An ABC construction of number fields

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I. An example.

II. Matrix step. $ABC = 1$
(Katz's theory of rigid local systems)

III. Polynomial step. $A(x) + B(x) + C(x) = 0$
(Theory of *dessins d'enfants*)

IV. Integer step. $ax^p + by^q + cz^r = 0$
(Along the lines of the ABC conjecture)

V. Further directions.

I. An example. The polynomial

$$f(x) = \\ x^{27} - 432\,x^{21} - 810\,x^{19} - 7056\,x^{18} \\ -39852\,x^{15} + 93312\,x^{13} - 254016\,x^{12} \\ -98415\,x^{11} + 625968\,x^{10} - 1168560\,x^{9} \\ +1705860\,x^{7} - 1796256\,x^{6} - 944784\,x^{5} \\ +979776\,x^{4} + 31104\,x^{3} - 571536\,x \\ -592704$$

is unusual in two ways:

- The Galois group of its splitting field is $PSp_4(\mathbf{F}_3).2$, which is nonsolvable of order $51,840 = 2^7 3^4 5$.

- The discriminant of the root field $\mathbf{Q}[x]/f(x)$ is $2^{20}3^{84}$, reflecting tame ramification at 2 and wild ramification at 3.

How can we systematically produce polynomials of this sort?

II. Matrix Step. Consider matrices A, B, $C \in GL_n(\overline{\mathbf{F}}_\ell)$ such that

- $ABC = I$

- $\langle A, B, C \rangle$ acts irreducibly on $\overline{\mathbf{F}}_\ell$.

- the sum of the centralizer dimensions of the matrices is maximal, namely

$$\text{cd}(A) + \text{cd}(B) + \text{cd}(C) = n^2 + 2.$$

Such a triple is **rigid** in the sense that the individual conjugacy classes $[A]$, $[B]$, $[C]$ determine the conjugacy class of the triple (A, B, C).

See (Katz, Rigid Local systems) for the very rich theory: rigid matrix triples are classified and they all come by reduction modulo ℓ from motivic monodromy representations.

Example of a rigid matrix triple in $GL_4(\mathbf{F}_3)$:

$$A = \begin{pmatrix} 0121 \\ 0102 \\ 1011 \\ 0100 \end{pmatrix} \sim \begin{pmatrix} 1100 \\ 0110 \\ 0011 \\ 0001 \end{pmatrix}$$

$$B = \begin{pmatrix} 0001 \\ 0020 \\ 0100 \\ 2000 \end{pmatrix} \sim \begin{pmatrix} i000 \\ 0i00 \\ 00\bar{i}0 \\ 000\bar{i} \end{pmatrix}$$

$$C = \begin{pmatrix} 0010 \\ 0002 \\ 1000 \\ 0200 \end{pmatrix} \sim \begin{pmatrix} 1000 \\ 0100 \\ 00i0 \\ 000\bar{i} \end{pmatrix}$$

$ABC = I$ holds by direct computation. Irreducibility holds because $\langle A, B, C \rangle = Sp_4(\mathbf{F}_3)$. The two sides of the rigidity condition are

$$(1+1+1+1)+(4+4)+(4+1+1) = 18$$

and

$$4^2 + 2 = 18$$

so the rigidity condition holds.

III. Polynomial step. Consider permutations A, B, $C \in S_N$ such that $ABC = e$. Such a triple determines a covering of algebraic curves over $\overline{\mathbf{Q}}$

$$F : X \to \mathbf{P}^1$$

ramified only above 0, 1, $\infty \in \mathbf{P}^1$.

Theorem. (1960's; Grothendieck) F has bad reduction within the primes dividing the order of the **global** monodromy group $\langle A, B, C \rangle$.

Theorem. (1990's; Katz) If A, B, C come from the rigid matrix situation of Part II via some representation $\langle A, B, C \rangle \to GL_n(\overline{\mathbf{F}}_\ell)$ then F has bad reduction within the primes dividing the orders of the **local** monodromy groups $\langle A \rangle$, $\langle B \rangle$, $\langle C \rangle$ and ℓ.

(Intuitively, "Katz three point covers" are extremal among all three point covers, and are very special, sharing some of the features of $X_0(\ell) \to j$-line.)

From degree 27 permutations corresponding to the matrices A, B, C of Part II, we computed

$$\begin{aligned} a(x) &= 2^{12}(3x^3-3x-1)^9 \\ b(x) &= f_{10}(x)^2 f_6(x) \\ c(x) &= (48x^3+108x^2+63x+11)g_6(x)^4 \end{aligned}$$

with

$$a(x)+b(x)+c(x)=0.$$

The corresponding cover is

$$F : \mathbf{P}^1 \to \mathbf{P}^1 : x \mapsto -\frac{a(x)}{c(x)},$$

The discriminant of $f(t,x)=a(x)+tc(x)$ is

$$D(t)=2^{336}3^{450}t^{24}(t-1)^{10},$$

illustrating the good reduction theorems.

(Summary so far: $f(t,x)$ is an analog of division polynomials corresponding to ℓ-torsion points on a general elliptic curve. Katz's theory gives a hierarchy of such polynomials, but at present they are hard to compute.)

IV. Integer step. Continuing with our example, for generic $\tau \in \mathbf{Q} - \{0, 1\}$, $f(\tau, x)$ is irreducible and

$$K_\tau = \mathbf{Q}[x]/f(\tau, x)$$

is a number field. If $\tau = -ax^9/cz^4$ with

$$ax^9 + by^2 + cz^4 = 0,$$

then K_τ is ramified within the primes dividing $6abc$. The specialization point $\tau = -48$, corresponding to

$$2^4 3 - 7^2 + 1 = 0$$

gives our field K_{-48}, which has the unusual property mentioned before of being only tamely ramified at 2.

V. Future directions. Systematically study the ramification in these "ABC number fields" $\mathbf{Q}[x]/f(\tau, x)$, as a function of the discrete group-theoretic data defining the Katz three point cover and the continuous parameter τ.