Questions

1. Using only algebraic methods, find the cubic function that satisfies the given table of values.

x	f(x)
-2	0
-1	24
1	0
5	0

2. Graph the functions using a calculator

$$f(x) = \frac{2x^3 - 3x^2 - 5x - 12}{x - 3}, \qquad g(x) = 2x^2 - 3x + 4, x \neq 3, \qquad h(x) = 2x^2 - 3x + 4.$$

How are these functions related? Include a discussion of the domain and continuity of each function.

3. Find the zeros of $f(x) = 2x^3 - 5x^2 - 9x + 18$ using the Rational Zero Theorem.

Solutions

1. Using only algebraic methods, find the cubic function with the given table of values. Check by sketching.

x	f(x)
-2	0
-1	24
1	0
5	0

A cubic polynomial has four coefficients, $f(x) = ax^3 + bx^2 + cx + d$. Since we are given the three zeros of the polynomial, we can write the polynomial in factored form, with only one coefficient left to determine:

$$f(x) = a(x - c_1)(x - c_2)(x - c_3) = a(x - (-2))(x - 1)(x - 5) = a(x + 2)(x - 1)(x - 5)$$

The fourth point given can be used to determine the coefficient a:

$$\begin{aligned} f(x) &= a(x+2)(x-1)(x-5) \\ f(-1) &= a((-1)+2)((-1)-1)((-1)-5) = 24 \\ a(1)(-2)(-6) &= 24 \\ a(12) &= 24 \\ a &= \frac{24}{12} = 2 \end{aligned}$$

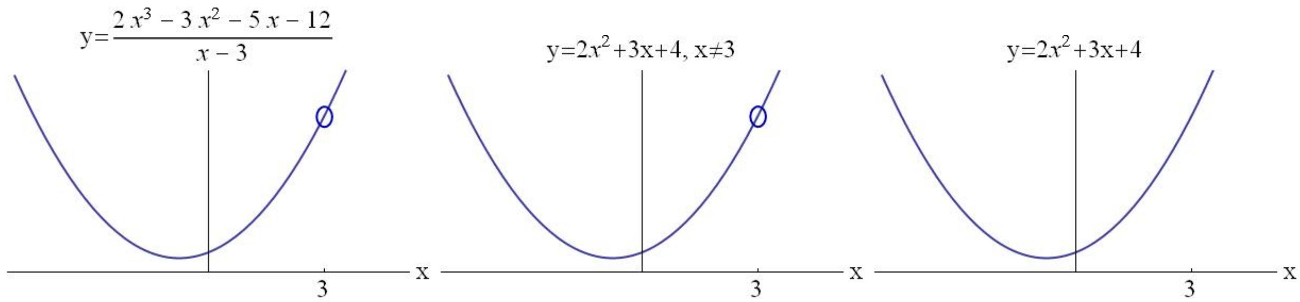
The cubic polynomial passing through the points is $f(x) = 2(x + 2)(x - 1)(x - 5)$.

2. Graph the functions using a calculator

$$f(x) = \frac{2x^3 - 3x^2 - 5x - 12}{x - 3}, \qquad g(x) = 2x^2 - 3x + 4, x \neq 3, \qquad h(x) = 2x^2 - 3x + 4.$$

How are these functions related? Include a discussion of the domain and continuity of each function.

I graphed these functions using a computer.



The functions on the left and center are exactly the same. They both have domain $x \in (-\infty, 3) \cup (3, \infty)$ and are discontinuous at $x = 3$ (the open circle on the sketch indicates the discontinuity).

The function $h(x) = 2x^2 - 3x + 4$ is different from f and g since it has domain $x \in (-\infty, \infty)$ and is continuous for all x.

It wasn't asked for, but you can show $f(x) = g(x)$ using algebra, but it requires a result we haven't seen yet.

$$
\begin{aligned}
f(x) &= \frac{2x^3 - 3x^2 - 5x - 12}{x - 3} \\
&= \frac{(2x^2 - 3x + 4)(x - 3)}{x - 3} \text{ factor–the graph above give us the hint about the factoring} \\
&= \frac{(2x^2 - 3x + 4)\cancel{(x - 3)}}{\cancel{x - 3}} \\
&= 2x^2 - 3x + 4, x - 3 \neq 0 \text{ looking ahead to Section Solving Equations in One Variable and Indeterminant Forms} \\
&= g(x)
\end{aligned}
$$

3. Find the zeros of $f(x) = 2x^3 - 5x^2 - 9x + 18$ using the Rational Zero Theorem.

Factors of $a_0 = 18$: $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18$

Factors of $a_4 = 2$: $\pm 1, \pm 2$.

Potential rational zeros of f are $\pm 1, \pm 2, \pm 3, \pm 6, \pm 9, \pm 18, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{9}{2}$.

$f(1) = 6$

$f(-1) = 20$

$f(2) = -4$

$f(-2) = 0 \leftarrow$ so $x - (-2) = x + 2$ will divide evenly into f.

$$
\begin{array}{r l}
 & 2x^2 - 9x + 9 \\
x+2 \,\big) & 2x^3 - 5x^2 - 9x + 18 \\
 & \underline{2x^3 + 4x^2} \\
 & \quad -9x^2 - 9x \\
 & \quad \underline{-9x^2 - 18x} \\
 & \qquad\quad 9x + 18 \\
 & \qquad\quad \underline{9x + 18} \\
 & \qquad\qquad 0
\end{array}
$$

$\rightarrow f(x) = (x+2)(2x^2 - 9x + 9)$

$\rightarrow$ Use Quadratic formula to factor the quadratic.

$$x = \frac{9 \pm \sqrt{81 - 4(2)(9)}}{4} = \frac{9 \pm 3}{4} = 3 \text{ or } \frac{3}{2}.$$

So $2x^2 - 9x + 9 = 2(x-3)(x-\frac{3}{2})$ ← remember to put the coefficient of x^2 in!

$$f(x) = (x+2)\,2(x-3)(x-\tfrac{3}{2}) = (x+2)(x-3)\,2(x-\tfrac{3}{2}) = (x+2)(x-3)(2x-3).$$

zeros of f, all of multiplicity 1, are

$x = -2, 3, 3/2$.

Using the Rational Zero Theorem isn't particularly hard, it just takes a while to implement since you have to check a bunch of possibilities to determine what might work as a factor, and there is no guarantee you'll find anything that works! For example, $f(x) = x^4 - 5x^2 + 6$ has no rational factors, yet it has four real roots.