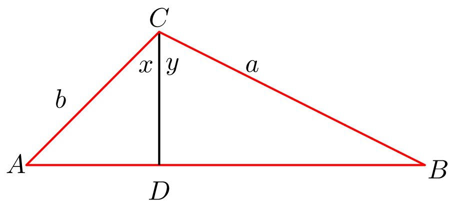
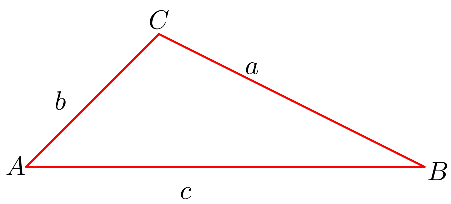
Questions

1. For the triangle given below, find $|\overline{BD}|$, $|\overline{CD}|$, $|\overline{DA}|$ in terms of a, b, x, and y.

Then use these results to prove $\sin(x + y) = \sin x \cos y + \cos x \sin y$.



2. Prove the area of a triangle can be found via the formula Area $= \dfrac{a^2 \sin B \sin C}{2 \sin A}$.

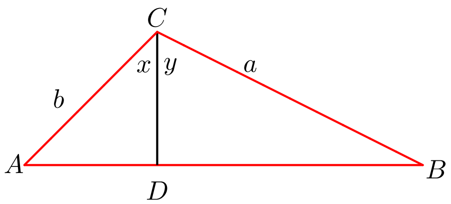


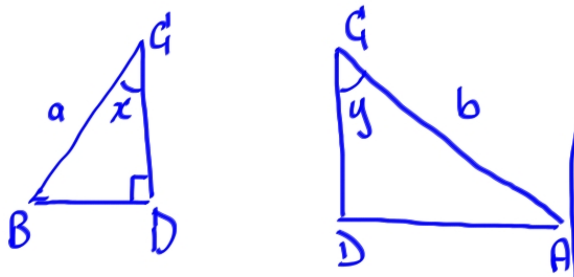
3. Find a formula for the area of a regular n-gon that is inscribed inside a circle of radius r. Express your answer in terms of n and r.

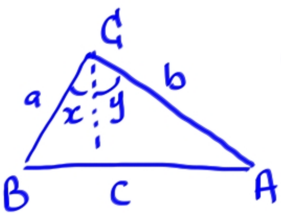
Solutions

1. For the triangle given below, find $|\overline{BD}|$, $|\overline{CD}|$, $|\overline{DA}|$ in terms of a, b, x, and y.

Then use these results to prove $\sin(x+y) = \sin x \cos y + \cos x \sin y$.







angle $C = x+y$

side $c = |\overline{BD}| + |\overline{DA}|$

$= a\sin x + b\sin y$

$\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{|\overline{BD}|}{a}$, $\sin y = \frac{|\overline{DA}|}{b}$

$\cos x = \frac{\text{adj}}{\text{hyp}} = \frac{|\overline{CD}|}{a}$, $\cos y = \frac{|\overline{CD}|}{b}$

$\Rightarrow |\overline{BD}| = a\sin x$

$|\overline{DA}| = b\sin y$

$|\overline{CD}| = a\cos x = b\cos y$

$\Rightarrow \frac{a}{b}\cos x = \cos y$ ✱

Law of Sines: $\frac{\sin C}{c} = \frac{\sin B}{b}$

$\Rightarrow \sin(x+y) = \frac{c}{b}\sin B$

Note: $\sin B = \frac{|\overline{CD}|}{a} = \cos x$

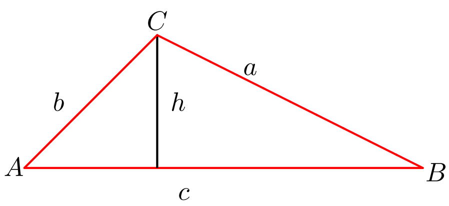
$= \frac{a\sin x + b\sin y}{b} \cdot \cos x$

$= \sin x\left(\frac{a}{b}\cos x\right) + \sin y\cos x$ use ✱

$\sin(x+y) = \sin x\cos y + \sin y\cos x$

Done!

2. Prove the area of a triangle can be found via the formula Area $= \dfrac{a^2 \sin B \sin C}{2 \sin A}$.

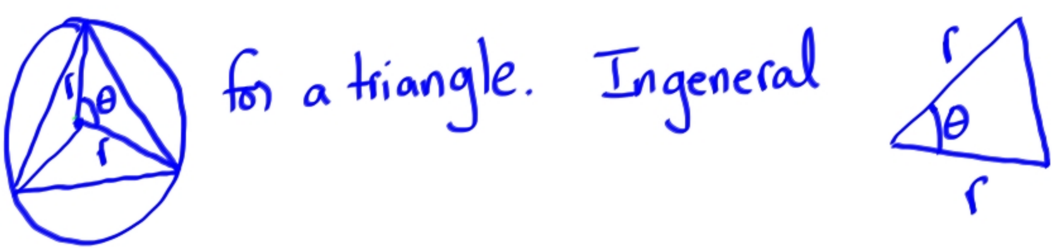


The area is usually found from the formula area $= \frac{1}{2}$(base)(perpendicular height). Let's start from there.

$$
\begin{aligned}
\text{Area } &= \frac{1}{2}ch \\
&= \frac{1}{2}cb\sin A \quad \text{since from the triangle on the left and SOH-CAH-TOA } \sin A = \frac{h}{b} \Rightarrow h = b\sin A \\
&= \frac{c^2 \sin B \sin A}{2\sin C} \quad \text{use Law of Sines } \frac{\sin C}{c} = \frac{\sin B}{b} \Rightarrow b = \frac{c \sin B}{\sin C} \\
&= \frac{(a^2 \sin^2 C \sin B \sin A}{2\sin^2 A \sin C} \quad \text{use Law of Sines } \frac{\sin C}{c} = \frac{\sin A}{a} \Rightarrow c^2 = \frac{a^2 \sin^2 C}{\sin^2 A} \\
&= \frac{(a^2 \sin C \sin B}{2 \sin A}
\end{aligned}
$$

3. Find a formula for the area of a regular n-gon that is inscribed inside a circle of radius r. Express your answer in terms of n and r.

This problem is asking us to work in general, so it is a bit difficult to draw an accurate sketch. Drawing a triangle $n = 3$ helps us get started on the problem.



An n-gon is made up of n congruent triangles, each triangle having two sides of length r. The angle between these two sides will be $\theta = 2\pi/n$.

$$\text{Area } = n \cdot \frac{1}{2}ab\sin C = \frac{nr^2}{2}\sin(2\pi/n).$$

The formula Area $= \frac{1}{2}ab\sin C$ was shown in question 2, and is known as Heron's Formula.