Concept: Sign Charts.

Solving inequality equations is more involved that solving equations.

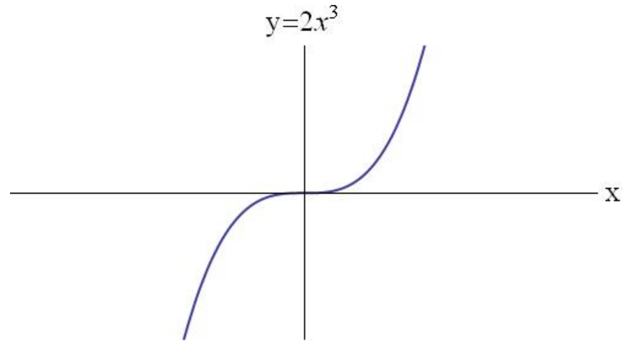
If you can sketch the function, then solving the inequality is easy. Factoring might be required if the function is not in a factored form.

Example Solve the inequality $f(x) = (x-2)(x-4)(2x+3) \geq 0$ by sketching the function $y = f(x)$.

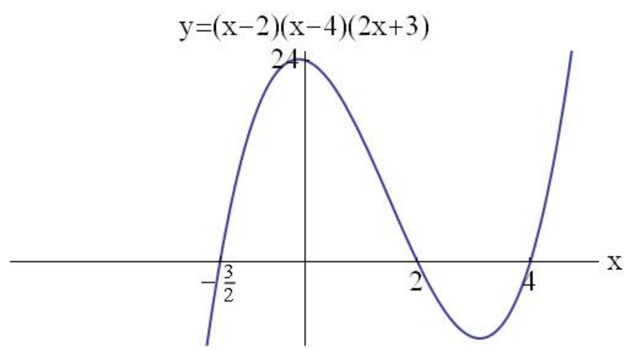
Since this is already factored for us, getting the sketch is easy.

Roots at $x = 2, 4, -3/2$. They are all multiplicity one, which is odd, so the function will cross the axis at these roots.

End behaviour: for $|x|$ large, we have $f(x) = (x-2)(x-4)(2x+3) \sim (x)(x)(2x) = 2x^3$.
Sketch of $y = 2x^3$:



From this, we can sketch the function:



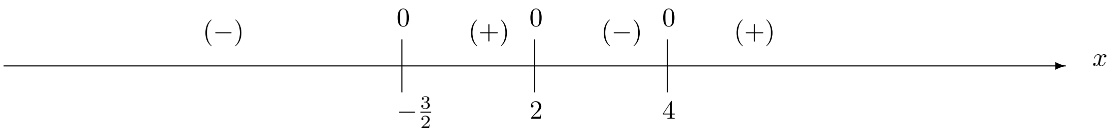
From the sketch, we can easily read off the solution to the inequality: $f(x) \geq 0$ for $x \in [-3/2, 2] \cup [4, \infty)$.

Sign Chart from end behaviour and multiplicity of roots

A sign chart simply lists the sign of the function you are interested in along the x-axis. It does not provide as much information as a sketch, but it provides enough to solve inequalities.

The same techniques that you use in sketching can allow you to create a sign chart. When you create a sign chart this way, you are creating it from the end behaviour and multiplicity of roots.

The sign chart from the above example looks like:



This can be constructed without sketching the graph. Notice the scale doesn't matter too much, you just need to identify the intervals correctly.

The $-3/2, 2, 4$ are where the function is zero (and could potentially change sign) and the 0 above the line represents that. If you have a rational function with vertical asymptotes, you can put an infinity sign over the value to remind you that the function isn't defined at that value.

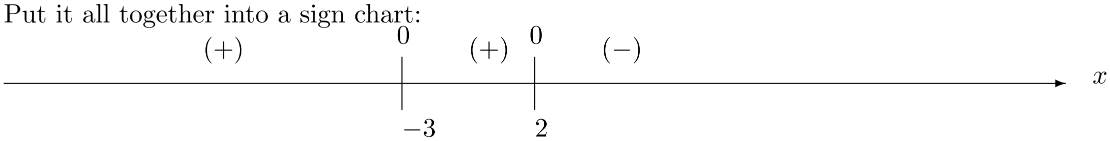
The $(+)$ means the function is positive in the interval, the $(-)$ means the function is negative in the interval.

Example Solve the inequality $f(x) = -2(x-2)^3(x+3)^2 < 0$ by constructing a sign chart by considering x-intercepts, multiplicity, and end behaviour.

The x-intercepts are at $x = 2$ and $x = -3$. The $x = 2$ intercept has multiplicity 3, which is odd, so the function f will cross the x-axis here (change sign). The $x = -3$ intercept has multiplicity 2, which is even, so the function f will not cross the x-axis here (will not change sign).

End behaviour: For $|x|$ large, $f(x) = -2(x-2)^3(x+3)^2 \sim -2(x)^3(x^2) = -2x^5$. This monomial will approach $-\infty$ if x is large, due to the minus sign of the coefficient. As x approaches $-\infty$, the monomial will approach ∞.

Put it all together into a sign chart:



From the sign diagram, we see that $f(x) < 0$ if $x \in (2, \infty)$.

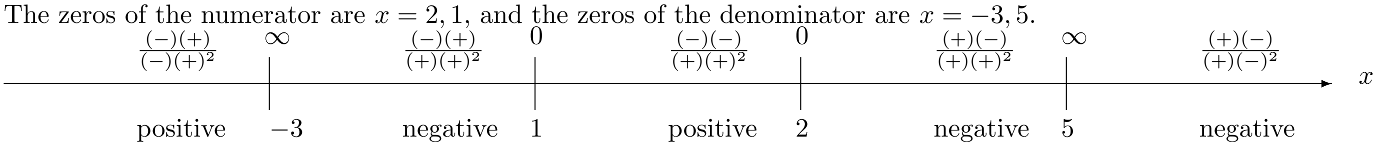
Sign Chart from sign of factors

Another way of creating a sign chart is by considering the signs of the factors. I actually prefer the previous way, but I wanted you to see this way as well. Use whichever one you prefer.

Example Create a sign chart for the rational function $f(x) = \dfrac{(x-2)(1-x)}{(x+3)(5-x)^2}$.

We will create this sign chart by looking at the zeros of the numerator and denominator, since these x values are were the function might possibly change sign.

The zeros of the numerator are $x = 2, 1$, and the zeros of the denominator are $x = -3, 5$.



When is $f(x) < 0$? Answer: When $x \in (-3, 1) \cup (2, 5) \cup (5, \infty)$.

Example Solve the inequality $(3x+5)^3|x-2| < 0$.

I will solve this using a sign chart, and examining the sign of the factors.

The roots of the function $f(x) = (3x+5)^3|x-2|$ are $x = -5/3, x = 2$.

$(-)\|-\|$	0	$(+)\|-\|$	0	$(+)\|+\|$	
negative	$-\frac{5}{3}$	positive	2	positive	x

From the sign diagram, we see that $(3x+5)^2|x-2| < 0$ if $x \in (-\infty, -5/3)$.

Example Solve the Inequality $\dfrac{(x-5)|x-2|}{\sqrt{2x-3}} \geq 0$.

I will solve this using a sign chart, and examining the sign of the factors.

The roots of the function $f(x) = \dfrac{(x-5)|x-2|}{\sqrt{2x-3}}$ are $x = 5, x = 2$.

The function $f(x) = \dfrac{(x-5)|x-2|}{\sqrt{2x-3}}$ will have a vertical asymptote at $x = 3/2$.

Because of the square root, the function is not defined if $2x - 3 < 0$.

$$\begin{aligned} 2x-3 &< 0 \\ -3 &< -2x \\ 3/2 &> x \text{ (direction of inequality changes when you multiply by something less than zero)} \end{aligned}$$

The function is not defined if $x < 3/2$.

$\frac{(-)\|-\|}{\sqrt{-}}$	∞	$\frac{(-)\|-\|}{\sqrt{+}}$	0	$\frac{(-)\|+\|}{\sqrt{+}}$	0	$\frac{(+)\|+\|}{\sqrt{+}}$	
undefined	$\frac{3}{2}$	negative	2	negative	5	positive	x

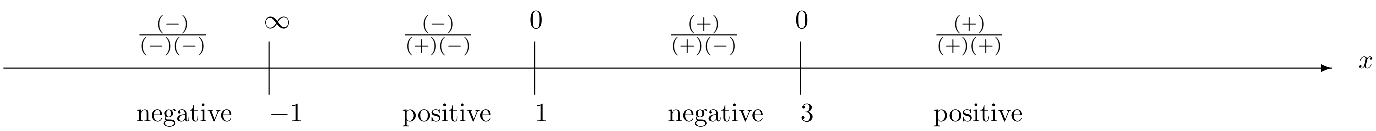
From the sign diagram, we see that $\dfrac{(x-5)|x-2|}{\sqrt{2x-3}} \geq 0$ if $x \in [5, \infty)$.

Example Solve the inequality $\dfrac{1}{x+1} + \dfrac{1}{x-3} \leq 0$.

We need to write this as a single rational function, rather than as a sum of rational functions, before we can construct our sign chart.

$$
\begin{aligned}
\frac{1}{x+1} + \frac{1}{x-3} &\leq 0 \\
\frac{1}{x+1}\left(\frac{x-3}{x-3}\right) + \frac{1}{x-3}\left(\frac{x+1}{x+1}\right) &\leq 0 \\
\frac{x-3}{(x+1)(x-3)} + \frac{x+1}{(x+1)(x-3)} &\leq 0 \\
\frac{(x-3)+(x+1)}{(x+1)(x-3)} &\leq 0 \\
\frac{2(x-1)}{(x+1)(x-3)} &\leq 0
\end{aligned}
$$

The numerator is zero if $x = 1$, the denominator is zero if $x = -1, 3$. These are the possible values where the function will change sign.



From the sign diagram, we see that $\dfrac{1}{x+1} + \dfrac{1}{x-3} \leq 0$ if $x \in (-\infty, -1) \cup [1, 3]$.

You MUST use sign charts or sketches to treat complicated inequalities correctly. If you try to do it with algebra you will get the wrong answer, as the following shows.

$$
\begin{aligned}
\frac{1}{x+1} + \frac{1}{x-3} &\leq 0 \\
\frac{1}{x+1} &\leq -\frac{1}{x-3} \\
x+1 &\geq -(x-3) \quad \textbf{(incorrect algebra at this step)} \\
x+1 &\geq -x+3 \\
2x &\geq 2 \\
x &\geq 1 \quad \textbf{(incorrect answer)}
\end{aligned}
$$

Why is the algebra wrong? Think of $-1/4 < 1/3$ (which is true), but inverting and change direction of inequality gives $-4 > 3$ which is not true. Algebra and inequalities are tricky! You can't just apply what you know works for positive numbers.