**Concepts:** Solving Quadratic Equations, Completing the Square, The Quadratic Formula, Sketching Quadratics.

## Solving Quadratic Equations

### Completing the Square

$$\begin{aligned}
ax^2+bx+c &= a\left(x^2+\frac{b}{a}x\right)+c && \text{Factor so the coefficient of } x^2 \text{ is 1. Coefficient of } x \text{ is } \tfrac{b}{a} \\
&= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c && \text{the blue terms add to zero; we haven't changed the equality} \\
&= a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2\right)+c && \text{the red terms can be collected together as a perfect square} \\
&= a\left(\left[x+\frac{b}{2a}\right]^2-\left(\frac{b}{2a}\right)^2\right)+c \\
&= a\left[x+\frac{b}{2a}\right]^2-a\left(\frac{b}{2a}\right)^2+c && \text{simplify} \\
&= a\left[x+\frac{b}{2a}\right]^2-\frac{b^2}{4a}+c \\
&= a\left[x+\frac{b}{2a}\right]^2-\frac{b^2-4ac}{4a}
\end{aligned}$$

Here is the process of completing the square in words:

- Factor so that there is just a 1 in front of the $x^2$ term.
- Identify the coefficient of the $x$ term.
- Take half of this coefficient and square, then add and subtract so you don't change the equation.
- Fold up the perfect square you have created.
- Simplify.

This is how you derive the quadratic formula, since from here we set this equal to zero and solve for $x$:

$$\begin{aligned}
ax^2+bx+c &= 0 \\
a\left[x+\frac{b}{2a}\right]^2-\frac{b^2-4ac}{4a} &= 0 \\
a\left[x+\frac{b}{2a}\right]^2 &= \frac{b^2-4ac}{4a} \\
\left[x+\frac{b}{2a}\right]^2 &= \frac{b^2-4ac}{4a^2} \\
x+\frac{b}{2a} &= \frac{\pm\sqrt{b^2-4ac}}{2a} \\
x &= \frac{-b\pm\sqrt{b^2-4ac}}{2a}
\end{aligned}$$

Right now, it looks like this doesn't do anything that the quadratic formula couldn't do, but trust me there are times when you will have to use competing the square instead of the quadratic formula.

**Example** Use completing the square to solve $2x^2 + 4x + 1 = 0$.

$$2x^2 + 6x + 1 = 0$$

We MUST have a coefficient of 1 in front of the $x^2$ before we complete the square.

$$x^2 + 3x + \frac{1}{2} = 0$$

$$x^2 - 3x + \frac{1}{2} = 0 \text{ To complete the square: } \left(\frac{3}{2}\right)^2.$$

$$\underline{x^2 + 3x + \left(\frac{3}{2}\right)^2} - \left(\frac{3}{2}\right)^2 + \frac{1}{2} = 0 \text{ underlined piece is a perfect square}$$

$$\underline{\left(x + \frac{3}{2}\right)^2} = \frac{9}{4} - \frac{1}{2}$$

$$\underline{\left(x + \frac{3}{2}\right)^2} = \frac{7}{4}$$

$$x + \frac{3}{2} = \pm\sqrt{\frac{7}{4}}$$

$$x = -\frac{3}{2} \pm \frac{\sqrt{7}}{2}$$

**Quadratic Formula**

**Memorize:** The two solutions to the equation $ax^2 + bx + c = 0$ are given by the *quadratic formula*:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You can use the quadratic formula to solve quadratic equations that previously you solved by factoring.

**Example** Solve $\frac{1}{15} + \frac{3}{y} = \frac{4}{y+1}$.

$$\frac{1}{15} + \frac{3}{y} = \frac{4}{y+1} \text{ (multiply by the LCD which is } 15y(y+1))$$

$$\frac{1}{\cancel{15}} \cdot \cancel{15}y(y+1) + \frac{3}{\cancel{y}} \cdot 15\cancel{y}(y+1) = \frac{4}{\cancel{y+1}} \cdot 15y\cancel{(y+1)} \text{ (simplify)}$$

$$y(y+1) + 45(y+1) = 60y$$

$$y^2 + y + 45y + 45 = 60y$$

$$y^2 - 14y + 45 = 0 \text{ (use quadratic formula–I always write it out)}$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{14 \pm \sqrt{(-14)^2 - 4(1)(45)}}{2(1)}$$

$$y = \frac{14 \pm \sqrt{16}}{2} = \frac{14 \pm 4}{2} = 7 \pm 2 = 9 \text{ or } y = 5$$

Since neither $y = 9$ nor $y = 5$ makes the LCD zero, these are both solutions.

**Sketching Quadratics**

For a quadratic function $y = f(x) = ax^2 + bx + c$, we can create a sketch by determining four things:

1. **$x$-intercepts:** determine these by using the quadratic formula $x_{\text{intercept}} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

   NOTE: If $b^2 - 4ac < 0$ then there are no $x$-intercepts (they are not real numbers).

2. **Vertex:** $(x_{\text{Vertex}}, y_{\text{Vertex}}) = \left(-\dfrac{b}{2a}, f\left(-\dfrac{b}{2a}\right)\right)$.

$$\begin{aligned} x_{\text{Vertex}} &= -\frac{b}{2a} \\ y_{\text{Vertex}} &= f\left(-\frac{b}{2a}\right) \end{aligned}$$

   How to remember this: The $x$-coordinate of the vertex will be right in the middle of the two $x$-intercepts, even if the $x$-intercepts are not real numbers!

$$\begin{aligned} x_{\text{Vertx}} &= \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}\right) \\ &= \frac{1}{2}\left(\frac{-2b}{2a}\right) \\ &= \frac{-b}{2a} \end{aligned}$$

3. **$y$-intercept:** evaluate at $x = 0$, so figure out what $f(0)$ is.

4. The function opens up if $a > 0$, and opens down if $a < 0$.

Note: The quantity $f$ in $y = f(x)$ is a *function*, which is something we will be studying in depth in the coming weeks. Basically, a function takes as input a value $x$ and spits out a new value $y$.

**The Vertex Form**

The form $f(x) = a(x - h)^2 + k$ is called the vertex form for a quadratic function. It is obtained from the standard form $f(x) = ax^2 + bx + c$ by completing the square.

The vertex of the parabola is $(h, k)$.
The axis of symmetry is $x = h$.
If $a > 0$, the parabola opens up, if $a < 0$ the parabola opens down.

You can also sketch a parabola by writing it in the vertex form and identifying the vertex, determining if it opens up/down, and finding any $x$-intercepts.

**Example** Sketch the parabola $y = 5x^2 + 4x - 12$. Label the vertex, $y$-intercept, and any $x$-intercepts on your sketch.
To get the sketch of a quadratic, we should do four things:

1. Determine if it opens up or down,
2. Determine the vertex,
3. Determine any $x$-intercepts (if they exist),
4. Determine the $y$-intercept.

A quadratic opens up if $a > 0$, and opens down if $a < 0$. Since $a = 5$ in this case, this quadratic opens up.
To get the vertex, identity $a = 5, b = 4$, and $c = -12$. Then the vertex is located at:

$$x = \frac{-b}{2a} = \frac{-4}{2(5)} = -\frac{2}{5}$$
$$y = f\left(\frac{-b}{2a}\right) = f\left(-\frac{2}{5}\right) = 5\left(-\frac{2}{5}\right)^2 + 4\left(-\frac{2}{5}\right) - 12 = -\frac{64}{5}$$

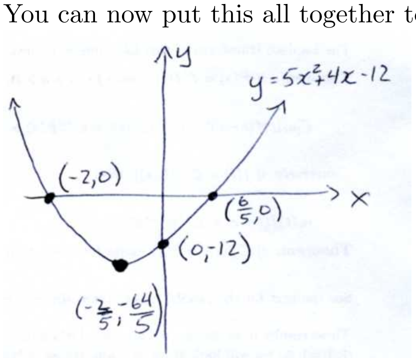
The vertex is at $\left(-\frac{2}{5}, -\frac{64}{5}\right)$.
To get the $x$-intercepts, use the quadratic formula:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-4 \pm \sqrt{4^2 - 4(5)(-12)}}{2(5)} \\ &= \frac{-4 \pm \sqrt{256}}{10} \\ &= \frac{-4 \pm 16}{10} \\ &= \frac{-4 + 16}{10} \text{ or } \frac{-4 - 16}{10} \\ &= \frac{12}{10} \text{ or } \frac{-20}{10} \\ &= \frac{6}{5} \text{ or } -2 \end{aligned}$$

To get the $y$-intercept, evaluate $f(0)$:

$$y = f(0) = 5(0)^2 + 4(0) - 12 = -12$$

You can now put this all together to get the sketch:



**Example** Solve $(2x+7)^2 - \frac{3}{4} = 14$.

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= 14 \\
(2x+7)^2 &= 14 + \frac{3}{4} \\
(2x+7)^2 &= \frac{59}{4} \\
\sqrt{(2x+7)^2} &= \pm\sqrt{\frac{59}{4}} \\
2x+7 &= \pm\frac{\sqrt{59}}{2} \\
2x &= -7 \pm \frac{\sqrt{59}}{2} \\
x &= -\frac{7}{2} \pm \frac{\sqrt{59}}{4}
\end{aligned}$$

You have to check if these are solutions by substituting back:

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} + \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 + \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

$$\begin{aligned}
(2x+7)^2 - \frac{3}{4} &= \left(2\left(-\frac{7}{2} - \frac{\sqrt{59}}{4}\right) + 7\right)^2 - \frac{3}{4} \\
&= \left(-7 - \frac{\sqrt{59}}{4} + 7\right)^2 - \frac{3}{4} \\
&= \left(-\frac{\sqrt{59}}{2}\right)^2 - \frac{3}{4} \\
&= \frac{59}{4} - \frac{3}{4} = \frac{56}{4} = 14
\end{aligned}$$

So both are solutions. $x = -\frac{7}{2} \pm \frac{\sqrt{59}}{4}$.