Note: Sketching the modified 12 basic functions $y = \sqrt{x-1}$ and $y = \frac{1}{x-1}$ by hand will be easier when we look at graphical transformations. Have a try at sketching them now, but realize sketching is a topic we will revisit in more detail in the future.

Questions

1. Compute the average rate of change of $f(x) = x^2$ for the interval $(x, x+h)$. Simplify so substitution of $h = 0$ does not yield $\frac{0}{0}$. What is the value when $x = 2, h = 4$? Sketch the situation by hand.

2. Compute the average rate of change of $f(x) = \sqrt{x-1}$ for the interval $(x, x+h)$. Simplify so substitution of $h = 0$ does not yield $\frac{0}{0}$. What is the value when $x = 2, h = 4$? Sketch the situation by hand.

3. Compute the average rate of change of $f(x) = \frac{1}{x-1}$ for the interval $(x, x+h)$. Simplify so substitution of $h = 0$ does not yield $\frac{0}{0}$. What is the value when $x = 2, h = 4$? Sketch the situation by hand.

4. Given $f(x) = x^2 - 2$, simplify the quantity $\dfrac{f(x+h) - f(x-h)}{2h}$ so that substitution of $h = 0$ does not give $\frac{0}{0}$.

Note: The quantity $\dfrac{f(x+h) - f(x-h)}{2h}$ *is actually the average rate of change of* f *over the interval* $(x-h, x+h)$*. Do you think you could draw a sketch showing this situation? I'll include a sketch for* $x = 2$*,* $h = 4$ *in my solution if you want to try it.*

5. Simplify as much as possible the expression for the average rate of change of the function f over the interval $(x, x+h)$,

$$\text{Average Rate of Change } = \frac{f(x+h) - f(x)}{h},$$

where $f(x) = 3x^2 - 5x + 6$.

Solutions

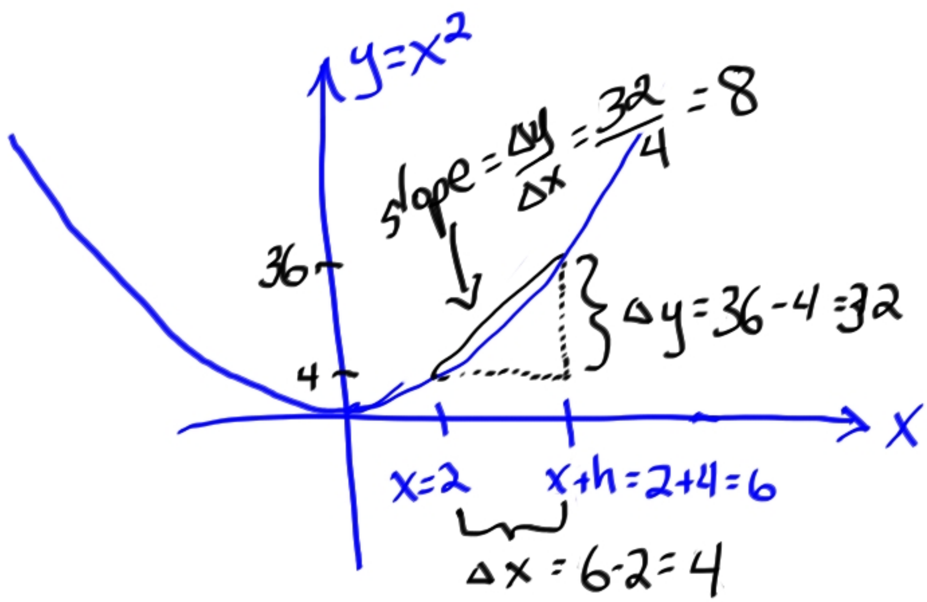
1. The function is a power function, so expect to factor out.

$$f(x) = x^2$$
$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2$$

$$\begin{aligned}\text{Average Rate of Change} &= \frac{f(x+h) - f(x)}{h}\\ &= \frac{x^2 + 2xh + h^2 - x^2}{h}\\ &= \frac{2xh + h^2}{h}\\ &= \frac{h(2x+h)}{h}\\ &= 2x + h\end{aligned}$$

When $x = 2$ and $h = 4$, the average rate of change on the interval $(x, x+h) = (2, 6)$ is $2(2) + (4) = 8$.

Sketch:



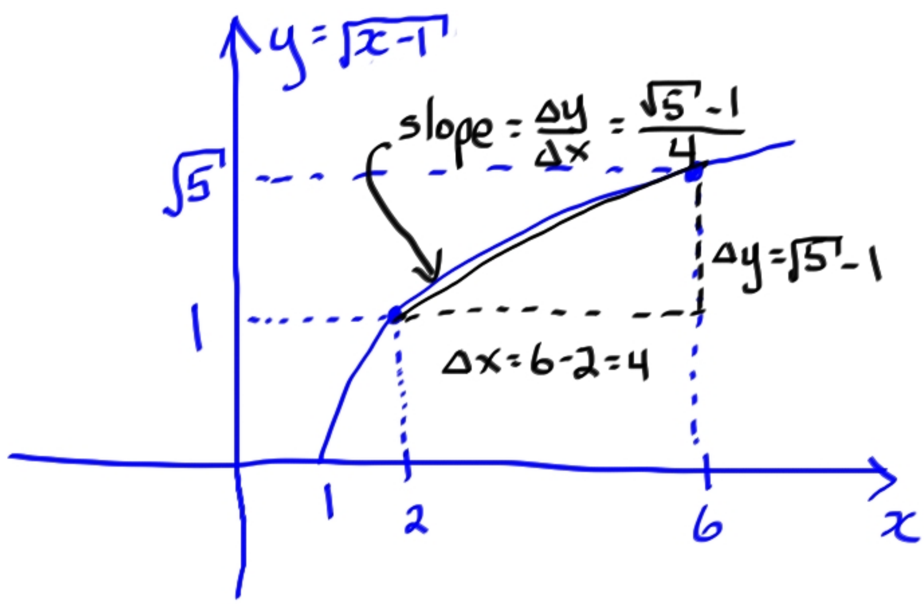
2. The function is a square root function, so expect to rationalize.

$$
\begin{aligned}
f(x) &= \sqrt{x-1} \\
f(x+h) &= \sqrt{(x+h)-1} = \sqrt{x+h-1}
\end{aligned}
$$

$$
\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \\
&= \frac{\sqrt{x+h-1}-\sqrt{x-1}}{h} \cdot \frac{\sqrt{x+h-1}+\sqrt{x-1}}{\sqrt{x+h-1}+\sqrt{x-1}} \\
&= \frac{(x+h-1)-(x-1)}{h(\sqrt{x+h-1}+\sqrt{x-1})} \\
&= \frac{\cancel{x}+h\cancel{-1}-\cancel{x}\cancel{+1}}{h(\sqrt{x+h-1}+\sqrt{x-1})} \\
&= \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h-1}+\sqrt{x-1})} \\
&= \frac{1}{\sqrt{x+h-1}+\sqrt{x-1}}
\end{aligned}
$$

When $x = 2$ and $h = 4$, the average rate of change on the interval $(x, x+h) = (2, 6)$ is $\frac{1}{\sqrt{(2)+(4)-1}+\sqrt{2-1}} = \frac{1}{\sqrt{5}+1}$.

Sketch:



Notice that this is the correct average rate of change, since:

$$
\begin{aligned}
\frac{1}{\sqrt{5}+1} &= \frac{1}{\sqrt{5}+1} \cdot \frac{\sqrt{5}-1}{\sqrt{5}-1} \text{ rationalize denominator} \\
&= \frac{\sqrt{5}-1}{5-1} = \frac{\sqrt{5}-1}{4}
\end{aligned}
$$

Either answer is acceptable.

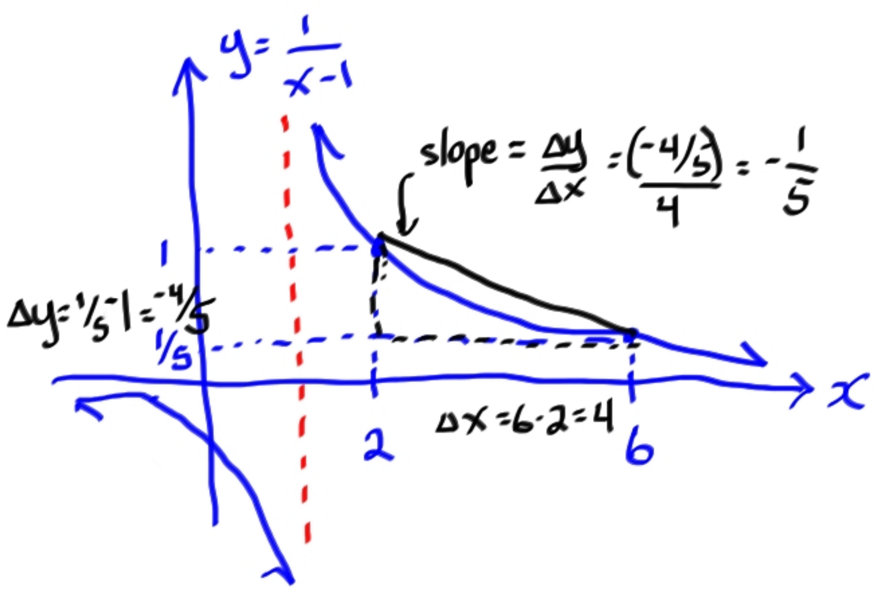
3. The function has a fraction (it's what we will call a *rational function*), so expect to find a common denominator.

$$f(x) = \frac{1}{x-1}$$
$$f(x+h) = \frac{1}{(x+h)-1} = \frac{1}{x+h-1}$$

$$\begin{aligned}
\text{Average Rate of Change} &= \frac{f(x+h)-f(x)}{h} \\
&= \frac{\frac{1}{x+h-1} - \frac{1}{x-1}}{h} \\
&= \frac{1}{h}\left(\frac{1}{x+h-1} - \frac{1}{x-1}\right) \\
&= \frac{1}{h}\left(\frac{x-1}{(x+h-1)(x-1)} - \frac{x+h-1}{(x+h-1)(x-1)}\right) \\
&= \frac{1}{h}\left(\frac{\cancel{x}-\cancel{1}-\cancel{x}-h+\cancel{1}}{(x+h-1)(x-1)}\right) \\
&= \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{(x+h-1)(x-1)}\right) \\
&= \frac{-1}{(x+h-1)(x-1)}
\end{aligned}$$

When $x=2$ and $h=4$, the average rate of change on the interval $(x, x+h) = (2,6)$ is $\frac{-1}{(x+h-1)(x-1)} = \frac{-1}{((2)+(4)-1)((2)-1)} = -\frac{1}{5}$.

Sketch:



4. The function is a power function, so expect to factor out.

$$
\begin{aligned}
f(x) &= x^2 - 2 \\
f(x+h) &= (x+h)^2 - 2 = x^2 + 2xh + h^2 - 2 \\
f(x-h) &= (x-h)^2 - 2 = x^2 - 2xh + h^2 - 2
\end{aligned}
$$

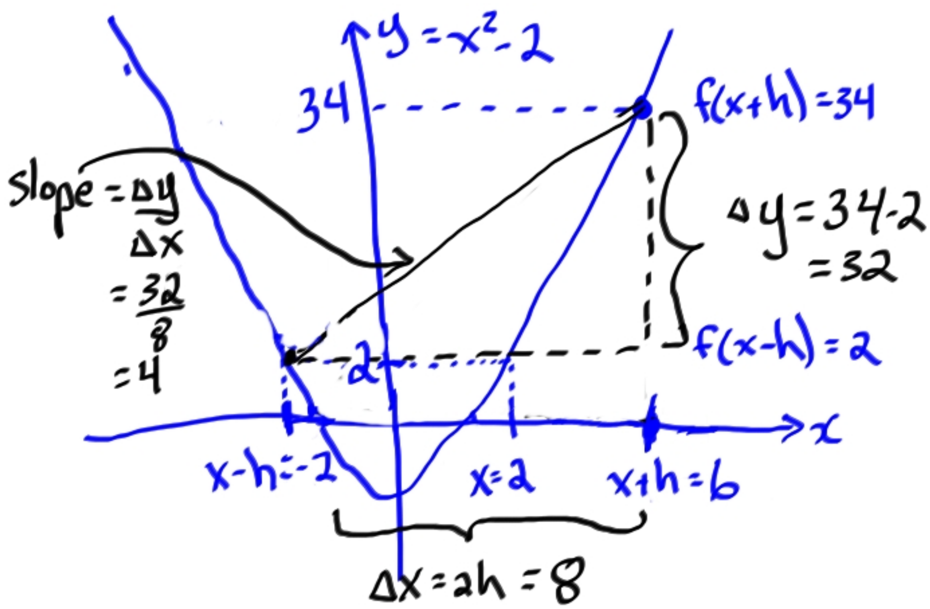
$$
\begin{aligned}
\frac{f(x+h) - f(x-h)}{2h} &= \frac{(x^2 + 2xh + h^2 - 2) - (x^2 - 2xh + h^2 - 2)}{2h} \\
&= \frac{x^2 + 2xh + h^2 - 2 - x^2 + 2xh - h^2 + 2}{2h} \\
&= \frac{\cancel{x^2} + 2xh + \cancel{h^2} - \cancel{2} - \cancel{x^2} + 2xh - \cancel{h^2} + \cancel{2}}{2h} \\
&= \frac{4xh}{2h} \\
&= 2x
\end{aligned}
$$

Notice it doesn't, in this case depend on h!

You weren't asked to do this, but here is a sketch of the situation:

When $x = 2$ and $h = 4$, the average rate of change on the interval $(x - h, x + h) = (2 - 4, 2 + 4) = (-2, 6)$ is $2(2) = 4$.

Sketch:



5. Simplify as much as possible the expression for the average rate of change of the function f over the interval $(x, x+h)$,

$$\text{Average Rate of Change } = \frac{f(x+h) - f(x)}{h},$$

where $f(x) = 3x^2 - 5x + 6$.

First, let's simplify some of the quantities we need, and get the functional substitution done correctly. The tricky one is $f(x+h)$, which we find as:

$$\begin{aligned} f(\qquad) &= 3(\qquad)^2 - 5(\qquad) + 6 \\ f(x+h) &= 3(x+h)^2 - 5(x+h) + 6 \\ &= 3(x^2 + 2xh + h^2) - 5x - 5h + 6 \\ &= 3x^2 + 6xh + 3h^2 - 5x - 5h + 6 \end{aligned}$$

Now we can simplify the average rate of change.

$$\begin{aligned} \text{Average Rate of Change } &= \frac{f(x+h) - f(x)}{h} \\ &= \frac{(3x^2 + 6xh + 3h^2 - 5x - 5h + 6) - (3x^2 - 5x + 6)}{h} \\ &= \frac{3x^2 + 6xh + 3h^2 - 5x - 5h + 6 - 3x^2 + 5x - 6}{h} \\ &= \frac{6xh + 3h^2 - 5h}{h} \\ &= \frac{h(6x + 3h - 5)}{h} \\ &= 6x + 3h - 5 \end{aligned}$$