Questions

1. Sketch $f(x) = \dfrac{4x^3 + 13x^2 - 32x - 15}{x^2 + 9}$.

Solutions

1. Sketch $f(x) = \dfrac{4x^3 + 13x^2 - 32x + 15}{x^2 - 9}$.

Examine the end behaviour (the leading term is dominant in the numerator and denominator):

$$f(x) = \frac{4x^3 + 13x^2 - 32x + 15}{x^2 - 9} \sim \frac{4x^3}{x^2} = 4x \text{ if } |x| \text{ is large.}$$

This tells us $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, and also that f has a slant asymptote with slope 4. If we want the actual slant asymptote we would need to divide $x^2 + 9$ into $4x^3 + 13x^2 - 32x - 15$ to determine the y-intercept of the slant asymptote.

Factor the denominator: $g(x) = x^2 - 9 = (x + 3)(x - 3)$.

We now need to factor the numerator, which is complicated looking so try the Rational Zero Theorem to get started:

Factors of $a_0 = 15$: $\pm1, \pm3, \pm5, \pm15$.
Factors of $a_3 = 4$: $\pm1, \pm2, \pm4$.
Potential rational zeros: $\pm1, \pm3, \pm5, \pm15, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \pm\frac{15}{2}, \pm\frac{1}{4}, \pm\frac{3}{4}, \pm\frac{5}{4}, \pm\frac{15}{4}$.

$$f(1) \quad = \quad 4(1)^3 + 13(1)^2 - 32(1) + 15 = 0$$

Therefore, $x = 1$ is a root. Let's factor out $x - 1$ using long division.

$$\begin{array}{r|l}
 & 4x^2 + 17x - 15 \\
\hline
x-1 & 4x^3 + 13x^2 - 32x + 15 \\
 & \underline{4x^3 - 4x^2} \\
 & \quad 17x^2 - 32x + 15 \\
 & \quad \underline{17x^2 - 17x} \\
 & \qquad -15x + 15 \\
 & \qquad \underline{-15x + 15} \\
 & \qquad\qquad 0
\end{array}$$

So
$$4x^3 + 13x^2 - 32x + 15 = (x-1)(4x^2 + 17x - 15)$$

Use the quadratic formula to factor the remaining quadratic:

Factor $4x^2+17x-15$:

$$x = \frac{-17 \pm \sqrt{17^2-4(4)(-15)}}{2(4)}$$

$$= \frac{-17 \pm \sqrt{529}}{8}$$

$$= \frac{-17 \pm 23}{8}$$

$$= -\frac{40}{8} \text{ or } \frac{6}{8}$$

$$= -5 \text{ or } 3/4$$

Remember to put a 4 in front to get coefficient of leading term correct:

$$4x^2+17x-15 = 4(x+5)(x-3/4)$$

$$= (x+5)(4x-3)$$

factored!

What we've learned from the factoring is that we can write

$$f(x) = \frac{(x-1)(x+5)(4x-3)}{(x+3)(x-3)}$$

Zeros:

- $x = 1$ multiplicity 1 (odd), so f changes sign.
- $x = -5$ multiplicity 1 (odd), so f changes sign.
- $x = 3/4$ multiplicity 1 (odd), so f changes sign.

Vertical Asymptotes:

- $x = 3$ multiplicity 1 (odd), so f changes sign.
- $x = -3$ multiplicity 1 (odd), so f changes sign.

The y-intercept is $f(0) = -\frac{15}{9}$.

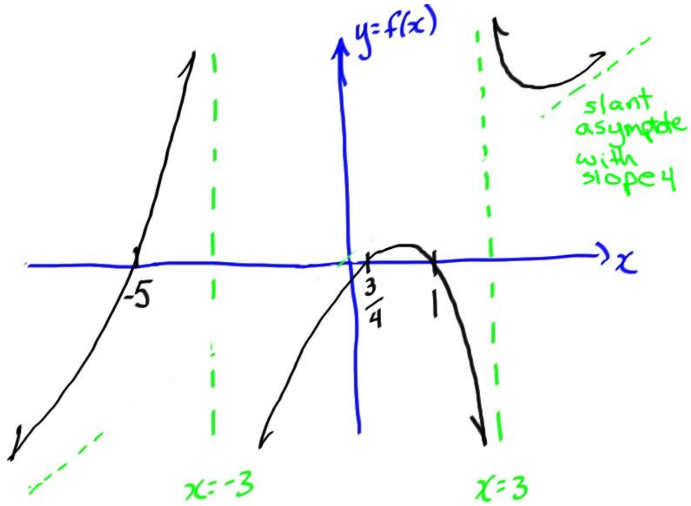
We now know all the interesting features of the graph!

First, put in the vertical asymptotes at $x = \pm 3$.
Then, put in the zeros at $x = 1, -5, 3/4$.
Then, the slant asymptote with slope 4.

Sketch by starting on the far right, where we know f is approaching $+\infty$.

As you move to the left, the function first encounters a vertical asymptote, so it must go off to $+\infty$ (it can't go to minus infinity, since there is no zero for it to pass through).

As you continue to move to the left, use the information you have obtained to get the sketch.



The scale doesn't matter so much, what matters is that it is well labeled so we can see all the interesting features. Using a computer to sketch this it would be hard to see the zeros at $x = 3/4$ and $x = 1$ unless you zoomed in.

You should note that the slant asymptote has no affect on the sketch near $x = 0$, it only represents the line $f(x)$ approaches when $|x|$ is large.

From the sketch, we can write things like the following:

$$\begin{aligned}
\text{Vertical Asymptotes: } & \lim_{x\to 3^+} f(x) = \infty \\
& \lim_{x\to 3^-} f(x) = -\infty \\
& \lim_{x\to -3^+} f(x) = -\infty \\
& \lim_{x\to -3^-} f(x) = \infty \\
\text{End Behaviour (slant asymptote): } & \lim_{x\to\infty} f(x) = \infty \\
& \lim_{x\to -\infty} f(x) = -\infty
\end{aligned}$$

If you want the equation of the slant asymptote, this is how you can get it:

$$\begin{array}{r}
4x + 13 \\
x^2+0x+9 \overline{\smash{)}\, 4x^3 + 13x^2 - 32x - 15} \\
\underline{4x^3 + 0x^2 + 36x} \\
13x^2 - 68x - 15 \\
\underline{13x^2 + 0x + 117} \\
-68x - 132
\end{array}$$

$$\text{So } f(x) = 4x + 13 + \frac{(-68x - 132)}{x^2+9}$$

$$\text{As } x \to \infty, \quad f(x) \sim 4x + 13$$

The slant asymptote is $y = 4x + 13$.