Concepts: Indeterminant Forms, Extraneous Solutions.

Solving the polynomial equation $f(x) = 0$, where f is a polynomial, is the same as finding the x-intercepts of the graph of $y = f(x)$. This can sometimes be done using the factoring techniques we saw in the Section Real Zeros of Polynomial Functions. Remember that for a general polynomial with irrational roots it can be difficult to find these roots.

Solving the rational equation $\dfrac{f(x)}{g(x)} = 0$ where f and g have no common factors involves solving the polynomial equation $f(x) = 0$.

Solving rational equations that are more complicated than $\dfrac{f(x)}{g(x)} = 0$, say for example $\dfrac{f(x)}{g(x)} + \dfrac{q(x)}{r(x)} = 0$, involves finding the *least common denominator (LCD)* and multiplying the equation by the LCD. This will result in a polynomial equation $s(x) = 0$ which needs to be solved.

The process of multiplying through by the LCD can lead to *extraneous solutions*. The extraneous solutions are solutions to the polynomial equation $s(x) = 0$ but not solutions of the original rational equation $\dfrac{f(x)}{g(x)} + \dfrac{q(x)}{r(x)} = 0$.

The extraneous solutions are introduced when you cancel a fraction as $\dfrac{h(x)}{h(x)} = 1$, a result which is not true if $h(x) = 0$!

For this reason, when solving rational equations, you must check that the solutions you find are solutions of the original rational equation.

Example of Horribly Incorrect Math

Begin: $1 = 1$
Let: $x = 1$
Square both sides: $x^2 = 1$
Subtract 1 from both sides: $x^2 - 1 = 0$
Factor as difference of squares: $(x + 1)(x - 1) = 0$
Divide equation by $(x - 1)$: $\dfrac{(x + 1)(x - 1)}{(x - 1)} = \dfrac{0}{(x - 1)}$
Simplify: $\dfrac{(x + 1)\cancel{(x - 1)}}{\cancel{(x - 1)}} = 0$
Result: $x + 1 = 0$
Replace x with 1 again: $1 + 1 = 0$
Result: $2 = 0$.

Go home, all of math is a lie.

Well, not really. We have not followed the laws of algebra, and that leads to an incorrect statement.

When we simplified $\dfrac{(x - 1)}{(x - 1)} = 1$, we have to exclude the possibility that $\dfrac{(x - 1)}{(x - 1)} = \dfrac{0}{0}$, which is an *indeterminate form*. This means the resulting equation, $x + 1 = 0$, is not valid at $x = 1$, which is, of course, where we promptly evaluated it to get the result $2 = 0$.

Indeterminant forms are studied extensively in calculus, but it is worthwhile for us to get a bit of an understanding of what they are in precalculus. We have already seen them when we looked at Average and Instantaneous Rate of Change.

Indeterminant Forms

Adding a large number to a large number produces a large number. That is a form we can understand.

But what about dividing a very small number by a very small number? Is that a small number, or a large number? The problem is that we think any real number divided by a very small number should be a very large number; but any real number multiplied by a very small number should be a very small number (this isn't quite correct).

So I ask again, is $\frac{0}{0}$ a very large number or a very small number? Or something in between?

The fact is that when we encounter a form like $\frac{0}{0}$ we don't know what the quotient reduces to. It could be literally anything. It is an indeterminant quotient. It depends on how the numerator and denominator are becoming zero.

What was said above should be written as: any real number *except zero* divided by a very small number should be a very large number; any real number *except zero* multiplied by a very small number should be a very small number.

Indeterminant quotients are immensely important to calculus since the derivative of a function is an indeterminant form.

There are other indeterminant forms such as

1^{∞}: (indeterminant power)
0^0 (indeterminant power)
$0 \cdot \infty$ (indeterminant product)

Saying a form is indeterminant simply means we have to do some work to figure out how to determine it. You will see much more about indeterminant forms in calculus, when you study limits in more depth.

Indeterminate Forms and Extraneous Solutions

In precalculus, the thing we must remember is that $\frac{h(x)}{h(x)} = 1$ only if $h(x) \neq 0$. The extraneous solutions arise by not including the restriction $h(x) \neq 0$. But, since the $h(x)$ is formed from the lowest common denominator in the original rational equation, we will always get division by zero in the original equation if we evaluate at values for which $h(x) = 0$.

Since we typically don't include the restriction, we must check all the solutions we find by back substitution into the original equation.

Example Solve $\dfrac{3x}{x+1}+\dfrac{5}{x-2}=\dfrac{15}{x^2-x-2}$.

$$\begin{aligned}
\frac{3x}{x+1}+\frac{5}{x-2} &= \frac{15}{(x-2)(x+1)} \\
(x-2)(x+1)\left[\frac{3x}{x+1}+\frac{5}{x-2}\right. &= \left.\frac{15}{(x-2)(x+1)}\right] && \text{multiply by LCD } (x-2)(x+1) \\
3x(x-2)+5(x+1) &= 15 \\
3x^2-6x+5x+5-15 &= 0 \\
3x^2-x-10 &= 0 \\
(3x+5)(x-2) &= 0
\end{aligned}$$

The possible solutions to the original rational equation are $x=-5/3$ and $x=2$.
Check which are solutions by substitution:
$x=-5/3$:

$$\begin{aligned}
\frac{3(-\frac{5}{3})}{-\frac{5}{3}+1}+\frac{5}{-\frac{5}{3}-2} &= \frac{15}{(-\frac{5}{3})^2-(-\frac{5}{3})-2} \\
\frac{-5}{(-\frac{2}{3})}+\frac{5}{(-\frac{11}{3})} &= \frac{15}{\frac{25}{9}+\frac{5}{3}-2} \\
-5\left(-\frac{3}{2}\right)+5\left(-\frac{3}{11}\right) &= \frac{15}{\left(\frac{50+30-36}{18}\right)} \\
\frac{15}{2}-\frac{15}{11} &= \frac{15}{\left(\frac{44}{18}\right)} \\
\frac{15}{2}-\frac{15}{11} &= 15\left(\frac{18}{44}\right) \\
\frac{9}{22} &= \frac{9}{22}
\end{aligned}$$

So $x=-5/3$ is a solution of the original rational equation.
$x=2$:

$$\frac{3(2)}{2+1}+\frac{5}{2-2} = \frac{15}{2^2-2-2}$$

Since $\dfrac{5}{2-2}$ is infinite, $x=2$ is not a solution of the original rational equation.
Notice that the extraneous solutions will result in division by zero in the original equation. This makes sense given our discussion of indeterminate forms.

An alternate solution:

$$\begin{aligned}
\frac{3x}{x+1}+\frac{5}{x-2} &= \frac{15}{(x-2)(x+1)} \\
(x-2)(x+1)\left[\frac{3x}{x+1}+\frac{5}{x-2}\right. &= \left.\frac{15}{(x-2)(x+1)}\right] && \text{multiply by LCD } (x-2)(x+1) \\
3x(x-2)+5(x+1) &= 15 && x\neq 2,-1 \\
3x^2-6x+5x+5-15 &= 0 && x\neq 2,-1 \\
3x^2-x-10 &= 0 && x\neq 2,-1 \\
(3x+5)(x-2) &= 0 && x\neq 2,-1
\end{aligned}$$

The possible solutions to the original rational equation are $x=-5/3$.

Example Consider all rectangles with an area of 200 m^2. Let x be the length of one side of such a rectangle.
(a) Express the perimeter P as a function of x.
(b) Find the dimensions of a rectangle whose perimeter is 70m.

The rectangle will have a side of length x (given) and another side of length y (my choice).
The area of the rectangle is $A = xy = 200$, which means that $y = 200/x$.
The perimeter of the rectangle is $P = 2x + 2y = 2x + 400/x$.
To find the dimensions of the rectangle with perimeter 70 m, we must solve the equation

$$\begin{aligned}
P = 70 &= 2x + 400/x \\
x\,[70 &= 2x + 400/x] \qquad \text{multiply by LCD } x \\
70 &= 2x^2 + 400 \qquad x \neq 0 \\
2x^2 - 70x + 400 &= 0 \qquad x \neq 0 \\
2(x^2 - 35x) + 400 &= 0 \qquad x \neq 0 \text{ solve by completing the square} \\
2\left(x^2 - 35x + \left(\frac{-35}{2}\right)^2 - \left(\frac{-35}{2}\right)^2\right) + 400 &= 0 \qquad x \neq 0 \\
2\left(\left(x - \frac{35}{2}\right)^2 - \left(\frac{-35}{2}\right)^2\right) + 400 &= 0 \qquad x \neq 0 \\
2\left(x - \frac{35}{2}\right)^2 - 2\left(\frac{1225}{4}\right) + 400 &= 0 \qquad x \neq 0 \\
2\left(x - \frac{35}{2}\right)^2 &= +\frac{1225}{2} - \frac{800}{2} \qquad x \neq 0 \\
2\left(x - \frac{35}{2}\right)^2 &= \frac{425}{2} \qquad x \neq 0 \\
\left(x - \frac{35}{2}\right)^2 &= \frac{425}{4} \qquad x \neq 0 \\
x - \frac{35}{2} &= \pm\frac{5\sqrt{17}}{2} \qquad x \neq 0 \\
x &= \frac{35}{2} \pm \frac{5\sqrt{17}}{2} \qquad x \neq 0 \\
&= 27.8078 \text{ or } 7.19224
\end{aligned}$$

If $x = 27.8078$, then $y = 200/27.8078 = 7.19224$.
If $x = 7.19224$, then $y = 200/7.19224 = 27.8078$.
The dimensions of the rectangle with the required properties is 7.19224 by 27.8078.