Concepts: Sketching Circles, Ellipses, Hyperbolas (implicit functions).

Although there are many interesting properties of the conic section, we will focus on sketching by hand in this section. This will be an important skill in future math and science classes.

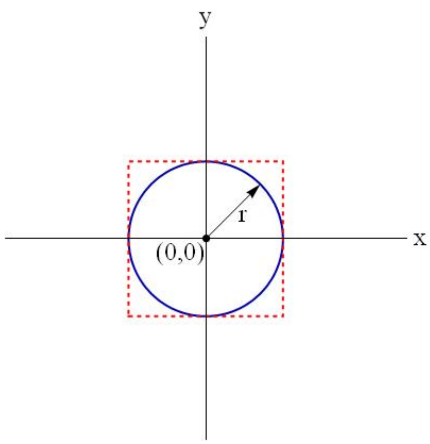
See the text if you want to see how the definitions lead to the particular equations in each case. I will not be asking you to derive the equations, but you should be able to sketch them!

Circles

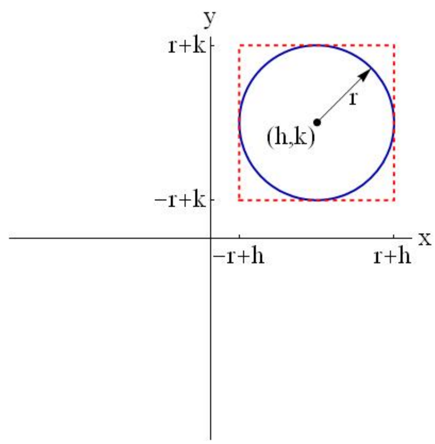
A *circle* is the set of all points in a plane equidistant from a particular point.

The standard form for the equation of a circle is $x^2 + y^2 = r^2$.

The transformed form is $(x - h)^2 + (y - k)^2 = r^2$.



Standard Form: $x^2 + y^2 = r^2$



Transformed Form: $(x - h)^2 + (y - k)^2 = r^2$

I draw the box around the circle since we will need one for ellipses and hyperbolas; it is not necessary to include the box for circles.

To sketch a circle, find h and k (you might have to complete the square in x and then again in y!) to get the center, and then sketch based on the transformed form.

The equation $x^2 + y^2 = 1$ is called an *implicit function* since it implicitly defines two *explicit functions*, $y = f_1(x) = \sqrt{1 - x^2}$ and $y = f_2(x) = -\sqrt{1 - x^2}$ (the top and bottom of the circle separately pass the vertical line test we used to determine if we had an explicit function).

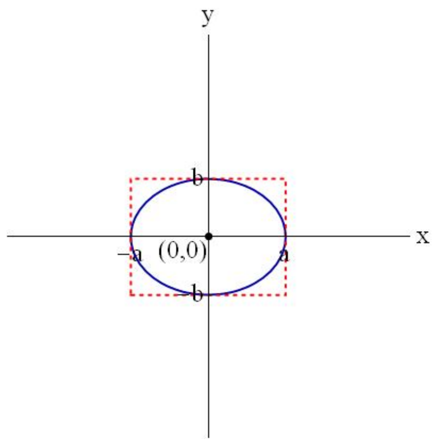
Ellipses

An *ellipse* is the set of all points in a plane whose distance from two fixed points in the plane have a constant sum. These two fixed points are called the *foci*. The *focal axis* is the lines through the foci, and the *center* is the point midway between the foci.

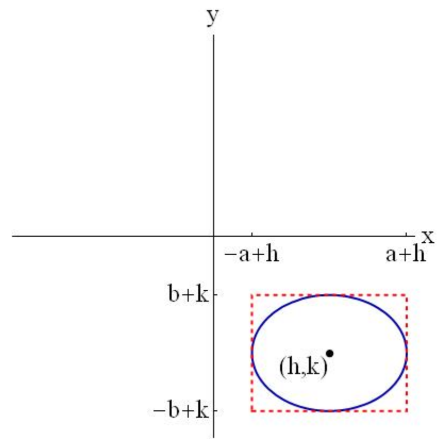
The standard form for the equation of an ellipse is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The transformed form is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$.

Notice that an ellipse becomes a circle when $a = b = r$.



Standard Form: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Transformed Form: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$

Notice that the ellipse is inside the box. Hyperbolas are created from the same box, but will be outside the box.

To sketch an ellipse, first sketch the box, then draw the ellipse inside.

You needn't memorize the numbers for the box, work it out each time:

For example, to sketch $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$:

Evaluate at $x = h \Rightarrow \dfrac{(y-k)^2}{b^2} = 1 \Rightarrow y = k \pm b$.

Evaluate at $y = k \Rightarrow \dfrac{(x-h)^2}{a^2} = 1 \Rightarrow x = h \pm a$.

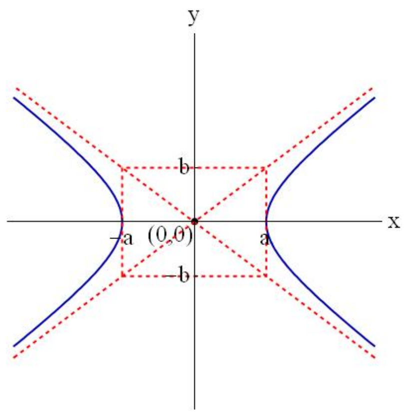
This gives you the sides of the box.

Hyperbolas

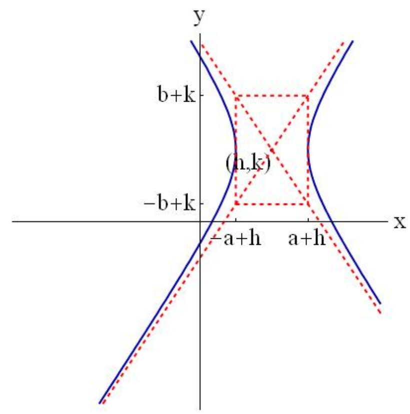
An *hyperbola* is the set of all points in a plane whose distance from two fixed points in the plane have a constant difference. These two fixed points are called the *foci*.

The standard form for the equation of an hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ or $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

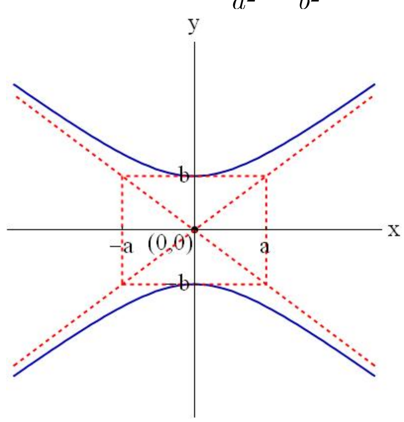
The transformed form is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$.



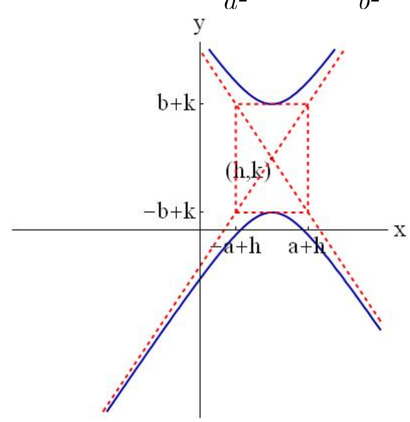
Standard Form: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



Transformed Form: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$



Standard Form: $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$



Transformed Form: $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$

Notice the hyperbola is outside the box, and has slant asymptotes given by $y - k = \pm\frac{b}{a}(x-h)$.

To sketch a hyperbola, sketch the box as if you had an ellipse. Draw lines through the corners of the box.

Figure out if the hyperbola opens up/down or left/right by figuring out what the values are when $x = h$ and $y = k$. For example, for $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ when $y = k$ there is no solution to $-\frac{(x-h)^2}{a^2} = 1$, but when $x = h$, then $\frac{(y-k)^2}{b^2} = 1$ has two solutions, $y = b \pm k$, so the hyperbola must open up/down.

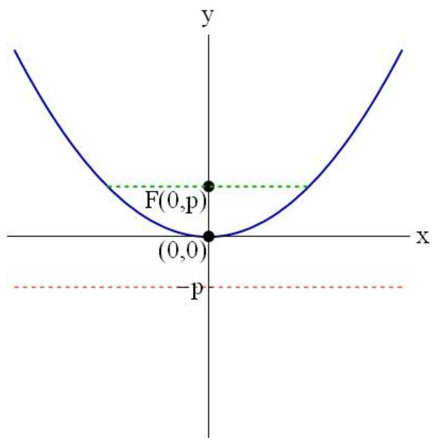
Parabolas

I've included the precise definition of a parabola here for completeness, but often people will call $y = x^2$ a parabola, and we've already seen how to sketch this function, as well as $y = ax^2 + bx + c$.

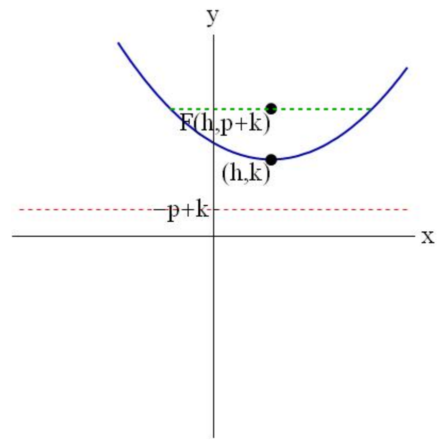
A *parabola* is the set of all points in a plane equidistant from a particular line (the directrix) <u>and</u> a particular point (the focus).

The standard form for the equation of a parabola is $x^2 = 4yp$ or $y^2 = 4xp$.

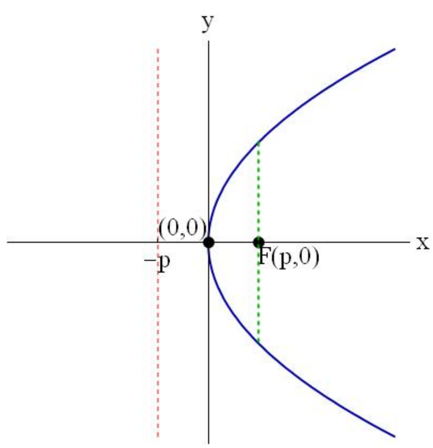
The transformed form is $y - k = 4p(x - h)^2$ or $x - h = 4p(y - k)^2$.



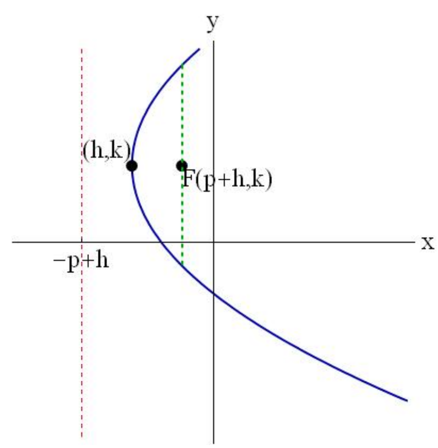
Standard Form: $x^2 = 4yp$



Transformed Form: $(x - h)^2 = 4(y - k)p$



Standard Form: $y^2 = 4xp$



Transformed Form: $y - k = 4p(x - h)^2$

The point F is the focus, and the green dashed line has length $|4p|$ (the focal length). The red dashed line is the directrix.

Parabolic mirrors with this particular geometry can be used to focus light at the focal point, F.