Concepts: Graphical Solution, Algebraic: Method of Substitution, Algebraic: Method of Elimination.

A solution of a system of two equations in two variables is an ordered pair or real numbers that is a solution of each equation.

There are graphical and algebraic methods of solving systems of equations.

Graphical Solution

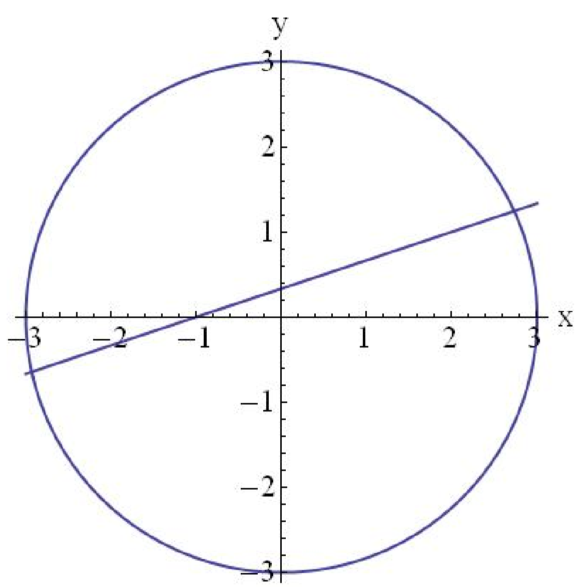
Graphically, the two equations are two curves in the xy-plane, and the solution to the system will be where the two curves intersect.

Example Solve the system of equations graphically.

$$\begin{aligned} x^2 + y^2 &= 9 \\ x - 3y &= -1 \end{aligned}$$

We need to graph the two equations. The first equation is a circle of radius 3 centered at $(0, 0)$.

Let's rewrite the second equation as $y = \frac{1}{3}(x+1) = \frac{x}{3} + \frac{1}{3}$. This is a straight line, with slope 1/3 and y-intercept 1/3.



We can see that there are two solutions, but we can only estimate what they are.

It looks like the solution to the system of equations is two points, which are approximately $(2.8, 1.2)$ and $(-2.9, -0.7)$. If we graphed these using a calculator and zoomed in on the points of intersection, we could get more accuracy.

Graphical solutions are good to verify a solution you have found, but cannot produce an exact solution in general. We need algebraic techniques.

The Method of Substitution

This method involves the following steps:

1. solve one of the equations for one of the unknown variables,
2. substitute the equation from step (1) into the other equation to produce a single equation in a single unknown variable,
3. solve this equation for the unknown variable,
4. substitute into the equation from step (1) to get the second unknown variable.

This procedure may not work if it is difficult to perform step (3).

Example Solve the system of equations using the method of substitution.

$$\begin{aligned} x^2 + y^2 &= 9 \\ x - 3y &= -1 \end{aligned}$$

We have a choice to make, which equation to begin with. If there is a linear equation, begin with that one.

$$\begin{aligned} x - 3y &= -1 \\ x &= -1 + 3y \end{aligned}$$

Now, substitute this into the other equation, and solve for y:

$$\begin{aligned} x^2 + y^2 &= 9 \\ (-1+3y)^2 + y^2 &= 9 \\ 1 + 9y^2 - 6y + y^2 &= 9 \\ 10y^2 - 6y - 8 &= 0 \\ 5y^2 - 3y - 4 &= 0 \\ y &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{3 \pm \sqrt{(-3)^2 - 4(5)(-4)}}{2(5)} \\ &= \frac{3 \pm \sqrt{9 + 80}}{10} \\ &= \frac{3 \pm \sqrt{89}}{10} \end{aligned}$$

Now, we substitute the two solutions we found for y to determine the corresponding x:

$$\begin{aligned} x &= -1 + 3y \\ &= -1 + 3\left(\frac{3 + \sqrt{89}}{10}\right) \\ &= \left(\frac{-10 + 9 + 3\sqrt{89}}{10}\right) = \left(\frac{-1 + 3\sqrt{89}}{10}\right) \end{aligned}$$

So a solution to the system is $\left(\frac{-1 + 3\sqrt{89}}{10}, \frac{3 + \sqrt{89}}{10}\right)$.

Get the other solution:

$$\begin{aligned} x &= -1+3y \\ &= -1+3\left(\frac{3-\sqrt{89}}{10}\right) \\ &= \left(\frac{-10+9-3\sqrt{89}}{10}\right) \\ &= \left(\frac{-1-3\sqrt{89}}{10}\right) \end{aligned}$$

So another solution to the system is $\left(\frac{-1-3\sqrt{89}}{10}, \frac{3-\sqrt{89}}{10}\right)$.

Compare with our graphical estimation:

$\left(\frac{-1+3\sqrt{89}}{10}, \frac{3+\sqrt{89}}{10}\right) \sim (2.73, 1.24)$.

$\left(\frac{-1-3\sqrt{89}}{10}, \frac{3-\sqrt{89}}{10}\right) \sim (-2.93, -0.64)$.

The Method of Elimination

This method works best on linear systems, although it can work in other instances.

The process:

1. rewrite one of the equations so that a coefficient of one of the variables is the opposite (different sign) from the other equation,
2. add the two equations, which will eliminate one of the variables,
3. solve the resulting equation for the unknown variable,
4. use one of the original equations to solve for the other unknown variable.

Example Solve the system of equations using elimination:

$$\begin{aligned} 2x+y &= 10 \\ x-2y &= -5 \end{aligned}$$

Rewrite the first equation by multiplying by 2, then add the equations

$$\begin{aligned} 4x+2y &= 20 \\ x-2y &= -5 \\ 5x &= 15 \end{aligned}$$

Solve the new equation for x: $x = 3$.

Use the first original equation to determine y:

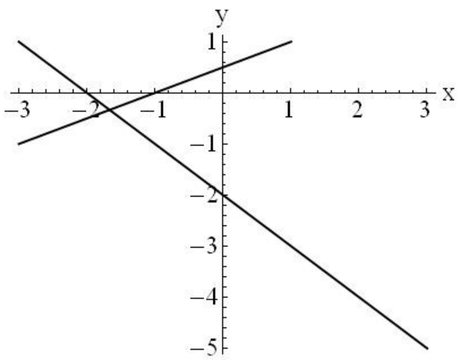
$$\begin{aligned} 2x+y &= 10 \\ 2(3)+y &= 10 \\ y &= 10-6 = 4 \end{aligned}$$

The solution to the system is $(3, 4)$, or $x = 3, y = 4$.

Linear Equations

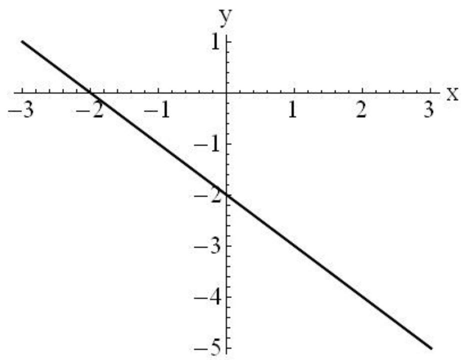
Linear equations are kind of special. If you are solving linear equations, then graphically you have two lines in the xy-plane, and the solution is where they intersect. The solution can be either a single point, an infinite number of solutions, or no solution. We can see why graphically.

$$x + y = -2$$
$$x - 2y = -1$$



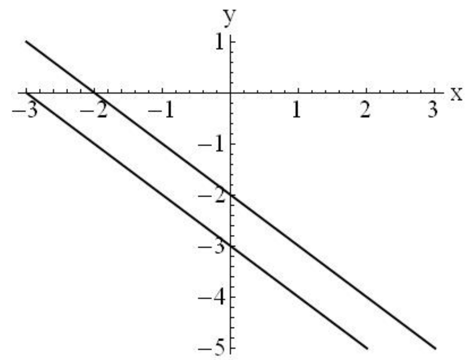
Two intersecting lines, one solution

$$x + y = -2$$
$$4x + 4y = -8$$



The same line, infinite number of solutions

$$x + y = -2$$
$$x + y = -3$$



Parallel lines, no solution