Concepts: Sum and Difference Trig Identities.

In this section we work out the cosine and sine of a sum and difference of two angles. I want you to be able to reproduce the proofs of where these identities come from, as well as know the identities we shall derive.

Another favourite quote from the text:

> *These identities provide clear examples of how different the algebra of functions can be from the algebra of real numbers.*

The idea is to prove one of the identities using geometry, and then use <u>that</u> identity to prove the next identity, and then use the new identity to prove the next, and so on.

What the Identities are <u>Not</u>

$$\cos(u+v) \neq \cos u + \cos v$$

This is easily shown to be a nonidentity if we evaluate both sides at $u = \pi/2$ and $v = \pi/2$.

$$\begin{aligned} \cos(\pi/2 + \pi/2) &= \cos\pi = -1 \\ \cos\pi/2 + \cos\pi/2 &= 0 + 0 = 0 \end{aligned}$$

Since $\cos(\pi/2 + \pi/2) \neq \cos\pi/2 + \cos\pi/2$, we know $\cos(u+v) \neq \cos u + \cos v$.

Similarly, we can show

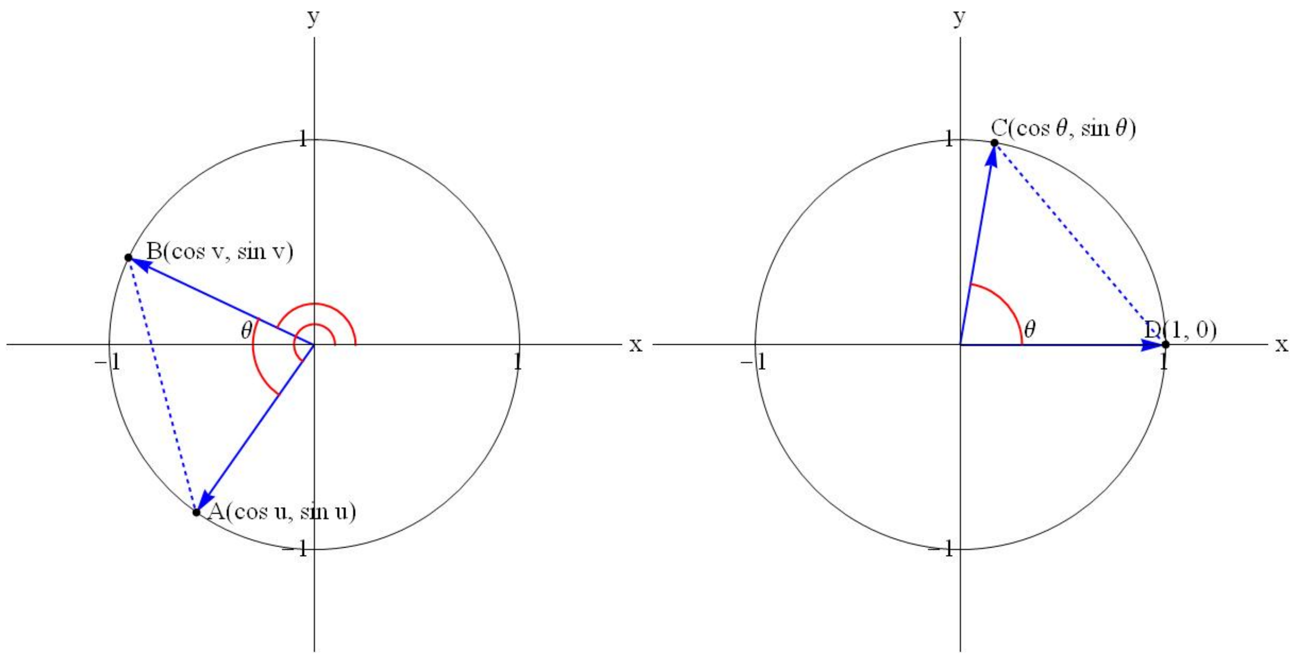
$$\begin{aligned} \cos(u-v) &\neq \cos u - \cos v \\ \sin(u \pm v) &\neq \sin u \pm \sin v \\ \tan(u \pm v) &\neq \tan u \pm \tan v \end{aligned}$$

The Cosine of a Difference Identity

To get the cosine of a difference, let's draw a diagram involving the unit circle and see what we can learn.

The angle u leads to a point $A(\cos u, \sin u)$ on the unit circle.
The angle v leads to a point $B(\cos v, \sin v)$ on the unit circle.
The angle $\theta = u - v$ is the angle between the the terminal sides of u and v.
The dotted line connects the points A and B.

This leads to two sketches:



The sketch on the left can be rotated so the angle θ is in standard position (initial side along the x axis). This gives the sketch on the right.

The dashed lines are the same length in both pictures. Therefore, we can use the distance between two points formula

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

and we can write:

$$\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2} = \sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2}$$

Now all we have to do is simplify this expression! Remember, $\theta = u - v$, so we want to solve this for $\cos\theta = \cos(u - v)$.

$$\begin{aligned}
(\sqrt{(\cos u - \cos v)^2 + (\sin u - \sin v)^2})^2 &= (\sqrt{(\cos\theta - 1)^2 + (\sin\theta - 0)^2})^2 \\
(\cos u - \cos v)^2 + (\sin u - \sin v)^2 &= (\cos\theta - 1)^2 + (\sin\theta - 0)^2 \\
(\cos^2 u + \cos^2 v - 2\cos u\cos v) + (\sin^2 u + \sin^2 v - 2\sin u\sin v) &= (\cos^2\theta + 1 - 2\cos\theta) + \sin^2\theta \\
(\cos^2 u + \sin^2 u) - 2\cos u\cos v + (\cos^2 v + \sin^2 v) - 2\sin u\sin v &= (\cos^2\theta + \sin^2\theta) + 1 - 2\cos\theta) \\
(1) - 2\cos u\cos v + (1) - 2\sin u\sin v &= (1) + 1 - 2\cos\theta \\
2 - 2\cos u\cos v - 2\sin u\sin v &= 2 - 2\cos\theta \\
2 - 2\cos u\cos v - 2\sin u\sin v &= 2 - 2\cos\theta \\
-2\cos u\cos v - 2\sin u\sin v &= -2\cos\theta \\
+\cos u\cos v + \sin u\sin v &= +\cos\theta \\
\cos\theta = \cos(u - v) &= \cos u\cos v + \sin u\sin v
\end{aligned}$$

We have arrived at the trig identity $\boxed{\cos(u - v) = \cos u\cos v + \sin u\sin v}$.

The Cosine of a Sum Identity

We can use the result we just found.

$$\begin{aligned} \cos(u-v) &= \cos u \cos v + \sin u \sin v \\ \cos(u+v) = \cos(u-(-v)) &= \cos u \cos(-v) + \sin u \sin(-v) && \text{cosine is even } \cos(-v) = \cos v \\ &= \cos u \cos(v) + \sin u(-\sin(v)) && \text{sine is odd } \sin(-v) = -\sin v \\ &= \cos u \cos v - \sin u \sin v \end{aligned}$$

We have arrived at the trig identity $\boxed{\cos(u+v) = \cos u \cos v - \sin u \sin v}$.

We can combine the two previous identities and write them as $\boxed{\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v}$.

Note the sign change.

The Sine of a Sum Identity

We can use the result we just found.

$$\begin{aligned} \sin(u+v) = \cos\left(\frac{\pi}{2} - (u+v)\right) &= \cos\left(\left(\frac{\pi}{2} - u\right) + v\right) && \text{cofunction identity with } \alpha = u+v \;\; \sin\alpha = \cos(\pi/2 - \alpha) \\ &= \cos\left(\frac{\pi}{2} - u\right)\cos v + \sin\left(\frac{\pi}{2} - u\right)\sin v && \text{cofunction identity } \sin(\pi/2 - u) = \cos u \\ &= \sin u \cos v + \cos u \sin v \end{aligned}$$

We have arrived at the trig identity $\boxed{\sin(u+v) = \sin u \cos v + \cos u \sin v}$.

The Sine of a Difference Identity

We can use the result we just found.

$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \cos u \sin v \\ \sin(u-v) = \sin(u+(-v)) &= \sin u \cos(-v) + \cos u \sin(-v) && \text{cosine is even } \cos(-v) = \cos v \\ &= \sin u \cos v - \cos u \sin v && \text{sine is odd } \sin(-v) = -\sin v \end{aligned}$$

We have arrived at the trig identity $\boxed{\sin(u-v) = \sin u \cos v - \cos u \sin v}$.

We can combine the two previous identities and write them as $\boxed{\sin(u \pm v) = \sin u \cos v \pm \cos u \sin v}$.

Note the sign does not change.

The Tangent of a Difference or Sum Identities

Since we have figured out what happens for the sine and cosine of a sum or difference, we can easily get the tangent of a sum or difference:

$$\boxed{\tan(u \pm v) = \frac{\sin(u \pm v)}{\cos(u \pm v)} = \frac{\sin u \cos v \pm \cos u \sin v}{\cos u \cos v \mp \sin u \sin v}}.$$

Example Prove the identity $\dfrac{\cos(x+h)-\cos x}{h} = \cos x\left(\dfrac{\cos h - 1}{h}\right) - \sin x\left(\dfrac{\sin h}{h}\right)$.

$$\begin{aligned}
\frac{\cos(x+h)-\cos x}{h} &= \frac{\cos x\cos h - \sin x \sin h - \cos x}{h}, \quad \text{use } \cos(u+v) = \cos u\cos v - \sin u \sin v \\
&= \frac{\cos x \cos h - \cos x}{h} - \frac{\sin x \sin h}{h}, \\
&= \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right).
\end{aligned}$$

Example Find the value of $\sin(\pi/12)$ exactly by using the sine of a difference identity.

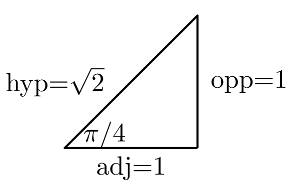
First, we need to figure out how to relate $\pi/12$ to some of our special angles. In Quadrant I, we have three special angles: $\dfrac{\pi}{3} = \dfrac{8\pi}{24}$, $\dfrac{\pi}{4} = \dfrac{6\pi}{24}$, $\dfrac{\pi}{6} = \dfrac{4\pi}{24}$.
We are told to use a difference formula, so the difference of two of our special angles should produce $\pi/12$.

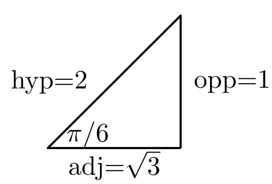
$$\frac{\pi}{12} = \frac{2\pi}{24} = \frac{6\pi - 4\pi}{24} = \frac{\pi}{4} - \frac{\pi}{6}.$$

Other choices are possible.

Therefore,

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{6}\right), \quad \text{use } \sin(u-v) = \sin u\cos v - \cos u \sin v \\
&= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) \\
&= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} \\
&= \frac{\sqrt{3}-1}{2\sqrt{2}}
\end{aligned}$$





Example Solve $\sin 2w + \sin w = 0$ algebraically for exact solutions in the interval $w \in (-\pi, \pi)$.

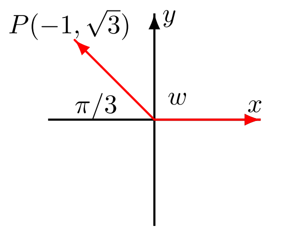
We want to get the angles to be all the same, so we want to use a trig identity to write the $\sin(2w)$ as some trig functions which depend only on w.
Use the sine of a sum to do this:

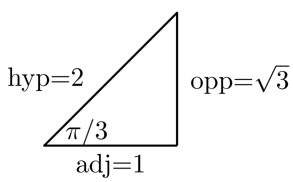
$$\begin{aligned} \sin(u+v) &= \sin u \cos v + \cos u \sin v \\ \sin(2w) &= \sin(w+w) \\ &= \sin w \cos w + \cos w \sin w = 2\cos w \sin w \end{aligned}$$

Now, we can rewrite our equation so all the angles are w, and no $2w$ appears:

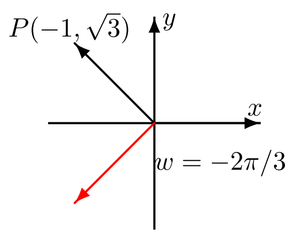
$$\begin{aligned} \sin 2w + \sin w &= 2\sin w \cos w + \sin w \\ &= (2\cos w + 1)\sin w \end{aligned}$$

So we need to solve $(2\cos w + 1)\sin w = 0$. This means either $2\cos w + 1 = 0$ or $\sin w = 0$.
$\underline{\sin w = 0}$: In the interval $(-\pi, \pi)$, this has solution $w = 0$.
$\underline{2\cos w + 1 = 0}$: This means $\cos w = -1/2$.
The equation $\cos w = -1/2 = x/r$ corresponds to the following sketches, and one of our special triangles appears:





So a solution is $w = \pi - \pi/3 = 2\pi/3$, which is in Quadrant II.
There is also a solution in Quadrant III, where $w = -2\pi/3$.



The solutions to $\sin 2w + \sin w = 0$ in the interval $(-\pi, \pi)$ are $w = 0, -\dfrac{2\pi}{3}, \dfrac{2\pi}{3}$.