Questions

1. State whether the function is an exponential growth or exponential decay, and describe its end behaviour using limits.

(a) $f(x) = 3^{-2x}$

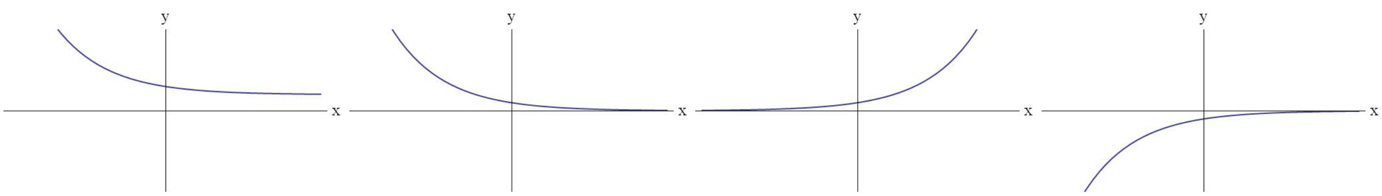
(b) $f(x) = 0.5^x$

(c) $f(x) = \left(\dfrac{1}{e}\right)^x$

(d) $f(x) = \left(\dfrac{1}{4}\right)^{-x}$

2. Match the function with its graph

(a) $y = e^{-x}$ (b) $y = e^x$ (c) $y = -e^{-x}$ (d) $y = e^{-x} + 2$



3. Graph the function $f(x) = -3e^{-x} + 2$ by transforming the basic function $y = e^x$, and analyze it for domain, range, continuity, increasing or decreasing behaviour, symmetry, boundedness, extrema, asymptotes, and end behaviour.

4. Sketch the Logistic function $f(x) = \dfrac{24}{1 + 3e^{-2x}}$. Label the y-intercept, and analyze the function for domain, range, continuity, increasing or decreasing behaviour, symmetry, boundedness, extrema, asymptotes, and end behaviour.

5. (calculus preview) In this problem we will investigate the instantaneous rate of change of the exponential function $f(x) = e^{2x-6}$.

Simplify the average rate of change $= \dfrac{f(x+h) - f(x)}{h}$ so that it involves a function of x times the quantity $\dfrac{e^{2h} - 1}{h}$.

The instantaneous rate of change $= \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$. Explain why can you not evaluate $\lim\limits_{h \to 0} \dfrac{e^{2h} - 1}{h}$ by direct substitution.

Evaluate $\lim\limits_{h \to 0} \dfrac{e^{2h} - 1}{h}$ using the following ideas.

First, sketch e^{2h} using a calculator near $h = 0$. Pick a domain that makes the function e^{2h} look like a straight line, and then determine the equation of the straight line $ah + b$ for which $e^{2h} \sim ah + b$.

Then, evaluate the limit using

$$\lim_{h \to 0} \frac{e^{2h} - 1}{h} \sim \lim_{h \to 0} \frac{(ah + b) - 1}{h}$$

Finally, write down an expression for the instantaneous rate of change of e^{2x-6}.

Solutions

1. State whether the function is an exponential growth or exponential decay, and describe its end behaviour using limits.

(a) $f(x) = 3^{-2x} = \left(\frac{1}{3}\right)^{2x} = \left(\frac{1}{9}\right)^{x}$

Base $b = 1/9 < 1$, so we have exponential decay. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$.

(b) $f(x) = 0.5^x = \left(\frac{1}{2}\right)^{x}$

Base $b = 1/2 < 1$, so we have exponential decay. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$.

(c) $f(x) = \left(\frac{1}{e}\right)^{x}$

Base $b = 1/e < 1$, so we have exponential decay. $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = \infty$.

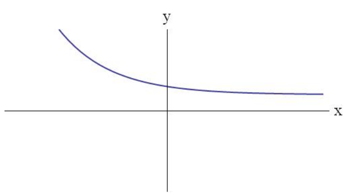
Alternately, rewrite as $f(x) = \left(\frac{1}{e}\right)^{x} = (e^{-1})^x = e^{-x}$, so we have exponential decay.

(d) $f(x) = \left(\frac{1}{4}\right)^{-x} = 4^x$

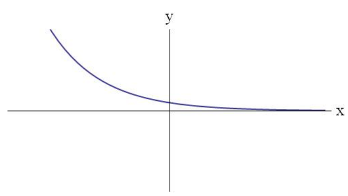
Base $b = 4 > 1$, so we have exponential growth. $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = 0$.

2. Match the function with its graph

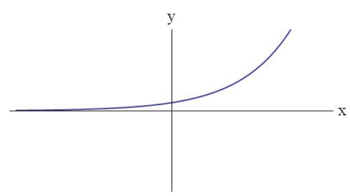
(a) $y = e^{-x}$ (b) $y = e^{x}$ (c) $y = -e^{-x}$ (d) $y = e^{-x} + 2$



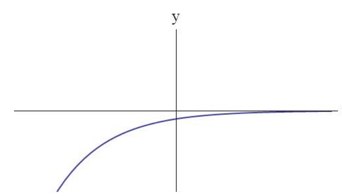
(d)



(a)



(b)

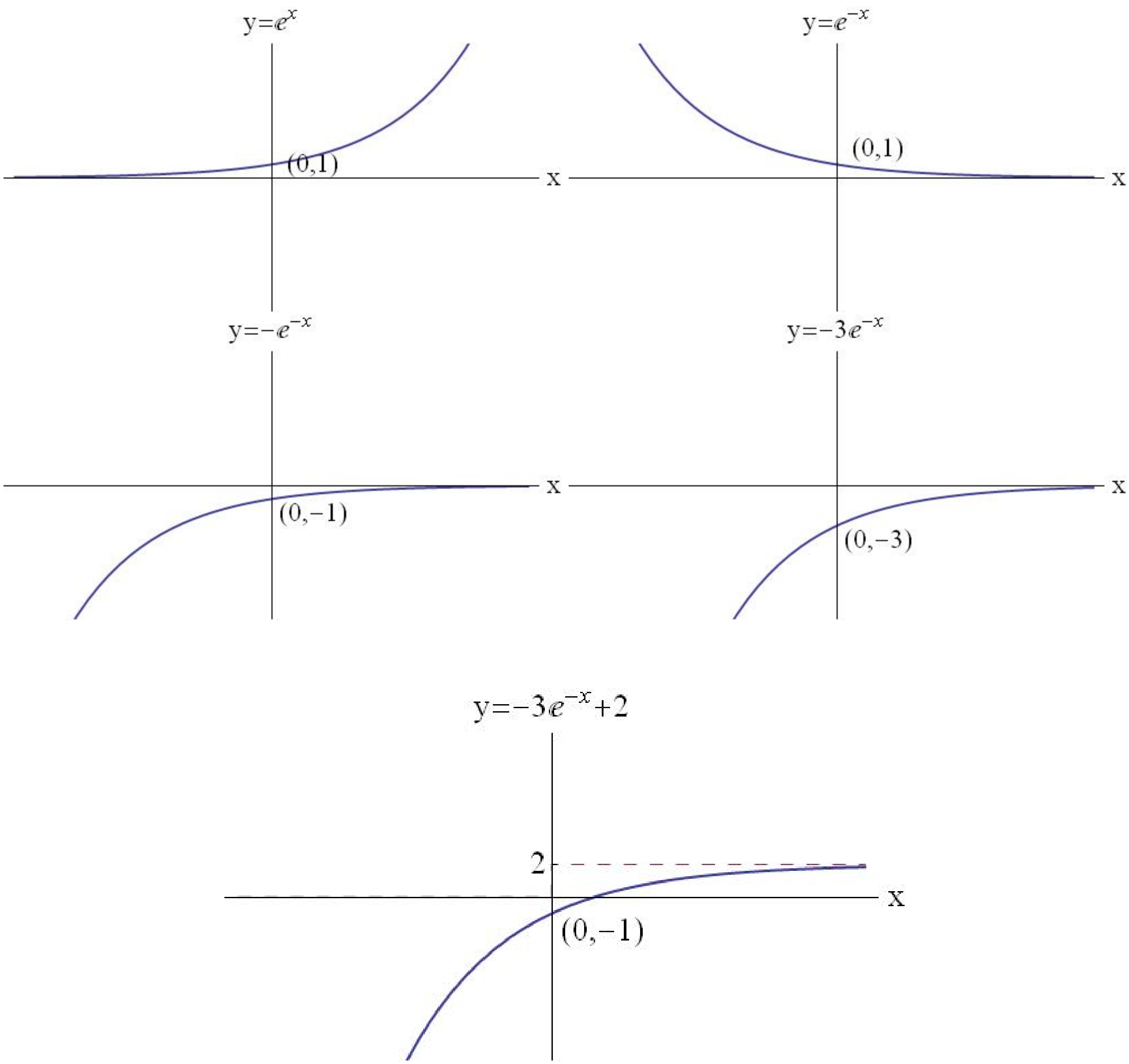


(c)

3. Graph the function $f(x) = -3e^{-x} + 2$ by transforming the basic function $y = e^x$, and analyze it for domain, range, continuity, increasing or decreasing behaviour, symmetry, boundedness, extrema, asymptotes, and end behaviour.

Here are the algebraic representations of the transformations:
Basic function: $y = g(x) = e^x$.
Flip about the y-axis: $y = g(-x) = e^{-x}$.
Flip about the x-axis: $y = -g(-x) = -e^{-x}$.
Stretch vertically by a factor of 3: $y = -3g(-x) = -3e^{-x}$.
Shift up 2 units: $y = -3g(-x) = -3e^{-x} + 2 = f(x)$.

Here are the sketches:



Domain: $x \in \mathbb{R}$
Range: $y \in (-\infty, 2)$ (it never reaches $y = 2$).
Continuity: continuous for all x
Increasing-decreasing behaviour: increasing for all x
Symmetry: none
Boundedness: bounded above
Local Extrema: none
Horizontal Asymptotes: $y = 2$
Vertical Asymptotes: none
End behaviour: $\lim_{x\to-\infty}(-3e^{-x}+2) = -\infty$ and $\lim_{x\to\infty}(-3e^{-x}+2) = 2$

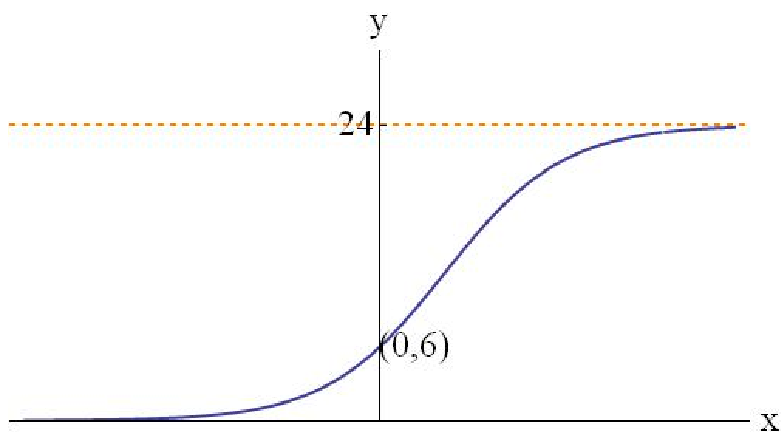
4. Sketch the Logistic function $f(x) = \dfrac{24}{1+3e^{-2x}}$. Label the y-intercept, and analyze the function for domain, range, continuity, increasing or decreasing behaviour, symmetry, boundedness, extrema, asymptotes, and end behaviour.

We know the basic shape of a logistic function, so to get the sketch we just have to figure out the value of the upper vertical asymptote. We can use the fact that $\lim_{x\to\infty} e^{-2x} = 0$ to do this.

$$\lim_{x\to\infty} \frac{24}{1+3e^{-2x}} = \frac{24}{1+3(0)} = 24$$

The y-intercept is at

$$f(0) = \frac{24}{1+3e^{0}} = \frac{24}{1+3(1)} = 6.$$



Domain: $x \in \mathbb{R}$
Range: $y \in (0, 24)$
Continuity: continuous for all x
Increasing-decreasing behaviour: increasing for all x
Symmetry: none
Boundedness: bounded above and below
Local Extrema: none
Horizontal Asymptotes: $y = 0$ and $y = 24$
Vertical Asymptotes: none
End behaviour: $\lim_{x\to-\infty} \dfrac{24}{1+3e^{-2x}} = 0$ and $\lim_{x\to\infty} \dfrac{24}{1+3e^{-2x}} = 24$

5.

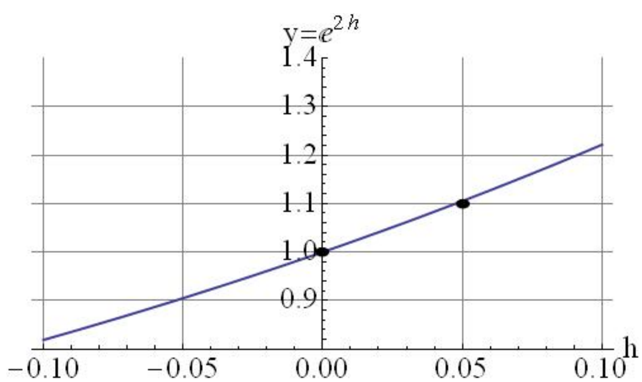
$$
\begin{aligned}
\text{average rate of change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{e^{2(x+h)-6}-e^{2x-6}}{h} \\
&= \frac{e^{2x+2h-6}-e^{2x-6}}{h} \\
&= \frac{e^{2x-6}e^{2h}-e^{2x-6}}{h} \\
&= e^{2x-6}\times\frac{e^{2h}-1}{h} \\
\text{instantaneous rate of change} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} e^{2x-6}\times\frac{e^{2h}-1}{h} \\
&= e^{2x-6}\lim_{h\to 0}\frac{e^{2h}-1}{h}
\end{aligned}
$$

If we try to use direct substitution to evaluate this limit, we get an indeterminant form:

$$\lim_{h\to 0}\frac{e^{2h}-1}{h}=\frac{e^0-1}{0}=\frac{1-1}{0}=\frac{0}{0}.$$

Remember, an indeterminant form just means we don't know what the number is, and have to do some more work to figure it out. You will learn how to treat problems like this in calculus without making an approximation like we do below. For now, we have to do some estimating to get an answer.

Here is the sketch:



It looks like the e^{2h} passes through two points, $(0, 1)$ and $(0.05, 1.1)$. The first point it passes through exactly, the second it only comes close too. From here on we are using an approximation.

The equation of the straight line through two points is:

$$
\begin{aligned}
\frac{y-y_1}{y_2-y_1} &= \frac{h-h_1}{h_2-h_1} \\
\frac{y-1}{1.1-1} &= \frac{h-0}{0.05-0} \\
\frac{y-1}{0.1} &= \frac{h}{0.05} \\
y &= \frac{0.1h}{0.05}+1 \\
&= 2h+1
\end{aligned}
$$

So we have shown $e^{2h} \sim 2h+1$ when $|h| \sim 0$.

Let's use this to work out the instantaneous rate of change:

$$
\begin{aligned}
\text{instantaneous rate of change} &= e^{2x-6}\lim_{h\to 0}\frac{e^{2h}-1}{h} \\
&\sim e^{2x-6}\lim_{h\to 0}\frac{(2h+1)-1}{h} \\
&= e^{2x-6}\lim_{h\to 0}\frac{2h}{h} \\
&= e^{2x-6}\lim_{h\to 0}2 \\
&= e^{2x-6}(2) \\
&= 2e^{2x-6}
\end{aligned}
$$

Therefore, we have shown that the instantaneous rate of change of e^{2x-6} is $2e^{2x-6}$.

You will learn a way of showing this using derivatives when you take calculus, that does not rely on the approximations we have made.