;
D. store (n, G, B) to indicate that the database is complete for degree n fields K with Galois group G such that $\mathrm{grd}(K) \leq B$.

In each case, database entries include the degree, so individual Galois groups can be stored by their T-number (a small integer). In the third case, we store the list L by an integer whose bits indicate which T-numbers are included in the set. For example, there are 50 T-numbers in degree 8, so a list of Galois groups in that degree is a subset of $S \subseteq \{1, \ldots, 50\}$ which we represent by the integer $\sum_{t \in S} 2^{t-1}$. These integers are too large to store in the database as integers, so they are stored as strings, and converted to multiprecision integers in Perl. The list of primes in the third table is simply stored as a string consisting of the primes and separating characters.

To start checking for completeness, we first check that there are only finitely many degrees involved, and that the search request contains an upper bound on at least one of: $|D|$, $\mathrm{rd}(K)$, $\mathrm{grd}(K)$, or the largest ramifying prime. We then loop over the degrees in the user's search. We allow for the possibility that a search is known to be complete by some combination of completeness criteria. So throughout the check, we maintain a list of Galois groups which need to be checked, and the discriminant values to check. If one check shows that some of the Galois groups for the search are known to be complete, they are removed from the list. If that list drops to being empty, then the search in that degree is known to be complete. Discriminant values are treated analogously.

For each degree, bounds on $|D|$ and $\mathrm{rd}(K)$ are clearly equivalent. Less obviously, bounds between $\mathrm{rd}(K)$ and $\mathrm{grd}(K)$ are related. In particular, we always have $\mathrm{rd}(K) \leq \mathrm{grd}(K)$, but also have for each Galois group, $\mathrm{grd}(K) \leq \mathrm{rd}(K)^{\alpha(G)}$ where $\alpha(G)$ is a rational number depending only on G (see [JRb]).

We then perform the following checks.

- We compare the request with Tables A, B, and D for discriminant bound restrictions.
- Remove Galois groups from the list to be checked based on grd.
- If there are at most ten discriminants not accounted for, check each individually against Table C.

- If there is a bound on the set of ramifying primes, which could arise from the user checking "Only these primes ramify", or from a bound on the maximum ramifying prime, check Table C.

4. Summarizing tables

The tables of this section summarize all fields in the database of degree ≤ 11. Numbers on tables which are known to be correct are given in regular type. Numbers which are merely the bounds which come from perhaps incomplete lists of fields are given in italics. The table has a line for each group nTj, sorted by degree n and the index j. A more descriptive name is given in the second column.

Degree 2									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\|\mathcal{K}[G,\Omega]\|$	Tot
1	2	7	7	3	15	1.73	1.73	1220	1216009

Degree 3									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\|\mathcal{K}[G,\Omega]\|$	Tot
1	3	1	0	1	1	3.66	3.66	47	1004
2	S_3	8	1	5	31	2.84	4.80	610	856373

Degree 4									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\|\mathcal{K}[G,\Omega]\|$	Tot
1	4	4	12	2	24	3.34	3.34	228	9950
2	2^2	7	7	1	35	3.46	3.46	2421	52469
3	D_4	28	24	0	176	3.29	6.03	2850	764341
4	A_4	1	0	0	1	7.48	10.35	59	28786
5	S_4	22	3	1	143	3.89	13.56	527	281089

Degree 5									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\|\mathcal{K}[G,\Omega]\|$	Tot
1	5	0	1	1	1	6.81	6.81	7	181
2	D_5	0	4	2	8	4.66	6.86	146	11516
3	F_5	1	19	7	82	8.11	11.08	102	1645
4	A_5	0	5	6	62	7.14	18.70	78	95337
5	S_5	5	38	22	1353	4.38	24.18	192	598542

The next four columns represent a main focus of the database, complete lists of fields ramified within a given set of primes. As a matter of notation, we write e.g. $\mathcal{K}(G, -^*p^*q^*)$ to denote the union of all $\mathcal{K}(G, -^s p^a q^b)$. The database contains completeness results for many other prime combinations beyond those given in the table; §5-§8 give examples of these further completeness results.

The next column gives minimal values of root discriminants. More refined minima can easily be obtained from the database. For example, for S_5, minimal discriminants for $s = 0$, 1, and 2 complex places are respectively $(61 \cdot 131)^{1/5} \approx 7.53$, $(13 \cdot 347)^{1/5} \approx 5.38$, and $1609^{1/5} \approx 4.38$. Completeness is typically known well past the minimum.

In understanding root discriminants, the Serre-Odlyzko constant $\Omega = 8\pi e^\gamma \approx 44.76$ mentioned in the introduction plays an important role as follows. First, if K

Degree 6									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	$\mathrm{rd}(K)$	$\mathrm{grd}(K)$	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	6	7	0	3	15	5.06	5.06	399	5291
2	S_3	8	1	5	31	4.80	4.80	610	8353
3	D_6	48	6	10	434	4.93	8.06	3590	147965
4	A_4	1	0	0	1	7.32	10.35	59	1357
5	$3 \wr 2$	8	0	5	31	4.62	10.06	254	2169
6	$2 \wr 3$	7	0	0	15	5.61	12.31	243	62484
7	S_4^+	22	3	1	143	5.69	13.56	527	242007
8	S_4	22	3	1	143	6.63	13.56	527	18738
9	S_3^2	22	0	4	375	7.89	15.53	445	9721
10	$3^2:4$	4	0	2	44	8.98	23.57	34	396
11	$2 \wr S_3$	132	18	2	2002	4.65	16.13	2196	323148
12	$PSL_2(5)$	0	5	6	62	8.12	18.70	78	275
13	$3^2:D_4$	50	0	0	624	4.76	21.76	274	27049
14	$PGL_2(5)$	5	38	22	1353	11.01	24.18	192	11519
15	A_6	8	2	4	*540*	8.12	31.66	10	670
16	S_6	54	30	42	*8334*	4.95	33.50	26	21594

Degree 7									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	$\mathrm{rd}(K)$	$\mathrm{grd}(K)$	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	7	0	0	0	0	17.93	17.93	4	117
2	D_7	0	0	0	0	6.21	8.43	80	496
3	$7:3$	0	0	0	0	21.03	31.64	2	56
4	$7:6$	0	0	1	5	12.10	15.99	94	189
5	$SL_3(2)$	0	0	0		7.95	32.25	*36*	618
6	A_7	0	2	3	*204*	8.74	39.52	*1*	331
7	S_7	10	24	14	*4391*	5.65	*40.49*	*1*	8357

has root discriminant $< \Omega/2$, then its maximal unramified extension K' has finite degree over K. Second, if $\mathrm{rd}(K) < \Omega$, then the generalized Riemann hypothesis implies the same conclusion $[K' : K] < \infty$. Third, suggesting that there is a modestly sharp qualitative transition associated with Ω, the field $\mathbb{Q}(e^{2\pi i/81})$ with root discriminant $3^{3.5} \approx 46.77$ has $[K' : K] = \infty$ by [Hoe09].

The next two columns of the tables again represent a main focus of the database, complete lists of fields with small Galois root discriminant. We write $\mathcal{K}[G, B]$ for the set of all fields with Galois group G and $\mathrm{grd}(K) \leq B$. The tables give first the minimal Galois root discriminant. They next give $|\mathcal{K}[G, \Omega]|$. For many groups, the database is complete for cutoffs well past Ω. For example, the set $\mathcal{K}[9T17, \Omega]$ is empty, and not adequate for the purposes of [JRa]. However the database identifies $|\mathcal{K}[9T17, 200]| = 36$ and this result is adequate for the application.

The last column in a table gives the total number of fields in the database for the given group. Note that one could easily make this number much larger in any case. For example, a regular family over $\mathbb{Q}(t)$ for each group is given in [MM99, App. 1], and one could simply specialize at many rational numbers t. However we do not do this: all the fields on our database are there only because discriminants met one criterion or another for being small. The fluctuations in this column should not be

Degree 8									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	8	4	8	0	16	11.93	11.93	23	5777
2	4×2	6	18	1	84	5.79	5.79	581	15412
3	2^3	1	1	0	15	6.93	6.93	908	10687
4	D_4	14	12	0	88	6.03	6.03	1425	24370
5	Q_8	2	0	0	8	18.24	18.24	7	778
6	D_8	20	20	0	104	6.71	9.75	708	29740
7	$8:\{1,5\}$	6	20	1	88	9.32	9.32	55	8040
8	$8:\{1,3\}$	22	10	0	120	10.09	10.46	121	10826
9	$D_4\times 2$	28	24	0	528	6.51	10.58	5908	175572
10	$2^2:4$	8	24	0	160	6.09	9.46	620	29649
11	$Q_8:2$	18	18	0	312	6.51	9.80	921	17350
12	$SL_2(3)$	0	0	0	0	12.77	29.84	4	681
13	$A_4\times 2$	7	0	0	15	8.06	12.31	243	26637
14	S_4	22	3	1	143	9.40	13.56	527	7203
15	$8:8^\times$	42	42	0	928	8.65	13.79	818	60490
16	$1/2[2^4]4$	8	24	0	176	7.45	13.56	76	15545
17	$4\wr 2$	16	72	0	480	5.79	13.37	252	42018
18	$2^2\wr 2$	24	8	0	608	7.04	16.40	2544	216411
19	$E(8):4$	8	24	0	192	9.51	14.05	220	24380
20	$[2^3]4$	4	12	0	96	8.46	14.05	110	13631
21	$1/2[2^4]E(4)$	4	12	0	96	8.72	14.05	110	10091
22	$E(8):D_4$	0	0	0	204	8.43	18.42	882	19733
23	$GL_2(3)$	128	24	4	912	8.31	16.52	388	6304
24	$S_4\times 2$	132	18	2	2002	6.04	16.13	2196	45996
25	$2^3:7$	0	0	0	0	12.50	17.93	1	20
26	$1/2[2^4]eD(4)$	64	24	0	1872	7.23	20.37	840	135840
27	$2\wr 4$	16	48	0	448	5.95	19.44	160	86501
28	$1/2[2^4]dD(4)$	16	48	0	448	8.67	19.44	160	47150
29	$E(8):D_8$	48	24	0	1296	6.58	19.41	1374	170694
30	$1/2[2^4]cD(4)$	16	48	0	448	8.25	19.44	140	48249
31	$2\wr 2^2$	16	8	0	432	5.92	19.41	458	54843
32	$[2^3]A_4$	0	0	0	0	13.56	34.97	24	29970
33	$E(8):A_4$	6	0	0	14	13.73	30.01	24	3240
34	$E(4)^2:D_6$	11	1	0	132	14.16	27.28	55	3907
35	$2\wr D(4)$	168	72	0	5568	5.83	22.91	1464	729730
36	$2^3:7:3$	0	0	0	0	14.37	31.64	4	298
37	$PSL_2(7)$	0	0			*21.00*	32.25	*18*	352
38	$2\wr A_4$	24	0	0	112	10.66	37.27	46	67160
39	$[2^3]S_4$	168	24	0	2496	6.73	32.35	84	24625
40	$1/2[2^4]S(4)$	216	24	0	3872	7.67	29.71	98	12796
41	$E(8):S_4$	90	12	0	2282	8.38	28.11	222	11929
42	$A_4\wr 2$	12	0	0	83	7.68	32.18	14	3432
43	$PGL_2(7)$	*4*			*8*	*11.96*	*27.35*	*27*	1495
44	$2\wr S_4$	656	96	0	22944	5.84	31.38	336	440683
45	$[1/2.S_4^2]2$	110	0	0	*836*	9.28	29.35	39	7732
46	$1/2[S(4)^2]2$	28	0	0	*54*	11.35	49.75	0	216
47	$S_4\wr 2$	542	0	0	*2185*	6.74	35.05	15	28765
48	$2^3:SL_3(2)$	0	0			*11.36*	*39.54*	*6*	495
49	A_8	*2*	*4*	*1*	*55*	*15.24*	*72.03*		90
50	S_8	*72*	*30*	*9*	*1728*	*11.33*	*43.99*	*1*	4026

Degree 9									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\|\mathcal{K}[G,\Omega]\|$	Tot
1	9	1	0	1	1	13.70	13.70	3	52
2	3^2	0	0	0	0	15.83	15.83	9	189
3	D_9	6	0	4	20	9.72	12.19	105	705
4	$S_3 \times 3$	8	0	5	31	8.38	10.06	254	10139
5	$3^2 : 2$	1	0	1	15	14.29	15.19	48	373
6	$1/3[3^3]3$	0	0	0	0	17.63	31.18	2	85
7	$3^2 : 3$	0	0	0	0	26.09	50.20	0	90
8	$S_3 \times S_3$	22	0	4	375	8.93	15.53	445	7055
9	$E(9) : 4$	2	0	1	22	19.92	23.57	17	142
10	$[3^2]S(3)_6$	22	0	17	171	9.57	17.01	69	1066
11	$E(9) : 6$	6	0	4	20	14.67	16.83	64	880
12	$[3^2]S(3)$	12	0	12	180	8.92	16.72	148	13929
13	$E(9) : D_6$	6	0	4	20	10.98	16.83	64	642
14	$3^2 : Q_8$	4	0	0	19	21.52	29.72	2	47
15	$E(9) : 8$	5	1	0	18	17.74	25.41	3	40
16	$E(9) : D_8$	25	0	0	312	9.19	21.76	137	434
17	$3 \wr 3$	0	0	0	0	14.93	75.41	0	1274
18	$E(9) : D_{12}$	80	0	8	1380	8.53	22.06	290	9260
19	$E(9) : 2D_8$	60	1	0	*124*	17.89	23.41	33	624
20	$3 \wr S_3$	18	0	12	60	7.83	29.89	30	7989
21	$1/2.[3^3 : 2]S_3$	54	0	54	1296	9.82	24.90	126	4282
22	$[3^3 : 2]3$	18	0	12	60	10.27	26.46	51	784
23	$E(9) : 2A_4$	0	0	0	0	16.48	49.57	0	40
24	$[3^3 : 2]S(3)$	321	0	48	8307	9.15	30.64	111	17973
25	$[1/2.S(3)^3]3$	4	0	0	4	12.89	29.96	4	303
26	$E(9) : 2S_4$	250	2	10	*362*	12.79	27.88	51	866
27	$PSL_2(8)$	*4*			*4*	*16.25*	*30.31*	*15*	19
28	$S_3 \wr 3$	28	0	0	90	8.18	33.56	7	6738
29	$[1/2.S(3)^3]S(3)$	45	0	1	512	9.38	40.81	2	1255
30	$1/2[S(3)^3]S(3)$	232	1	40	*1637*	6.86	30.37	35	5026
31	$S_3 \wr S_3$	616	0	5	19865	6.83	36.26	15	112887
32	$\Sigma L_2(8)$	*64*			*240*	*16.09*	*34.36*	*15*	1141
33	A_9	*13*		*2*	*314*	*14.17*	*62.12*		627
34	S_9	*46*	*1*	*1*	*1507*	*7.84*	*53.19*		3189

viewed as significant, as the criteria depend on the group in ways driven erratically by applications.

There are a number of patterns on the summarizing tables which hold because of relations between transitive groups. For example the groups $5T4 = A_5$, $6T12 = PSL_2(5)$, and $10T7$ are all isomorphic. Most of the corresponding lines necessarily agree. Similarly A_5 is a quotient of $10T11$, $10T34$, and $10T36$. Thus the fact that $\mathcal{K}(A_5, -^*2^*3^*) = \emptyset$ immediately implies that also $\mathcal{K}(10Tj, -^*2^*3^*) = \emptyset$ for $j \in \{11, 34, 36\}$.

Almost all fields in the database come from complete searches of number fields carried out by the authors. In a few cases, we obtained polynomials from other sources, notably for number fields of small discriminant: those compiled by the Bordeaux group [Bor], which in turn were computed by several authors, and the

Degree 10

T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	rd(K)	grd(K)	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	10	0	7	3	15	8.65	8.65	69	360
2	D_5	0	4	2	8	6.86	6.86	146	822
3	D_{10}	0	24	4	112	8.08	10.91	768	857
4	$1/2[F(5)]2$	1	19	7	82	10.23	11.08	102	178
5	$F_5 \times 2$	6	114	14	1148	9.48	14.50	584	1611
6	$[5^2]2$	0	8	4	16	6.84	18.02	32	175
7	A_5	0	5	6	62	12.35	18.70	78	146
8	$[2^4]5$	0	3	0	3	12.75	24.98	18	36
9	$[1/2.D(5)^2]2$	0	12	2	56	*12.71*	24.72	34	87
10	$1/2[D(5)^2]2$	0	22	12	126	*14.02*	24.00	22	144
11	$A_5 \times 2$	0	35	18	930	9.42	22.24	179	1177
12	$S_5(10a)$	5	38	22	1353	12.04	24.18	192	1560
13	$S_5(10d)$	5	38	22	1353	9.16	24.18	192	1712
14	$2 \wr 5$	0	21	0	45	9.32	26.08	45	2050
15	$[2^4]D(5)$	0	60	0	360	9.33	25.15	72	620
16	$1/2[2^5]D(5)$	0	60	0	360	9.46	25.15	72	509
17	$[5^2:4]2$	0	59	0	916	*17.46*	26.65	34	1094
18	$[5^2:4]2_2$	0	*16*	0	*17*	*19.75*	35.98	3	22
19	$[5^2:4_2]2$	0	17	0	*63*	*16.96*	28.08	18	111
20	$[5^2:4_2]2_2$	0	0	0	*31*	*27.36*	48.25	0	43
21	$D_5 \wr 2$	0	34	0	*118*	7.54	28.08	36	235
22	$S_5 \times 2$	30	228	44	18942	7.06	26.99	570	26851
23	$2 \wr D_5$	0	360	0	5040	7.26	26.26	240	24024
24	$[2^4]F(5)$	7	173	0	1250	14.13	27.62	30	1491
25	$1/2[2^5]F(5)$	7	173	0	1250	13.84	27.62	30	1491
26	$PSL_2(9)$	4	1	2	*270*	20.20	31.66	5	334
27	$[1/2.F_5^2]2$	0	*56*	0	*652*	*13.40*	40.43	18	1052
28	$1/2[F_5^2]2$	*1*	*24*	*2*	*57*	*15.16*	*32.71*	*8*	59
29	$2 \wr F_5$	42	1038	0	17500	11.44	32.17	90	19112
30	$PGL_2(9)$	*11*	*5*	*1*	*55*	*22.64*	*34.42*	*6*	149
31	M_{10}	*20*	*4*	*13*	*83*	*27.73*	*53.50*		198
32	S_6	27	15	21	*4166*	*14.74*	33.50	13	6913
33	$F_5 \wr 2$	0	*177*	0	*484*	9.93	*35.41*	*3*	485
34	$[2^4]A_5$	0	35	0	1322	10.82	35.81	5	1388
35	$P\Gamma L_2(9)$	*100*	*32*	*15*	*1666*	*17.98*	*38.61*	*15*	3531
36	$2 \wr A_5$	0	245	0	19830	10.39	36.60	8	20660
37	$[2^4]S_5$	91	450	8	42059	7.80	38.11	17	60029
38	$1/2[2^5]S_5$	91	450	8	42059	9.41	38.11	17	42851
39	$2 \wr S_5$	546	2700	16	588826	6.79	38.11	30	1095840
40	$A_5 \wr 2$	*12*	*63*	*29*	*1093*	9.48	*41.90*	*1*	1124
41	$[A_5:2]2$	*28*	*139*	*11*	*9435*	9.30	*43.89*	*1*	9677
42	$1/2[S_5^2]2$	*18*	*68*	*31*	*850*	*14.35*	*45.93*		882
43	$S_5 \wr 2$	*185*	*422*		*20743*	6.82	*48.97*		31847
44	A_{10}	*23*	*16*	*6*	*801*	*19.37*	*51.68*		1201
45	S_{10}	*1*	*12*	*3*	*2585*	*7.77*	*70.36*		4944

the tables of totally real fields of Voight [Voi08, Voi]. In addition, we include fields found by the authors in joint work with others [DJ10, JW12].

Degree 11									
T	G	$\{2,3\}$	$\{2,5\}$	$\{3,5\}$	$\{2,3,5\}$	$\mathrm{rd}(K)$	$\mathrm{grd}(K)$	$\lvert\mathcal{K}[G,\Omega]\rvert$	Tot
1	11	0	0	0	0	17.30	17.30	1	18
2	D_{11}	0	0	0	0	10.24	12.92	32	55
3	11 : 5	0	0	0	0	*88.82*	*105.74*	0	2
4	11 : 10					*17.01*	20.70	*4*	55
5	$PSL_2(11)$					*15.36*	*42.79*	*2*	91
6	M_{11}				*1*	*96.24*	*270.83*		10
7	A_{11}				*4*	*21.15*	*146.24*		71
8	S_{11}	*5*	*4*	*1*	*123*	*7.72*	*91.50*		931

To compute cubic fields, we used the program of Belabas [Bel04, Bel]. Otherwise, we obtain complete lists by using traditional and targeted Hunter searches [JR99, JR03] or the class field theory functions in pari/gp [PAR13]. For larger nonsolvable groups where completeness results are currently out of reach, we obtain most of our fields by specializations of families at carefully chosen points to keep ramification small in various senses.

5. S_5 QUINTICS WITH DISCRIMINANT $-^s2^a3^b5^c7^d$

One of our longest searches determined $\mathcal{K}(S_5, -^*2^*3^*5^*7^*)$, finding it to consist of 11279 fields. In this section, we consider how this set interacts with mass heuristics.

In general, mass heuristics [Bha07, Mal02] give one expectations as to the sizes $|\mathcal{K}(G,D)|$ of the sets contained in the database. Here we consider these heuristics only in the most studied case $G = S_n$. The mass of a $\mathbb{Q}_v$-algebra K_v is by definition $1/|\mathrm{Aut}(K_v)|$. Thus the mass of $\mathbb{R}^{n-2s}\mathbb{C}^s$ is

$$\mu_{n,-^s} = \frac{1}{(n-2s)!s!2^s}. \tag{5.1}$$

For p a prime, similarly let μ_{n,p^c} be the total mass of all p-adic algebras with degree n and discriminant p^c. For $n < p$, all algebras involved are tame and

$$\mu_{n,p^c} = |\{\text{Partitions of } n \text{ having } n-c \text{ parts}\}|. \tag{5.2}$$

For $n \geq p$, wild algebras are involved. General formulas for μ_{n,p^c} are given in [Rob07].

The mass heuristic says that if the discriminant $D = -^s\prod_p p^{c_p}$ in question is a non-square, then

$$|\mathcal{K}(S_n, -^s\prod_p p^{c_p})| \approx \delta_n \mu_{n,-^s}\prod_p \mu_{n,p^{c_p}}. \tag{5.3}$$

Here $\delta_n = 1/2$, except for the special cases $\delta_1 = \delta_2 = 1$ which require adjustment for simple reasons. The left side is an integer and the right side is often close to zero because of (5.1) and (5.2). So (5.3) is intended only to be used in suitable averages.

For $n \leq 5$ fixed and $|D| \to \infty$, the heuristics are exactly right on average, the case $n = 3$ being the Davenport-Heilbronn theorem and the cases $n = 4$, 5 more recent results of Bhargava [Bha05, Bha10]. For a fixed set of ramifying primes S and $n \to \infty$, the mass heuristic predicts no fields after a fairly sharp cutoff $N(S)$, while in fact there can be many fields in degrees well past this cutoff [Rob]. Thus the regime of applicability of the mass heuristic is not clear.

To get a better understanding of this regime, it is of interest to consider other limits. Let c_n be the number of elements of order ≤ 2 in S_n. Let μ_{n,p^*} be the total mass of all $\mathbb{Q}_p$-algebras of degree n. Thus μ_{n,p^*} is the number of partitions of n if $n < p$. Then, for $k \to \infty$, the mass heuristic predicts the asymptotic equivalence

$$|\mathcal{K}(S_n, -^*2^* \cdots p_k^*\})| \sim \delta_n \mu_{n,-^*} \prod_{j=1}^{k} \mu_{n,p^*}. \tag{5.4}$$

Both sides of (5.4) are 1 for all k when $n = 1$. For $n = 2$ and $k \geq 1$, the statement becomes $2^{k+1} - 1 \sim 2^{k+1}$ which is true. Using the fields in the database as a starting point, we have carried out substantial calculations suggesting that, after removing fields with discriminants of the form $-3u^2$ from the count on the left, (5.4) holds also for $n = 3$ and $n = 4$.

In this section, we focus on the first nonsolvable case, $n = 5$. For $k \geq 3$, (5.4) becomes

$$|\mathcal{K}(S_5, -^*2^* \cdots p_k^*)| \sim \frac{1}{2} \cdot \frac{26}{120} \cdot 40 \cdot 19 \cdot 27 \cdot 7^{k-3} \approx 6.48 \cdot 7^k. \tag{5.5}$$

Through the cutoff $k = 4$, there are fewer fields than predicted by the mass heuristic:

p_k	2	3	5	7
$\lvert\mathcal{K}(S_5, -^*2^* \cdots p_k^*)\rvert$	0	5	1353	11279 .
	0%	6%	61%	72%

For comparison, the ratio $11279/(6.48 \cdot 7^4) \approx 72\%$ is actually larger than the ratios at $k = 4$ for cubic and quartic fields with discriminants $-3u^2$ removed, these being respectively $64/(1.33 \cdot 3^4) \approx 47\%$ and $740/(3.30 \cdot 5^k) \approx 36\%$. As remarked above, these other cases experimentally approach 100% as k increases. This experimental finding lets one reasonably argue that (5.5) may hold too, with the small percentage 72% being a consequence of a small discriminant effect.

$v \setminus c$	0	1	2	3	4	5	6	7	8	9	10	11	Total
∞	1	10	15										26
	0.71	9.52	15.77										
2	1		2	2	5	4	6		4	4	4	8	40
	0.73		1.66	1.48	4.71	3.83	5.66		4.47	4.37	4.15	8.94	
3	1	1	1	3	5	5	3						19
	0.76	0.85	0.78	2.89	5.24	5.13	3.43						
5	1	1	2	2		4	4	4	4	5			27
	0.37	0.39	0.96	1.32		4.07	4.17	4.70	4.65	6.38			
7	1	1	2	2	1								7
	0.84	0.88	1.92	2.12	1.24								

TABLE 5.1. Local masses $120\mu_{5,-^c}$ and μ_{5,p^c}, compared with frequencies of local discriminants from $\mathcal{K}(S_5, -^*2^*3^*5^*7^*)$.

Table 5.1 compares local masses with frequencies of actually occurring local discriminants, inflated by the ratio $(6.58 \cdot 7^4)/11279$ to facilitate direct comparison. Thus, e.g., the 7-adic discriminants $(7^0, 7^1, 7^2, 7^3, 7^4)$ are predicted by the mass heuristic to occur with relatively frequency $(1, 1, 2, 2, 1)$. They actually occur with relative frequency $(0.84, 0.88, 1.92, 2.12, 1.24)$. Here and for the other four places, trends away from the predicted asymptotic values are explained by consistent underrepresentation of fields with small discriminant. The consistency of the data with the mass heuristic on this refined level provides further support for (5.5).

6. Low degree nonsolvable fields with discriminant $-^s p^a q^b$

Our earliest contributions to the general subject of number field tabulation were [JR99] and [JR03]. These papers respectively found that there are exactly 398 sextic and 10 septic fields with discriminant of the form $-^s 2^a 3^b$. On the lists from these papers, the nonsolvable groups $PSL_2(5) \cong A_5$, $PGL_2(5) \cong A_6$, A_6, S_6 and $SL_3(2)$, A_7, S_7 respectively arise 0, 5, 8, 54, and 0, 0, 10 times. In this section, we summarize further results from the database of this form, identifying or providing lower bounds for $|\mathcal{K}(G, -^* p^* q^*)|$.

	2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53	59	61	67	71	73	79	83	89	97	T
2	•	5	38	2	2	4	3	2	5	6	3	6	9	14	10	11	8	13	13	8	5	11	10	13	4	205
3		•	22				1	4			1	2	2	6	3	5	3	3	2	8	3	2	4	3	2	81
5	5	6	•	4	5	9	12	8	8	8	9	13	12	11	8	14	8	15	13	14	9	14	11	11	14	290
7				•	1				1	2			2	2					1	3		1		1		20
11		2			•			1	1			1		1	1		1			1	1		1	1		18
13			1			•		1		2		3	2		1						1			1	1	25
17	1	1					•	1		3		3	4	1		1	1			1	2			1	3	37
19	1	3						•	2		3		1		1	1	1	1	3	2		1	1		2	36
23		1							•	1		1		2			1	1					2	1	2	28
29	2	3				1	2		2	•		2	3	1	1		2	3	3	1	4	2		2	2	48
31	1	3	1			2	1				•	1		4	1			2	2	1			2		1	30
37			1									•	2	3		2	2	2				3	4	1	1	52
41	2	2	2				1	1	1				•	1		2		3		1	1	2	1	3	5	56
43	1	3	1		1								1	•			2	1	1	2	2	4	1	2		61
47															•	1	4		3			1	3			38
53			2		1				1							•	3	3		1	2	1	3	2		52
59	1	3	2			1	2				1						•			1	1	2	1	3	1	45
61	1	1					1			1		1	1					•	2	3	2	1			1	56
67	2	1	1												1				•	4		2	2	1	2	54
71	1	2	1					1		1	4	2					2	1		•		4	1	3		59
73	1		1	1						1			2	1	1			1			•			2	3	38
79	4	4	2			1		2					3			1	1					•		2	1	54
83		1														1			1			2	•	4		51
89	1	3	2							1	1	1	2			2		1	1	1		2		•	1	58
97		1				1							1				1		1			1		1	•	46
T	24	40	28	1	4	7	9	8	5	14	14	5	19	8	2	8	14	9	8	16	9	23	5	19	7	

Table 6.1. $|\mathcal{K}(A_5, -^* p^* q^*)|$ beneath the diagonal and $|\mathcal{K}(S_5, -^* p^* q^*)|$ above the diagonal. It is expected that A_5 totals are smaller for primes $p \equiv 2, 3$ (5) because in this case p^4 is not a possible local discriminant.

The format of our tables exploits the fact that in the range considered for a given group, there are no fields ramified at one prime only. In fact [JR08], the smallest prime p for which $\mathcal{K}(G, -^*p^*)$ is nonempty is as follows:

G	A_5	S_5	A_6	S_6	$SL_3(2)$	A_7	S_7	$PGL_2(7)$
p	653	101	1579	197	227	>227	191	53

.

Here $7T5 = SL_3(2)$ is abstractly isomorphic to $8T37 = PSL_2(7)$ and thus has index two in $8T43 = PGL_2(7)$.

	2	3	5	7	11	13	17	19	23	29	31	T
2	•	54	30			2	2	2	4	4	6	104
3	8	•	42		4		8		2	12	2	124
5	2	4	•	2	2	2		6	8	2	4	98
7		2		•								2
11					•							6
13		2				•						4
17			2				•		2			12
19		2				2		•				8
23	2	2							•			16
29		4						2		•		18
31	4	6								2	•	12
T	16	30	8	2		4	2	6	4	8	12	

Table 6.2. $|\mathcal{K}(A_6, -^*p^*q^*)|$ beneath the diagonal and $|\mathcal{K}(S_6, -^*p^*q^*)|$ above the diagonal. All entries are even because contributing fields come in twin pairs.

$SL_3(2)$ and $PGL_2(7)$

	2	3	5	7	11	13
2	•	*4*	*0*	*51*	*0*	*0*
3	0	•	*0*	*28*	*0*	*0*
5	0	0	•	*4*	*0*	*0*
7	*44*	*12*	*4*	•	*4*	*6*
11	4	0	*0*	*6*	•	*0*
13	0	*0*	*0*	*0*	0	•

A_7 and S_7

	2	3	5	7	11	13
2	•	10	24	*55*	0	0
3	0	•	14	*44*	2	*0*
5	2	3	•	*18*	*0*	*0*
7	*0*	*7*	*5*	•	*5*	*0*
11	0	0	*1*	*0*	•	0
13	0	*0*	*0*	*0*	0	•

Table 6.3. Determinations or lower bounds for $|\mathcal{K}(G, -^*p^*q^*)|$ for four G. The entries $|\mathcal{K}(SL_3(2), -^*p^*q^*)|$ are all even because contributing fields come in twin pairs.

Restricting to the six groups G of the form A_n or S_n, our results on $|\mathcal{K}(G, -^*p^*q^*)|$ compare with the mass heuristic as follows. First, local masses μ_{n,v^*} are given in the middle six columns:

(6.1)

n	∞	2	3	5	7	tame	μ
5	26/120	40	19	27		7	5.31 .
6	76/720	145	83	31		11	6.39
7	232/5040	180	99	55	57	15	5.18

The column μ contains the global mass $0.5\mu_{n,-^*}\mu_{n,p^*}\mu_{n,q^*}$ for two tame primes p and q. When one or both of the primes are wild, the corresponding global mass is substantially larger.

Tables 6.1, 6.2, and 6.3 clearly show that there tend to be more fields when one or more of the primes p, q allows wild ramification, as one would expect from (6.1). To make plausible conjectures about the asymptotic behavior of the numbers $|\mathcal{K}(G, -^*p^*q^*)|$, one would have to do more complicated local calculations than those summarized in (6.1). These calculations would have to take into account various secondary phenomena, such as the fact that s is forced to be even if $p \equiv q \equiv 1$ (4). Tables 6.1, 6.2, and 6.3 each reflect substantial computation, but the amount of evidence is too small to warrant making formal conjectures in this setting.

7. Nilpotent octic fields with odd discriminant $-^s p^a q^b$

The database has all octic fields with Galois group a 2-group and discriminant of the form $-^s p^a q^b$ with p and q odd primes < 250. There are $\binom{52}{2} = 1326$ pairs $\{p, q\}$ and the average size of $\mathcal{K}(\mathrm{NilOct}, -^*p^*q^*)$ in this range is about 12.01. In comparison with the nonsolvable cases discussed in the previous two sections, there is much greater regularity in this setting. We exhibit some of the greater regularity and explain how it makes some of the abstract considerations of [BE11, BP00] more concrete.

Twenty-six of the fifty octic groups have 2-power order. Table 7.1 presents the nonzero cardinalities, so that e.g. $|\mathcal{K}(8Tj, -^*5^*29^*)| = 4, 2, 2$ for $j = 19$, 20, 21. The repeated proportion $(2, 1, 1)$ for these groups and other similar patterns are due to the sibling phenomenon discussed in Section 4. Only the sixteen 2-groups generated by ≤ 2 elements actually occur. Columns s_3, #, ν, T, and s are all explained later in this section.

The main phenomenon presented in Table 7.1 is that the multiplicities presented are highly repetitious, with e.g. the multiplicities presented for $(5, 29)$ occurring for all together eleven pairs (p, q), as indicated in the # column. The repetition is even greater than indicated by the table itself. Namely if (p_1, q_1) and (p_2, q_2) correspond to the same line, then not only are the numbers $\mathcal{K}(8Tj, -^*p_i^*q_i^*)$ independent of i, but the individual $\mathcal{K}(8Tj, -^*p_i^a q_i^b)$ and even further refinements are also independent of i.

The line corresponding to a given pair (p, q) is almost determined by elementary considerations as follows. Let U be the order of q in $(\mathbb{Z}/p)^\times$ and let V be the order of p in $(\mathbb{Z}/q)^\times$. Let $u = \gcd(U, 4)$ and $v = \gcd(V, 4)$. Then all (p, q) on a given line have the same u, v, and a representative is written (p_u, q_v) in the left two columns. Almost all lines are determined by their datum $\{[p]_u, [q]_v\}$, with $[\cdot]$ indicating reduction modulo 8. The only exceptions are $\{[p]_u, [q]_v\} = \{5_4, 1_4\}$ and $\{[p]_u, [q]_v\} = \{1_4, 1_4\}$ which have two lines each. The column headed by # gives the number of occurrences in our setting $p, q < 250$. In the five cases where this number is less than 10 we continued the computation up through $p, q < 500$ assuming GRH.

p	q	1	2	4	5	6	7	8	10	16	17	19	20	21	27	28	30	s_3	#	ν	T	s
3, 7	3, 7																			1/4		
7_2	3_1			1				1										4	193	1/8	○	4
11_2	7_1			1		2												4	185	1/8	●	≥ 5
3, 7	5																			1/4		
3_1	5_1		1				1											4	219	1/8	*i*	4
11_4	5_2		1	2					2	2	4							6	87	1/16	*ii*	6
19_2	5_2		1	2		2		1	2		2	2	1	1	2	2	4	7	86	1/16	*iii*	19
3, 7	1																			1/4		
3_1	17_1	2	1															4	162	1/8		
19_2	17_2	2	1	2		2	1	1	2		2							6	66	1/16		
23_4	41_2	2	1	2		4	1	2	2	2	4	2	1	1	4	4	4	8	52	1/16		
5	5																			1/16		
5_1	13_1		3				2											5	42	1/32	*I*	6
5_2	29_2		3	3	1				6		8	4	2	2	4	4		9	11	1/128	*II*	27
13_4	53_4		3	3	1	2		2	6		8	12	6	6	12	12	16	11	13	1/128	*III*	≥ 17
13_2	29_4		3	3	1				6	2	12	4	2	2	2	2	4	9	25	1/64	*IV*	≥ 30
5	1																			1/8		
5_1	17_1	4	3															5	76	1/16		
13_4	17_2	4	3	3	1		2		6	2	8	4	2	2	4	4		9	17	1/64		
5_2	41_4	4	3	3	1		2		6		4	4	2	2	2	2	4	9	22	1/64		
53_2	17_2	4	3	3	1	2	2	2	6		12	4	2	2	2	2	4	9	18	1/64		
109_4	73_4	4	3	3	1	4	2	4	6	2	16	12	6	6	12	12	16	11	6	1/128		
101_4	97_4	4	3	3	1	4	2	4	6	2	16	12	6	6	16	16	24	12	1	1/128		
1	1																			1/16		
17_1	41_1	12	3															6	27	1/32		
41_4	73_2	12	3	3	1		6		6	2		4	2	2	4	4		9	12	1/64		
41_2	241_2	12	3	3	1	4	6	4	6	2	16	4	2	2	4	4	4	10	2	1/128		
73_4	89_4	12	3	3	1	6	6	6	6	6	24	12	6	6	16	16	16	12	2	1/256		
73_4	137_4	12	3	3	1	6	6	6	6	6	24	12	6	6	24	24	24	13	2	1/256		

TABLE 7.1. Nonzero cardinalities $|\mathcal{K}(8Tj, -^*p^*q^*)|$ for $8Tj$ an octic group of 2-power order

We expect that all possibilities are accounted for by the table, and they occur with asymptotic frequencies given in the column headed by ν. Assuming these frequencies are correct, the average size of $\mathcal{K}(\mathrm{NilOct}, -^*p^*q^*)$ is exactly 15.875, substantially larger than the observed 12.01 in the $p, q < 250$ setting.

The connection with [BE11, BP00] is as follows. Let $L(p,q)_k \subset \mathbb{C}$ be the splitting field of all degree 2^k fields with Galois group a 2-group and discriminant $-^*p^*q^*$. The Galois group $\mathrm{Gal}(L(p,q)_k/\mathbb{Q})$ is a 2-group and so all ramification at the odd primes p and q is tame. Let $L(p,q)$ be the union of these $L(p,q)_k$. The group $\mathrm{Gal}(L(p,q)/\mathbb{Q})$ is a pro-2-group generated by the tame ramification elements τ_p

and τ_q. The central question pursued in [BE11, BP00] is the distribution of the $\mathrm{Gal}(L(p,q)/\mathbb{Q})$ as abstract groups.

Table 7.1 corresponds to working at the level of the quotient $\mathrm{Gal}(L(p,q)_3/\mathbb{Q})$. The fact that this group has just the two generators τ_p and τ_q explains why only the sixteen 2-groups having 1 or 2 generators appear. One has $|\mathrm{Gal}(L(p,q)_3/\mathbb{Q})| = 2^{s_3}$ where s_3 is as in Table 7.1. The lines with an entry under T are pursued theoretically in [BE11]. The cases marked by ○-●, *i*-*iii*, and *I*-*IV* are respectively treated in §5.2, §5.3, and §5.4 there. The entire group $\mathrm{Gal}(L(p,q),\mathbb{Q})$ has order 2^s, with $s = \infty$ being expected sometimes in Case *IV*.

Some of the behavior for $k > 3$ is previewed by 2-parts of class groups of octic fields. For example, in Case *ii* all 87 instances behave the same: the unique fields in $\mathcal{K}(8T2, -^4p^3q^7)$, $\mathcal{K}(8T4, -^4p^4q^6)$, $\mathcal{K}(8T17, -^4p^6q^5)$, and the two fields in $\mathcal{K}(8T17, -^4p^6q^7)$ all have 2 exactly dividing the class number; the remaining six fields all have odd class number. In contrast, in Case *iii* the 86 instances break into two types of behaviors, represented by $(p,q) = (19,5)$ and $(p,q) = (11,37)$. These patterns on the database reflect the fact [BE11, §5.3] that in Case *ii* there is just one possibility for $(\mathrm{Gal}(L(p,q)/\mathbb{Q}); \tau_p, \tau_q)$ while in Case *iii* there are two.

8. NILPOTENT OCTIC FIELDS WITH DISCRIMINANT $-^s2^aq^b$

The database has all octic fields with Galois group a 2-group and discriminant of the form $-^s2^aq^b$ with $q < 2500$. The sets $\mathcal{K}(\mathrm{NilOct}, -^*2^*q^*)$ average 1711 fields, the great increase from the previous section being due to the fact that now there are many possibilities for wild ramification at 2. As in the previous section, there is great regularity explained by identifications of relevant absolute Galois groups [Koc02]. Again, even more so this time, there is further regularity not explained by theoretical results.

Continuing with the notation of the previous section, consider the Galois extensions $L(2,q) = \cup_{k=1}^{\infty} L(2,q)_k$ and their associated Galois groups $\mathrm{Gal}(L(2,q)/\mathbb{Q}) = \varprojlim \mathrm{Gal}(L(2,q)_k/\mathbb{Q})$. As before, octic fields with Galois group a 2-group let one study $\mathrm{Gal}(L(2,q)_3/\mathbb{Q})$. Table 8.1 presents summarizing data for $q < 2500$ in a format parallel to Table 7.1 but more condensed. Here the main entries count Galois extensions of $\mathbb{Q}$. Thus an entry m in the 19^2 20 21 column corresponds to m Galois extensions of $\mathbb{Q}$ having degree 32. Each of these Galois extensions corresponds to four fields on our database, of types $8T19$, $8T19$, $8T20$, and $8T21$.

In the range studied, there are thirteen different behaviors in terms of the cardinalities $|\mathcal{K}(8Tj, -^*2^*q^*)|$. As indicated by Table 7.1, these cardinalities depend mainly on the reduction of q modulo 16. However classes 1, 9, and 15 are broken into subclasses. The biggest subclasses have size $|1A| = 23$, $|9A| = 24$, and $|15A| = 28$. The remaining subclasses are

$$\begin{aligned}
1B &= \{113, 337, 353, 593, 881, 1249, 1777, 2113, 2129, 2273\},\\
1C &= \{257, 1601\},\\
1D &= \{577, 1201, 1217, 1553, 1889\},\\
1E &= \{1153\},\\
9B &= \{73, 281, 617, 1033, 1049, 1289, 1753, 1801, 1913, 2281, 2393\},\\
9C &= \{137, 409, 809, 1129, 1321, 1657, 1993, 2137\},\\
15B &= \{31, 191, 383, 607, 719, 863, 911, 991, 1103, 1231, 1327, 1471,
\end{aligned}$$

$$1487, 1567, 1583, 2063, 2111, 2287, 2351, 2383\}.$$

A prime $q \equiv 1$ (16) is in $1A$ if and only if $2 \notin \mathbb{F}_q^{\times 4}$. Otherwise we do not have a concise description of these decompositions.

	$\|G\| = 8$					$\|G\| = 16$						$\|G\| = 32$					$\|G\| = 64$					
																21						
																20		28^2	31^2			
q	1	2	3	4	5	6^2	7	8	9^4	10^2	11^3	15^2	16^2	17^2	18^8	19^2	26^4	27^2	29^6	30^4	35^8	Tot
$\mathbb{Q}_2$	24	18	1	18	6	16	36	36	9	12	16	38	12	48	4	24	24	48	16	24	48	1449
$1A$	24	18	1	30	2	42	36	44	15	36	12	64	36	96	16	48	80	104	32	52	72	2895
$1B$	”	”	”	”	”	54	36	60	”	”	”	84	60	144	”	96	144	256	80	128	312	6783
$1C$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	272	”	136	336	7071
$1D$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	336	”	168	384	7839
$1E$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	240	6687
$9A$	24	18	1	30	2	44	36	48	15	36	12	68	36	112	16	48	96	104	48	52	156	3807
$9B$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	132	3615
$9C$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	72	”	36	156	3615
$3, 11$	4	6	1	14	2	10	6	22	7	4	6	21	4	8	3	4	16	8	8	4	21	579
$5, 13$	8	18	1	12	0	10	20	10	6	12	6	21	12	36	1	12	6	24	4	2	9	621
7	4	6	1	20	0	30	6	16	10	12	4	34	12	24	8	12	44	24	20	12	60	1401
$15A$	4	6	1	20	0	32	6	16	10	12	4	36	16	24	8	20	52	64	24	32	96	2041
$15B$	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	”	84	1945

TABLE 8.1. The q-j entry gives the number of Galois extensions of $\mathbb{Q}$ with Galois group $8Tj$ and discriminant of the form $-^s 2^a q^b$. The number of Galois extensions of $\mathbb{Q}_2$ with Galois group $8Tj$ is also given.

Let $D_\infty = \{1, c\}$ where c is complex conjugation. Let $D_q \subseteq \mathrm{Gal}(L(2,q)/\mathbb{Q})$ be a q-decomposition group. Then, working always in the category of pro-2-groups, one has the presentation $D_q = \langle \tau, \sigma | \sigma^{-1}\tau\sigma = \tau^q \rangle$; here τ is a ramification element and σ is a Frobenius element. Representing a more general theory, for $q \equiv 3, 5$ (8) one has two remarkable facts [Koc02, Example 11.18]. First, the 2-decomposition group D_2 is all of $\mathrm{Gal}(L(2,q)/\mathbb{Q})$. Second, the global Galois group is a free product:

$$\mathrm{Gal}(L(2,q)/\mathbb{Q}) = D_\infty * D_q. \tag{8.1}$$

As a consequence, always for $q \equiv 3, 5$ (8), the quotients $\mathrm{Gal}(L(2,q)_k/\mathbb{Q})$ are computable as abstract finite groups and moreover depend only on q modulo 8. In particular, the counts on the lines 3,11 and 5,13 of Table 8.1 can be obtained purely group-theoretically. The other lines of Table 8.1 are not covered by the theory in [Koc02].

A important aspect of the situation is not understood theoretically, namely the wild ramification at 2. The database exhibits extraordinary regularity at the level $k = 3$ as follows. By 2-adically completing octic number fields $K \in \mathcal{K}(\mathrm{NilOct}, -^*2^*q^*)$, one gets 579 octic 2-adic fields if $q \equiv 3$ (8) and 621 octic 2-adic fields if $q \equiv 5$ (2). The regularity is that the subset of all 1499 nilpotent octic 2-adic fields which arise depends on q only modulo 8, at least in our range $q < 2500$. One can see some of this statement directly from the database: the cardinalities $|\mathcal{K}(8Tj, -^*2^a q^*)|$ for given (j, a) depend only on q modulo 8.

In the cases $q \equiv 3, 5\ (8)$, the group $\mathrm{Gal}(L(2,q)/\mathbb{Q}) = D_2$ has a filtration by higher ramification groups. From the group-theoretical description of $\mathrm{Gal}(L(2,q)/\mathbb{Q})$, one can calculate that the quotient group $\mathrm{Gal}(L(2,q)_3/\mathbb{Q})$ has size 2^{18}. The 18 slopes measuring wildness of 2-adic ramification work out to

$$\begin{array}{llllll} 3: & 0, & 2,\ 2,\ 2\frac{1}{2} & 3,\ 3,\ 3\frac{1}{2},\ 3\frac{1}{2},\ 3\frac{5}{8},\ 3\frac{3}{4}, & 4,\ 4,\ 4\frac{1}{4},\ 4\frac{1}{4},\ 4\frac{3}{8},\ 4\frac{1}{2},\ 4\frac{3}{4} & 5 \\ 5: & 0,\ 0, & 2,\ 2,\ 2,\ 2\frac{1}{2} & 3,\ 3,\ 3,\ 3\frac{1}{2},\ 3\frac{1}{2},\ 3\frac{3}{4}, & 4,\ 4\frac{1}{4},\ 4\frac{1}{2},\ 4\frac{3}{4},\ 4\frac{3}{4}, & 5. \end{array}$$

Most of these slopes can be read off from the octic field part of the database directly, via the automatic 2-adic analysis of fields given there. For example, the first four slopes for $q = 3$ all arise already from $\mathbb{Q}[x]/(x^8 + 6x^4 - 3)$, the unique member of $\mathcal{K}(8T8, -^3 2^{16} 3^7)$. A few of the listed slopes can only be seen directly by working with degree sixteen resolvents. A natural question, not addressed in the literature, is to similarly describe the slopes appearing in all of $\mathrm{Gal}(L(2,q)/\mathbb{Q})$.

9. Minimal nonsolvable fields with grd $\leq \Omega$

Our focus for the remainder of the paper is on Galois number fields, for which root discriminants and Galois root discriminants naturally coincide. As reviewed in the introduction, in [JR07] we raised the problem of completely understanding the set $\mathcal{K}[\Omega]$ of all Galois number fields $K \subset \mathbb{C}$ with grd at most the Serre-Odlyzko constant $\Omega = 8\pi e^{\gamma} \approx 44.76$. As in [JR07], we focus attention here on the interesting subproblem of identifying the subset $\mathcal{K}^{\mathrm{ns}}[\Omega]$ of K which are nonsolvable. Our last two sections explain how the database explicitly exhibits a substantial part of $\mathcal{K}^{\mathrm{ns}}[\Omega]$.

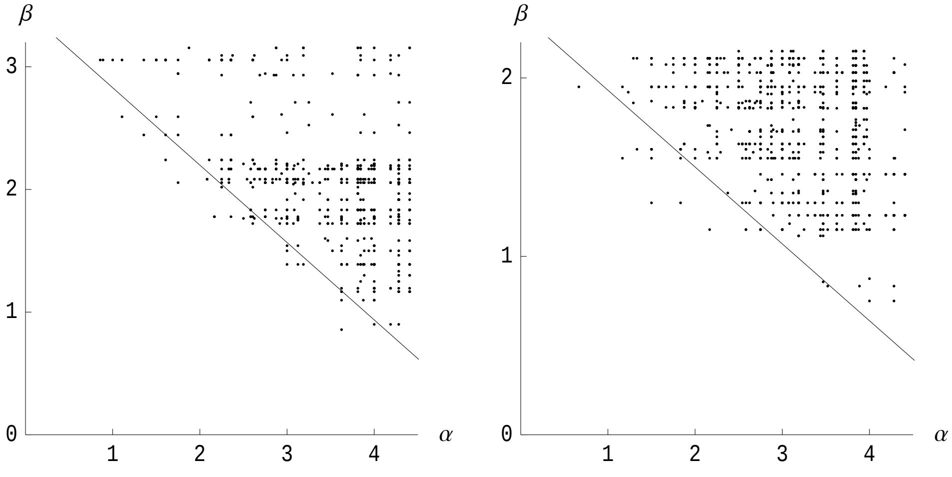

Figure 9.1. Galois root discriminants $2^{\alpha}3^{\beta}$ (left) and $2^{\alpha}5^{\beta}$ (right) arising from minimal nonsolvable fields of degree ≤ 11 in the database. The lines have equation $2^{\alpha}q^{\beta} = \Omega$.

We say a nonsolvable number field is *minimal* if it does not contain a strictly smaller nonsolvable number field. So fields with Galois group say S_n are minimal, while fields with Galois group say $C_p \times S_n$ or $C_p^k : S_n$ are not. Figure 9.1 draws a dot for each minimal nonsolvable field $K_1 \in \mathcal{K}^{\mathrm{ns}}_{\mathrm{min}}[\Omega]$ coming the degree ≤ 11 part of the database with grd of the form $2^\alpha 3^\beta$ or $2^\alpha 5^\beta$. There are 654 fields in the first case and 885 in the second. Of these fields, 24 and 17 have grd $\leq \Omega$. Figure 9.1 illustrates the extreme extent to which the low grd problem is focused on the least ramified of all Galois number fields.

Figure 9.1 also provides some context for the next section as follows. Consider the compositum $K = K_1 K_2$ of distinct minimal fields K_1 and K_2 contributing to the same half of Figure 9.1. Let $2^{\alpha_i} q^{\beta_i}$ be the root discriminant of K_i. The root discriminant $2^\alpha q^\beta$ of K satisfies $\alpha \geq \max(\alpha_1, \alpha_2)$ and $\beta \geq \max(\beta_1, \beta_2)$. The figure makes it clear that one must have almost exact agreement $\alpha_1 \approx \alpha_2$ and $\beta_1 \approx \beta_2$ for K to even have a chance of lying in $\mathcal{K}^{\text{ns}}[\Omega]$. As some examples where one has exact agreement, consider the respective splitting fields K_1, K_2, and K_3 of

$$
\begin{aligned}
f_1(x) &= x^5 - 10x^3 - 20x^2 + 110x + 116, \\
f_2(x) &= x^5 + 10x^3 - 10x^2 + 35x - 18, \\
f_3(x) &= x^5 + 10x^3 - 40x^2 + 60x - 32.
\end{aligned}
$$

All three fields have Galois group A_5 and root discriminant $2^{3/2}5^{8/5} \approx 37.14$. The first two completely agree at 2, but differ at 5, so that $K_1 K_2$ has root discriminant $2^{3/2}5^{48/25} \approx 62.17$. The other two composita also have root discriminant well over Ω, with $\text{grd}(K_2K_3) = 2^{9/4}5^{8/5} \approx 62.47$ and $\text{grd}(K_1K_3) = 2^{9/4}5^{48/25} \approx 104.55$. These computations, done automatically by entering $f_i(x)f_j(x)$ into the grd calculator of [JR06], are clear illustrations of the general difficulty of using known fields in $\mathcal{K}[\Omega]$ to obtain others.

In [JR07], we listed fields proving $|\mathcal{K}^{\text{ns}}_{\text{min}}[\Omega]| \geq 373$. Presently, the fields on the database show $|\mathcal{K}^{\text{ns}}_{\text{min}}[\Omega]| \geq 386$. In [JR07], we highlighted the fact that the only simple groups involved were the five smallest, A_5, $SL_3(2)$, A_6, $SL_2(8)$ and $PSL_2(11)$, and the eighth, A_7. The new fields add $SL_2(16)$, $G_2(2)'$, and A_8 to the list of simple groups involved. These groups are 10th, 12th, and tied for 19th on the list of all non-abelian simple groups in increasing order of size.

#	$\lvert H\rvert$	$G = H$	#	$G = H.Q$	#
1	60	A_5	**78**	S_5	**192**
2	168	$SL_3(2)$	*.18*	$PGL_2(7)$	*...23*
3	360	A_6	**5**	S_6, $PGL_2(9)$, M_{10}, $P\Gamma L_2(9)$	**13**, *....6*, *0*, *.15*
4	504	$SL_2(8)$	*15*	$\Sigma L_2(8)$	*15*
5	660	$PSL_2(11)$	*1*	$PGL_2(11)$	*0*
8	2520	A_7	*1*	S_7	*1*
1^2	3600	A_5^2		$A_5^2.2$, $A_5^2.V$, $A_5^2.C_4$, $A_5^2.D_4$	*1*, *.1*, *0*, *0*
10	4080	$SL_2(16)$	*.1*	$SL_2(16).2, SL_2(16).4$	*0*, *0*
12	6048	$G_2(2)'$	*0*	$G_2(2)$	*.1*
19	20160	A_8	*0*	S_8	*.1*

TABLE 9.1. Lower bounds on $|\mathcal{K}[G, \Omega]|$ for minimal nonsolvable groups G. Entries highlighted in bold are completeness results from [JR07]. Fields found since [JR07] are indicated by .'s.

Table 9.1 summarizes all fields on the database in $\mathcal{K}^{\text{ns}}_{\text{min}}[\Omega]$. It is organized by the socle $H \subseteq G$, which is a simple group except in the single case $H = A_5 \times A_5$. The .'s indicate that, for example, of the 23 known fields in $\mathcal{K}[PGL_2(7), \Omega]$, twenty are listed in [JR07] and three are new. The polynomial for the $SL_2(16)$ field was found by Bosman [Bos07], starting from a classical modular form of weight 2. We found polynomials for the new $SL_3(2)$ field and the three new $PGL_2(7)$ fields starting from Schaeffer's list [Sch12, App A] of ethereal modular forms of weight 1. Polynomials for the other new fields were found by specializing families. All fields summarized by Table 9.1 come from the part of the database in degree ≤ 11, except for Bosman's degree seventeen polynomial and the degree twenty-eight polynomial for $G_2(2)$. It would be of interest to pursue calculations with modular forms more systematically. They have the potential not only to yield new fields in $\mathcal{K}^{\mathrm{ns}}_{\mathrm{min}}[\Omega]$, but also to prove completeness for certain G.

10. General nonsolvable fields with grd $\leq \Omega$

We continue in the framework of the previous section, so that the focus remains on Galois number fields contained in $\mathbb{C}$. For $K_1 \in \mathcal{K}^{\mathrm{ns}}_{\mathrm{min}}[\Omega]$ such a Galois number field, let $\mathcal{K}[K_1;\Omega]$ be the subset of $\mathcal{K}^{\mathrm{ns}}[\Omega]$ consisting of fields containing K_1. Clearly

$$\mathcal{K}^{\mathrm{ns}}[\Omega] = \bigcup_{K_1} \mathcal{K}[K_1;\Omega]. \tag{10.1}$$

So a natural approach to studying all of $\mathcal{K}^{\mathrm{ns}}[\Omega]$ is to study each $\mathcal{K}[K_1;\Omega]$ separately.

The refined local information contained in the database can be used to find fields in $\mathcal{K}[K_1;\Omega]$. The set of fields so obtained is always very small, often just $\{K_1\}$. Usually it seems likely that the set of fields obtained is all of $\mathcal{K}[K_1;\Omega]$, and sometimes this expectation is provable under GRH. We sketch such a proof for a particular K_1 in the first example below. In the remaining examples, we start from other K_1 and now construct proper extensions $K \in \mathcal{K}[K_1;\Omega]$, illustrating several phenomena. Our examples are organized in terms of increasing degree $[K:\mathbb{Q}]$. The fields here are all extremely lightly ramified for their Galois group, and therefore worthy of individual attention.

Our local analysis of a Galois number field K centers on the notion of p-adic slope content described in [JR06, §3.4] and automated on the associated database. Thus a p-adic slope content of $[s_1,\ldots,s_m]_t^u$ indicates a wild inertia group P of order p^m, a tame inertia group I/P of order t, and an unramified quotient D/I of order u. Wild slopes $s_i \in \mathbb{Q}\cap(1,\infty)$ are listed in weakly increasing order and from [JR06, Eq. 7] the contribution p^α to the root discriminant of K is determined by

$$\alpha = \left(\sum_{i=1}^{m} \frac{p-1}{p^i} s_{n+1-i}\right) + \frac{1}{p^m}\frac{t-1}{t}.$$

The quantities t and u are omitted from presentations of slope content when they are 1.

Degree 120 *and nothing more from* S_5. The polynomial

$$f_1(x) = x^5 + x^3 + x - 1$$

has splitting field K_1 with root discriminant $\Delta_1 = 11^{2/3}37^{1/2} \approx 30.09$. Since $\Delta_1 2^{2/3} \approx 47.76$, $\Delta_1 3^{1/2} \approx 52.11$, $\Delta_1 11^{1/6} \approx 44.87$, and $\Delta_1 37^{1/4} \approx 74.20$ are all more than Ω, any $K \in \mathcal{K}[K_1;\Omega]$ has to have root discriminant $\Delta = \Delta_1$. The GRH bounds say that a field with root discriminant 30.09 can have degree at most 2400 [Mar82].

The main part of the argument is to use the database to show that most other *a priori* possible G in fact do not arise as $\mathrm{Gal}(K/\mathbb{Q})$ for $K \in \mathcal{K}[K_1;\Omega]$. For example, if there were an S_3 field K_2 with absolute discriminant $11^2 37$, then K_1K_2 would be in $\mathcal{K}[K_1;\Omega]$; there is in fact an S_3 field with absolute discriminant $11 \cdot 37^2$, but not one with absolute discriminant $11^2 37$. As an example of a group that needs a supplementary argument to be eliminated, the central extension $G = 2.S_5$ does not appear because the degree 12 subfield of K_1 fixed by $D_5 \subset S_5$ has root discriminant Δ_1 and class number 1.

Degree 1920 *from* A_5. The smallest root discriminant of any nonsolvable Galois field is $2^{6/7}17^{2/3} \approx 18.70$ coming from a field K_1 with Galois group A_5. This case is complicated because one can add ramification in several incompatible directions, so that there are different maximal fields in $\mathcal{K}[K_1;\Omega]$. One overfield is the splitting field $\tilde{K}_1$ of $f_-(x)$ where

$$f_\pm(x) = x^{10} + 2x^6 \pm 4x^4 - 3x^2 \pm 4.$$

In this direction, ramification has been added at 2 making the slope content there $[2,2,2,2,4]^6$ and the root discriminant $2^{39/16}17^{2/3} \approx 35.81$. The only solvable field K_2 on the database which is not contained in $\tilde{K}_1$ but has $\mathrm{rd}(\tilde{K}_1K_2) < \Omega$ is $\mathbb{Q}(i)$. The field $\tilde{K}_1K_2$ is the splitting field of $f_+(x)$ with Galois group $10T36$. There is yet another wild slope of 2, making the root discriminant $2^{79/32}17^{2/3} \approx 36.60$.

Degree 25080 *from* $PSL_2(11)$. The only known field K_1 with Galois group $PSL_2(11)$ and root discriminant less than Ω first appeared in [KM01] and is the splitting field of

$$f_1(x) = x^{11} - 2x^{10} + 3x^9 + 2x^8 - 5x^7 + 16x^6 - 10x^5 + 10x^4 + 2x^3 - 3x^2 + 4x - 1.$$

The root discriminant is $\Delta_1 = 1831^{1/2} \approx 42.79$, forcing all members of $\mathcal{K}[K_1;\Omega]$ to have root discriminant $1831^{1/2}$ as well.

The prime 1831 is congruent to 3 modulo 4, so that the associated quadratic field $\mathbb{Q}(\sqrt{-1831})$ is imaginary and its class number can be expected to be considerably larger than 1. This class number is in fact 19, and the splitting field of a degree 19 polynomial in the database is the corresponding Hilbert class field K_2. The field $K_1K_2 \in \mathcal{K}[K_1;\Omega]$ has degree $660 \cdot 38 = 25080$.

Degree 48384 *from* $SL_2(8).3$. The splitting field K_1 of

$$f_1(x) = x^9 - 3x^8 + 4x^7 + 16x^2 + 8x + 8$$

has Galois group $\mathrm{Gal}(K_1/\mathbb{Q}) = 9T32 = SL_2(8).3$ and root discriminant $2^{73/28}7^{8/9} \approx 34.36$. This root discriminant is the smallest known from a field with Galois group $SL_2(8).3$. In fact, it is small enough that it is possible to add ramification at both 2 and 7 and still keep the root discriminant less than Ω. Namely let

$$\begin{aligned} f_2(x) &= x^4 - 2x^3 + 2x^2 + 2, \\ f_3(x) &= x^4 - x^3 + 3x^2 - 4x + 2. \end{aligned}$$

The splitting fields K_2 and K_3 have Galois groups A_4 and D_4 respectively. Composing with K_2 increases degree by four and adds wild slopes 2 and 2 to the original 2-adic slope content $[20/7,20/7,20/7]_7^3$. Composing with K_3 then increases degrees by 8, adding another wild slope of 2 to the 2-adic slope content and increasing the 7-adic tame degree from 9 to 36. The root discriminant of $K_1K_2K_3$ is then $2^{153/56}7^{35/36} \approx 44.06$.

Degree 80640 *from* S_8. The largest group in Table 9.1 is S_8, and the only known field in $\mathcal{K}[S_8, \Omega]$ is the splitting field K_1 of

$$f_1(x) = x^8 - 4x^7 + 4x^6 + 8x^3 - 32x^2 + 32x - 20.$$

Here Galois slope contents are $[15/4, 7/2, 7/2, 3, 2, 2]^3$ and $[\ ]_7$ at 2 and 5 respectively, giving root discriminant $2^{111/32}5^{6/7} \approx 43.99$. The only field on the database which can be used to give a larger field in $\mathcal{K}[K_1; \Omega]$ is $K_2 = \mathbb{Q}(i)$. This field gives an extra wild slope of 2, raising the degree of K_1K_2 to 80640 and the root discriminant to $2^{223/64}5^{6/7} \approx 44.47$.

Degree 86400 *from* $A_5^2.V$. Another new field K_1 on Table 9.1, found by Driver, is the splitting field of

$$f_1(x) = x^{10} - 2x^9 + 5x^8 - 10x^6 + 28x^5 - 26x^4 - 5x^2 + 50x - 25.$$

Like in the previous example, this field K_1 is wildly ramified at 2 and tamely ramified at 5. Slope contents are $[23/6, 23/6, 3, 8/3, 8/3]_3$ and $[\ ]_6$ for a root discriminant of $2^{169/48}5^{5/6} \approx 43.89$. The splitting field K_2 of $x^3 - x^2 + 2x + 2$ has Galois group S_3, with 2-adic slope content $[3]$ and 5-adic slope content $[\ ]_3$. In the compositum K_1K_2, the extra slope is in fact 2 giving a root discriminant of $2^{85/24}5^{5/6} \approx 44.53$.

Degree 172800 *from* S_5 *and* S_6. Consider the $\binom{386}{2} = 74305$ composita K_1K_2, as K_1 and K_2 vary over distinct known fields in $\mathcal{K}^{\mathrm{ns}}_{\mathrm{min}}[\Omega]$. From our discussion of Figure 9.1, one would expect that very few of these composita would have root discriminant less than Ω. In fact, calculation shows that exactly one of these composita has $\mathrm{rd}(K_1K_2) \leq \Omega$, namely the joint splitting field of

$$\begin{aligned} f_1(x) &= x^5 - x^4 - x^3 + 3x^2 - x - 19, \\ f_2(x) &= x^6 - 2x^5 + 4x^4 - 8x^3 + 2x^2 + 24x - 20. \end{aligned}$$

Here $\mathrm{Gal}(K_1/\mathbb{Q}) = S_5$ and $\mathrm{Gal}(K_2/\mathbb{Q}) = S_6$. Both fields have tame ramification of order 2 at 3 and order 5 at 7. Both are otherwise ramified only at 2, with K_1 having slope content $[2, 3]^2$ and K_2 having slope content $[2, 2, 3]^3$. In the compositum K_1K_2, there is partial cancellation between the two wild slopes of 3, and the slope content is $[2, 2, 2, 2, 3]^6$. The root discriminant of K_1K_2 then works out to $2^{39/16}3^{1/2}7^{4/5} \approx 44.50$. The existence of this remarkable compositum contradicts Corollary 12.1 of [JR07] and is the only error we have found in [JR07].

The field discriminants of f_1 and f_2 are respectively $-^22^63^17^4$ and $-^22^93^17^4$. The splitting fields K_1 and K_2 thus contain distinct quadratic fields, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$ respectively. The compositum therefore has Galois group all of $S_5 \times S_6$, and so the degree $[K_1K_2 : \mathbb{Q}] = 120 \cdot 720 = 86400$ ties that of the previous example. But, moreover, $K_3 = \mathbb{Q}(\sqrt{-3})$ is disjoint from $\mathbb{Q}(\sqrt{3}, \sqrt{6})$ and does not introduce more ramification. So $K = K_1K_2K_3$ has the same root discriminant $2^{39/16}3^{1/2}7^{4/5} \approx 44.50$, but the larger degree $2 \cdot 86400 = 172800$.

The GRH upper bound on degree for a given root discriminant $\delta \in [1, \Omega)$ increases to infinity as δ increases to Ω (as illustrated by Figure 4.1 of [JR07]). However, we have only exhibited fields K here of degree ≤ 172800. Dropping the restriction that K is Galois and nonsolvable may let one obtain somewhat larger degrees, but there remains a substantial and intriguing gap between degrees of known fields and analytic upper bounds on degree.

References

[BE11] Nigel Boston and Jordan S. Ellenberg, *Random pro-p groups, braid groups, and random tame Galois groups*, Groups Geom. Dyn. **5** (2011), no. 2, 265–280. MR 2782173 (2012b:11172)

[Bel] Karim Belabas, *cubic v. 1.2*, `http://www.math.u-bordeaux1.fr/~kbelabas/research/software/cubic-1.2.tgz`.

[Bel04] ———, *On quadratic fields with large 3-rank*, Math. Comp. **73** (2004), no. 248, 2061–2074 (electronic). MR MR2059751 (2005c:11132)

[Bha05] Manjul Bhargava, *The density of discriminants of quartic rings and fields*, Ann. of Math. (2) **162** (2005), no. 2, 1031–1063. MR 2183288 (2006m:11163)

[Bha07] ———, *Mass formulae for extensions of local fields, and conjectures on the density of number field discriminants*, Int. Math. Res. Not. IMRN (2007), no. 17, Art. ID rnm052, 20. MR 2354798 (2009e:11220)

[Bha10] ———, *The density of discriminants of quintic rings and fields*, Ann. of Math. (2) **172** (2010), no. 3, 1559–1591. MR 2745272 (2011k:11152)

[BM83] Gregory Butler and John McKay, *The transitive groups of degree up to eleven*, Comm. Algebra **11** (1983), no. 8, 863–911.

[BMO90] A.-M. Bergé, J. Martinet, and M. Olivier, *The computation of sextic fields with a quadratic subfield*, Math. Comp. **54** (1990), no. 190, 869–884. MR 1011438 (90k:11169)

[Bor] *Bordeaux tables of number fields*, `http://pari.math.u-bordeaux.fr/pub/pari/packages/nftables/`.

[Bos07] Johan Bosman, *A polynomial with Galois group* $\mathrm{SL}_2(\mathbb{F}_{16})$, LMS J. Comput. Math. **10** (2007), 1461–1570 (electronic). MR 2365691 (2008k:12008)

[BP00] Nigel Boston and David Perry, *Maximal 2-extensions with restricted ramification*, J. Algebra **232** (2000), no. 2, 664–672. MR 1792749 (2001k:12005)

[Dah08] S. R. Dahmen, *Classical and modular methods applied to diophantine equations*, Ph.D. thesis, University of Utrecht, 2008.

[DJ10] Eric D. Driver and John W. Jones, *Minimum discriminants of imprimitive decic fields*, Experiment. Math. **19** (2010), no. 4, 475–479. MR 2778659 (2012a:11169)

[GAP06] The GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.4*, 2006, (`http://www.gap-system.org`).

[HKS06] Klaus Hulek, Remke Kloosterman, and Matthias Schütt, *Modularity of Calabi-Yau varieties*, Global aspects of complex geometry, Springer, Berlin, 2006, pp. 271–309. MR 2264114 (2007g:11052)

[Hoe09] Jing Long Hoelscher, *Infinite class field towers*, Math. Ann. **344** (2009), no. 4, 923–928. MR 2507631 (2010h:11185)

[Hun57] John Hunter, *The minimum discriminants of quintic fields*, Proc. Glasgow Math. Assoc. **3** (1957), 57–67. MR 0091309 (19,944b)

[JRa] John W. Jones and David P. Roberts, *Artin L-functions with small conductor*, in preparation.

[JRb] ———, *The tame-wild principle for discriminant relations for number fields*, to appear in *Algebra and Number Theory.*

[JR99] ———, *Sextic number fields with discriminant* $(-1)^j 2^a 3^b$, Number theory (Ottawa, ON, 1996), CRM Proc. Lecture Notes, vol. 19, Amer. Math. Soc., Providence, RI, 1999, pp. 141–172. MR 2000b:11142

[JR03] ———, *Septic fields with discriminant* $\pm 2^a 3^b$, Math. Comp. **72** (2003), no. 244, 1975–1985 (electronic). MR MR1986816 (2004e:11119)

[JR06] ———, *A database of local fields*, J. Symbolic Comput. **41** (2006), no. 1, 80–97, website: `http://math.asu.edu/~jj/localfields`.

[JR07] ———, *Galois number fields with small root discriminant*, J. Number Theory **122** (2007), no. 2, 379–407. MR 2292261 (2008e:11140)

[JR08] ———, *Number fields ramified at one prime*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 5011, Springer, Berlin, 2008, pp. 226–239. MR 2467849 (2010b:11152)

[JW12] John W. Jones and Rachel Wallington, *Number fields with solvable Galois groups and small Galois root discriminants*, Math. Comp. **81** (2012), no. 277, 555–567.

[KM01] Jürgen Klüners and Gunter Malle, *A database for field extensions of the rationals*, LMS J. Comput. Math. **4** (2001), 182–196 (electronic). MR 2003i:11184

[Koc02] Helmut Koch, *Galois theory of p-extensions*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002, With a foreword by I. R. Shafarevich, Translated from the 1970 German original by Franz Lemmermeyer, With a postscript by the author and Lemmermeyer. MR 1930372 (2003f:11181)

[Mal02] Gunter Malle, *On the distribution of Galois groups*, J. Number Theory **92** (2002), no. 2, 315–329. MR 1884706 (2002k:12010)

[Mar82] Jacques Martinet, *Petits discriminants des corps de nombres*, Number theory days, 1980 (Exeter, 1980), London Math. Soc. Lecture Note Ser., vol. 56, Cambridge Univ. Press, Cambridge, 1982, pp. 151–193. MR 84g:12009

[MM99] Gunter Malle and B. Heinrich Matzat, *Inverse Galois theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1999. MR 1711577 (2000k:12004)

[Odl90] A. M. Odlyzko, *Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results*, Sém. Théor. Nombres Bordeaux (2) **2** (1990), no. 1, 119–141. MR MR1061762 (91i:11154)

[OT05] Ken Ono and Yuichiro Taguchi, *2-adic properties of certain modular forms and their applications to arithmetic functions*, Int. J. Number Theory **1** (2005), no. 1, 75–101. MR 2172333 (2006e:11057)

[PAR13] The PARI Group, Bordeaux, *Pari/gp, version 2.6.2*, 2013.

[Poh82] Michael Pohst, *On the computation of number fields of small discriminants including the minimum discriminants of sixth degree fields*, J. Number Theory **14** (1982), no. 1, 99–117. MR 644904 (83g:12009)

[Rob] David P. Roberts, *Chebyshev covers and exceptional number vields*, In preparation. http://facultypages.morris.umn.edu/ roberts/.

[Rob07] ———, *Wild partitions and number theory*, J. Integer Seq. **10** (2007), no. 6, Article 07.6.6, 34. MR 2335791 (2009b:11206)

[Sch12] George Johann Schaeffer, *The Hecke Stability Method and Ethereal Forms*, ProQuest LLC, Ann Arbor, MI, 2012, Thesis (Ph.D.)–University of California, Berkeley. MR 3093915

[SPDyD94] A. Schwarz, M. Pohst, and F. Diaz y Diaz, *A table of quintic number fields*, Math. Comp. **63** (1994), no. 207, 361–376. MR 1219705 (94i:11108)

[Voi] John Voight, *Tables of totally real number fields*, `http://www.math.dartmouth.edu/~jvoight/nf-tables/index.html`.

[Voi08] ———, *Enumeration of totally real number fields of bounded root discriminant*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 5011, Springer, Berlin, 2008, pp. 268–281. MR 2467853 (2010a:11228)

School of Mathematical and Statistical Sciences, Arizona State University, PO Box 871804, Tempe, AZ 85287

E-mail address: `jj@asu.edu`

Division of Science and Mathematics, University of Minnesota Morris, Morris, MN 56267

E-mail address: `roberts@morris.umn.edu`