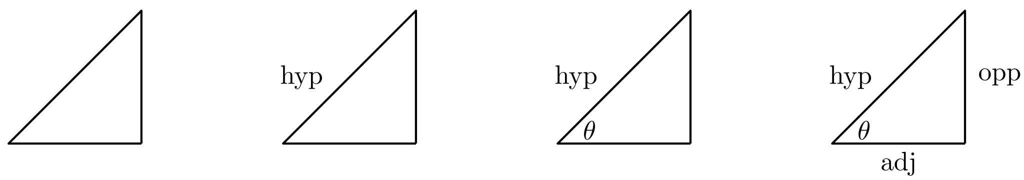
Concepts: Similar Triangles, Definitions of Sine, Cosine, Tangent, Cosecant, Secant, Cotangent (SOH-CAH-TOA); 45-45-90 triangle, 30-60-90 triangle.

Right Triangle Trigonometry

Two triangles are *similar* if they have the same shape. The shape is the same if the angles in one triangle are conguent (meaning the angles coincide when the triangles are superimposed) to the angles in the other, and the sides in one triangle are proportional to the sides in the other.

For right triangles, all we need for two triangles to be similar is that one of the acute angles in one triangle is equal to one of the acute triangles in the other. Remember, acute angles are angles which are less than $\pi/2$ radians.



The acute angle here is labeled θ. The hypotenuse is always the length of the side opposite the right angle in the triangle. Once the angle is labeled, the remaining sides of the triangle can be labeled as adjacent to the angle θ, and opposite to the angle θ (remember, these are lengths of the sides).

We can use the Pythagorean theorem to relate the lengths of the sides of the right triangle:

$$(\text{hypotenuse})^2 = (\text{opposite})^2 + (\text{adjacent})^2$$

The six basic trigonometric functions relate the angle θ to ratios of the length of the sides of the right triangle:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}, \quad \cos\theta = \frac{\text{adj}}{\text{hyp}}, \quad \tan\theta = \frac{\text{opp}}{\text{adj}}, \quad \csc\theta = \frac{\text{hyp}}{\text{opp}}, \quad \sec\theta = \frac{\text{hyp}}{\text{adj}}, \quad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

How to Remember these Six Relations: SOH CAH TOA

SOH: Sine is Opposite over Hypotenuse
CAH: Cosine is Adjacent over Hypotenuse
TOA: Tangent is Opposite over Adjacent

The other three are the multiplicative inverses (reciprocals), that is

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite}}, \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent}}, \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent}}{\text{opposite}}.$$

Remember that cosecant is one over sine since the first letter changes.
Remember that secant is one over cosine since the first letter changes.

Functional Notation

The sine function is just that, a function, meaning it takes elements from a domain as input and returns elements in the range as output. So, think of $\sin\theta$ as $\sin(\theta)$, which is similar to $f(\theta)$.

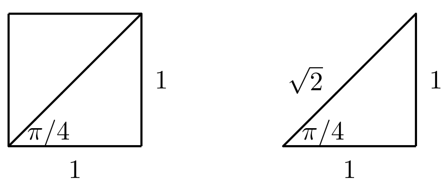
Do not think of the trigonometric functions as multiplication, and note that this means $\sin x \neq x \sin$ since the sine function is not acting on anything on right hand side–writing $x \sin$ instead of $\sin x$ is a serious error! For the same reason, we have $\sin(2x) \neq 2\sin x$ (just as we know $f(2x) \neq 2f(x)$ for a function f). To work with something like $\sin(2x)$ will require trig identities, which we will study in the next unit.

Special Triangles

The six basic trigonometric functions relate the angle θ to ratios of the length of the sides of the right triangle. For certain triangles, the trig functions of the angles can be found geometrically. These special triangles occur frequently enough that it is expected that you learn the value of the trig functions for the special angles.

A 45-45-90 Triangle

Consider the square given below.



The angle here must be $\pi/4$ radians ($45°$), since this triangle is half of a square of side length 1.

Now, we can write down all the trig functions for an angle of $\pi/4$ radians $= 45°$:

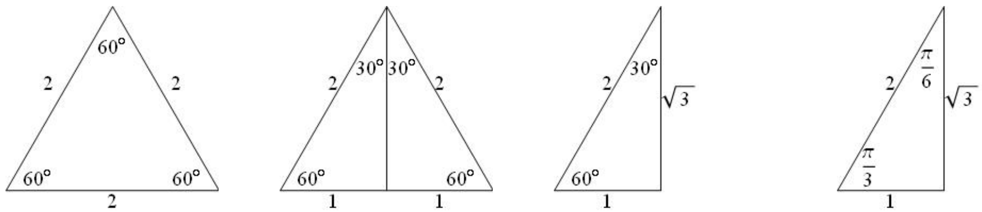
$$\sin\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \csc\left(\frac{\pi}{4}\right) = \frac{1}{\sin\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

$$\cos\left(\frac{\pi}{4}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{\sqrt{2}} \qquad \sec\left(\frac{\pi}{4}\right) = \frac{1}{\cos\left(\frac{\pi}{4}\right)} = \sqrt{2}$$

$$\tan\left(\frac{\pi}{4}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{1} = 1 \qquad \cot\left(\frac{\pi}{4}\right) = \frac{1}{\tan\left(\frac{\pi}{4}\right)} = 1$$

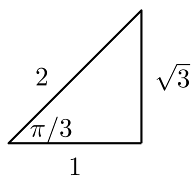
A 30-60-90 Triangle

Consider the equilateral triangle given below. Geometry allows us to construct a 30-60-90 triangle:



We can now determine the six trigonometric functions at two more angles! Note that the little reference triangles I have drawn below are not to scale.

$60^\circ = \dfrac{\pi}{3}$ radians:

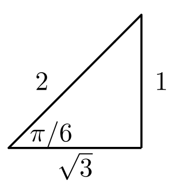


$$\sin\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{\pi}{3}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\cos\left(\frac{\pi}{3}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{1}{2} \qquad \sec\left(\frac{\pi}{3}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = 2$$

$$\tan\left(\frac{\pi}{3}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sqrt{3}}{1} = \sqrt{3} \qquad \cot\left(\frac{\pi}{3}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{1}{\sqrt{3}}$$

$30^\circ = \dfrac{\pi}{6}$ radians:



$$\sin\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{1}{2} \qquad \csc\left(\frac{\pi}{6}\right) = \frac{1}{\sin\left(\frac{\pi}{3}\right)} = 2$$

$$\cos\left(\frac{\pi}{6}\right) = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{3}}{2} \qquad \sec\left(\frac{\pi}{6}\right) = \frac{1}{\cos\left(\frac{\pi}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{\text{opposite}}{\text{adjacent}} = \frac{1}{\sqrt{3}} \qquad \cot\left(\frac{\pi}{6}\right) = \frac{1}{\tan\left(\frac{\pi}{3}\right)} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

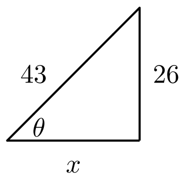
Other Angles

For other acute angles, where geometry cannot help us determine the right angle triangle associated with the angle, we can rely on a calculator to help us. Make sure your calculator is measuring angles in the correct units when you evaluate the trigonometric functions!

$$\sin(0.34) = 0.333487$$

$$\sin(\pi/10) = \frac{1}{4}(\sqrt{5} - 1) \sim 0.309017$$

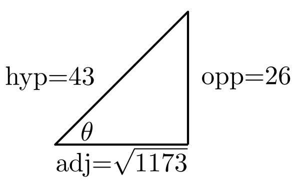
Example Find the value of all six of the trigonometric functions of the angle θ given the following right angle triangle. Note the triangle is not drawn to scale.



First, we need to use the Pythagorean theorem to find the length of the side adjacent to θ in the triangle, which I have labelled x:

$$\begin{aligned} x^2 + (26)^2 &= (43)^2 \\ x &= \sqrt{1173} \end{aligned}$$

The triangle can be labelled as



We need the trigonometric relations for the six trig functions, use SOH-CAH-TOA and reciprocals to get the correct ratios:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{26}{43} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{43}{26}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{1173}}{43} \qquad \sec\theta = \frac{1}{\cos\theta} = \frac{43}{\sqrt{1173}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{26}{\sqrt{1173}} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\sqrt{1173}}{26}$$