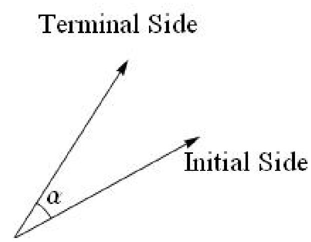
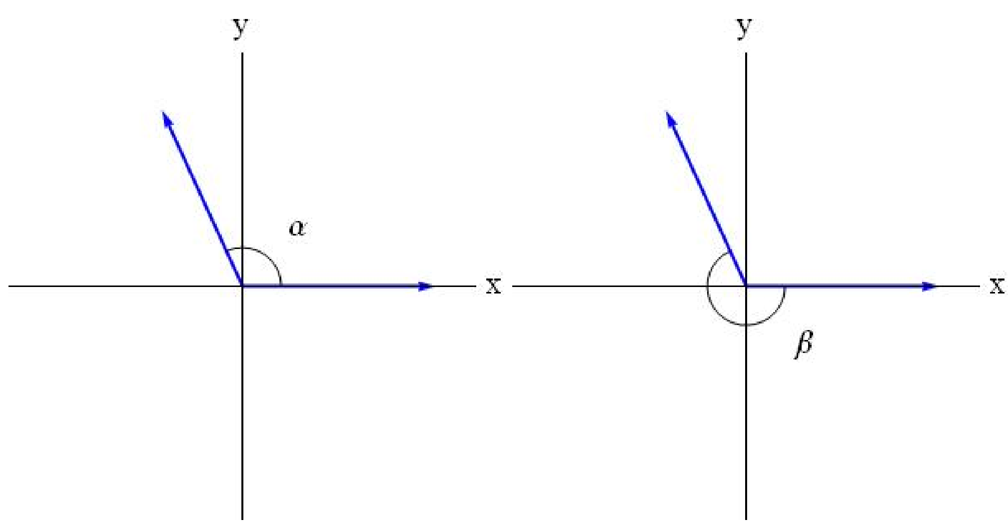
Concepts: nomenclature (initial side, terminal side, standard position, coterminal angles, quadrants), using coordinate system to solve trig problems, unit circle, quadrantal angles, new definitions for trigonometric functions that work for any angle.

Trigonometric Functions of Any Angle

Initial Side: Beginning position of the line which is to be rotated.
Vertex: endpoint of line, about which the line is rotated.
Terminal Side: The final position of the line which was rotated.
Positive Angles: rotate counterclockwise.
Negative Angles: rotate clockwise.



Standard position aligns the initial side along the positive direction of the x axis. This introduces a coordinate system into the notation. The angles are always measured from the x-axis (the initial side).

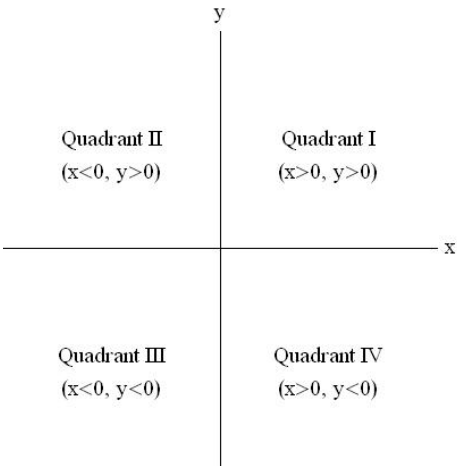


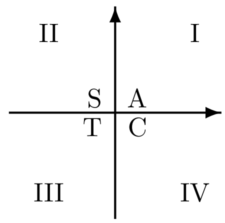
The angles $\alpha > 0$ and $\beta < 0$ both describe the same terminal side. Note that since $\beta < 0$, we have $\alpha - \beta = 2\pi$.

Coterminal angles are angles which, although different, have the same terminal sides. The angles α and β are therefore coterminal angles. Coterminal angles are most easily found by adding rotations which are multiples of 2π radians, since that will lead to the same terminal side.

$\alpha = \alpha + 2n\pi, \quad n = \ldots, -2, -1, 0, 1, 2, \ldots$ are all coterminal angles.

We divide the coordinate plane into four quadrants, each quadrant has different different signs for the (x, y) pair.





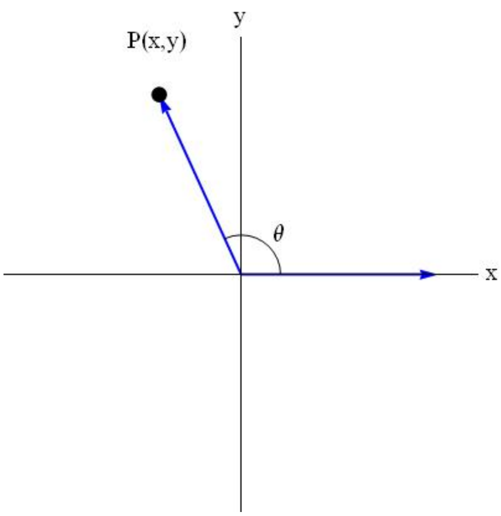
C: Cosine is positive in this Quadrant

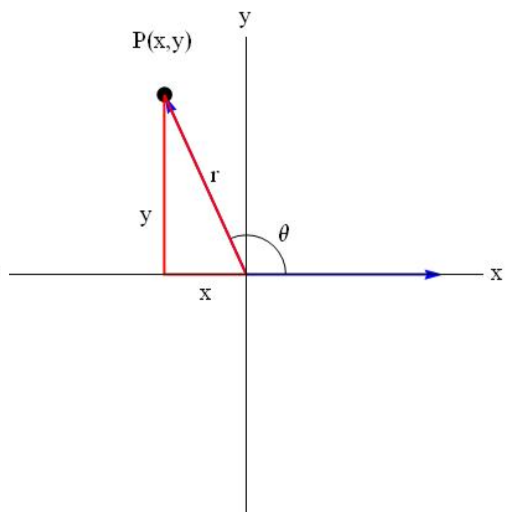
A: All are positive in this Quadrant

S: Sine is positive in this Quadrant

T: Tangent is positive in this Quadrant

What we need to know know are the signs of the trigonometric functions for angles that land in the different quadrants. For example, below we have a point $P(x, y)$ that is in Quadrant II. We can construct a triangle based on the (x, y) coordinates that locate P.





Let $r = \sqrt{x^2 + y^2}$, we can construct new definitions for the six trig functions and our diagram (memorize, these are as useful as SOH-CAH-TOA, which they reduce to if P is in Quadrant I):

$$\sin\theta = \frac{y}{r} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x},\ \ x \neq 0$$

$$\cos\theta = \frac{x}{r} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y},\ \ y \neq 0$$

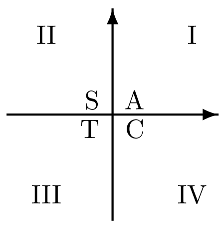
$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x},\ \ x \neq 0 \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y},\ \ y \neq 0$$

Note: These definitions lead to the concept of *Polar Coordinates*, which will will look at later, which defines new coordinates (r, θ), instead of (x, y), using $x = r\cos\theta$ and $y = r\sin\theta$.

We drew the point P in Quadrant II to help motivate the new expressions for the trigonometric functions, but the expressions hold no matter what quadrant the point P is in. Here's an example of how they can be used.

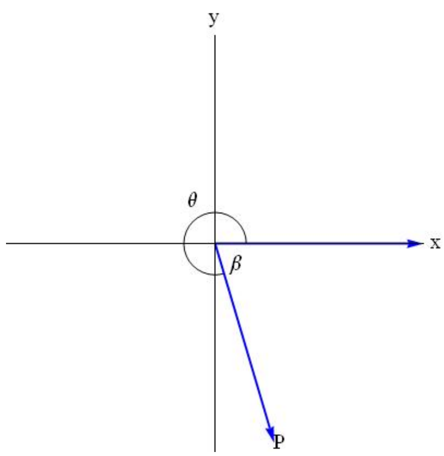
Example Find $\cos\theta$ and $\sin\theta$ if $\tan\theta = -\dfrac{17}{88}$ and $\sin\theta < 0$.

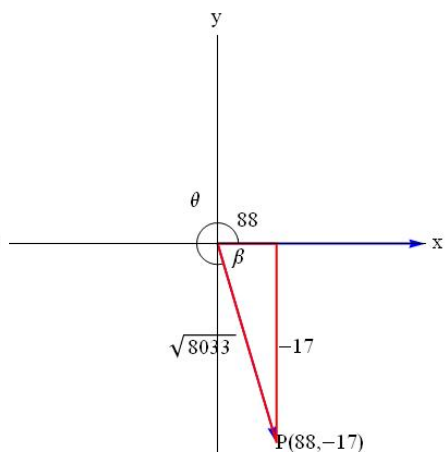
Let's solve this by drawing a coordinate system. First, we need to figure out what Quadrant P lies in.



When $\sin\theta = \dfrac{y}{r} < 0 \Rightarrow P$ is in either QIII or QIV.
When $\tan\theta = \dfrac{y}{x} < 0 \Rightarrow P$ is in either QII or QIV.

Therefore, we must be in Quadrant IV (so both above facts are true). Our angle has a terminal side in the fourth quadrant. Sketch!





In the second diagram, I have labelled the lengths of the sides, being careful to indicate quantities which are less than zero based on the quadrant the point P is in. I did this by using the fact that

$$\tan\theta = \frac{y}{x} = \frac{-17}{88} \Rightarrow x = 88, y = -17.$$

Basically, the diagrams helped us figure out that the minus sign went with the y coordinate.
Also, we have found the hypotenuse of the triangle using the Pythagorean theorem.

$$r = \sqrt{x^2 + y^2} = \sqrt{(-17)^2 + (88)^2} = \sqrt{8033}.$$

Now, we can use the new definitions

$$\cos\theta = \frac{x}{r} = \frac{88}{\sqrt{8033}}$$

$$\sin\theta = \frac{y}{r} = \frac{-17}{\sqrt{8033}}$$

You could solve the above example using reference triangles. I prefer the coordinate axis solution above, so will tend to stick with that type of solution.

The Unit Circle

The *unit circle* is a circle of radius 1 centered at the origin. Since the radius is $r = 1$, the coordinates of point along the unit circle relate to the sine and cosine since

$$\begin{aligned} x &= r\cos\theta = \cos\theta \\ y &= r\sin\theta = \sin\theta \end{aligned}$$

Note this is only true on the unit circle!

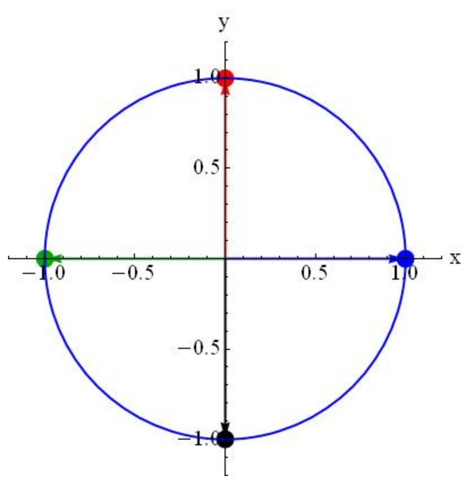
The unit circle is pictured in the text. I will not refer to the unit circle very often, since most of the time people use it to memorize a bunch of trig values, which is pointless. I prefer to understand how to use the 30-60-90 and 45-45-90 triangles, along with what quadrant the terminal side of the angle is in to determine the sign, than to memorize things that are easily forgotten or confused (the unit circle falls into that category).

This is in line with a saying that has been on the board in my office for years:
$\boxed{\text{CUBMA!} \Rightarrow \text{Conceptual Understanding Beats Memorization–Always!}}$

The unit circle can be useful of course, but try to understand it rather than simply memorize it. I find it useful to have memorized the first quadrant and work out the rest as you need them. If you forget what the values are in the first quadrant, you can use the 30-60-90 and 45-45-90 triangles to work it out!

The key fact to remember is that the x goes with the cosine, and the y goes with the sine.

Definition: *Quadrantal angles* are angles which result in a terminal side which aligns with the coordinate axes, so $\theta = 0, \pi/2, \pi, 3\pi/2$ are quadrantal angles. In these cases there is no reference triangle, but the trig functions can be easily determined by using the unit circle and the definitions $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ etc.



$$\cos 0 = \frac{x}{r} = \frac{1}{1} = 1, \quad \csc 3\pi/2 = \frac{1}{\sin 3\pi/2} = \frac{r}{y} = \frac{1}{-1} = -1, \quad \tan\pi/2 = \frac{y}{x} = \frac{1}{0} \text{ not defined, etc.}$$

Definition: A function f is *periodic with period* c if $f(x+c) = f(x)$ and c is the smallest number for which this relation holds.

The sine and cosine are periodic with period 2π. The tangent function has period π, which we shall see later.