Concepts: Strategy to Prove Trig Identities.

We want to be able to rewrite expressions using identities since different ways of writing an expression may lead you to be able to solve a problem which you could not solve using the original expression.

My current favourite quote from the text:

> *The changes at every step are accomplished by algebraic manipulations or identities, but the manipulations or identities should be sufficiently obvious as to require no additional justification. Since "obvious" is often in the eye of the beholder, it is usually safer to err on the side of including too many steps than too few.*

Proof strategies:

- The proof begins with the expression on one side of the identity.
- The proof ends with the expression on the other side of the identity.
- The proof in between consists of showing in sequence a series of expressions, each one easily seen to be equivalent to its preceding expression.
- Begin with the more complicated expression and work towards the less complicated expression.
- If no other move presents itself as a good choice, convert the entire expression to one involving only sines and cosines.
- Combine function by obtaining a common denominator.
- Factor functions.
- Use difference of squares $(a+b)(a-b) = a^2 - b^2$ (in either direction) to set up use of Pythagorean identities.
- Keep in my mind you final destination, and choose steps that bring you closer to the final form you seek.
- Work from both sides and see if you can meet in the middle.

Disprove non-identities by showing the non-identity is not true at a specific value of x.

Example Prove the identity $(1-\tan x)^2 = \sec^2 x - 2\tan x$.

$$\begin{aligned}(1-\tan x)^2 &= 1+\tan^2 x - 2\tan x && \text{multiply out}\\ &= 1+(\sec^2 x - 1) - 2\tan x, && \text{use } 1+\tan^2 x = \sec^2 x\\ &= \sec^2 x - 2\tan x\end{aligned}$$

Example Prove the identity $\sin^2 x\cos^3 x = (\sin^2 x - \sin^4 x)(\cos x)$.

$$\begin{aligned}\sin^2 x\cos^3 x &= \sin^2 x\cos^2 x(\cos x) && \text{factor a } \cos x \text{ since that is in final form}\\ &= \sin^2 x(1-\sin^2 x)(\cos x), && \text{use } \cos^2 x + \sin^2 x = 1\\ &= (\sin^2 x - \sin^4 x)(\cos x)\end{aligned}$$

Example Prove the identity $\dfrac{\cot v - 1}{\cot v + 1} = \dfrac{1-\tan v}{1+\tan v}$.

$$\begin{aligned}\frac{\cot v - 1}{\cot v + 1} &= \frac{\frac{1}{\tan v} - 1}{\frac{1}{\tan v} + 1} && \text{convert to } \tan v \text{ since final form has tangents}\\ &= \left(\frac{\frac{1}{\tan v} - 1}{\frac{1}{\tan v} + 1}\right)\cdot 1\\ &= \left(\frac{\frac{1}{\tan v} - 1}{\frac{1}{\tan v} + 1}\right)\cdot \frac{\tan v}{\tan v}\\ &= \frac{1-\tan v}{1+\tan v}\end{aligned}$$

Example Prove the identity $\tan^4 t + \tan^2 t = \sec^4 t - \sec^2 t$.

$$\begin{aligned}
\tan^4 t + \tan^2 t &= \left(\frac{\sin t}{\cos t}\right)^4 + \left(\frac{\sin t}{\cos t}\right)^2 && \text{convert to sines and cosines} \\
&= \left(\frac{\sin t}{\cos t}\right)^4 + \left(\frac{\sin t}{\cos t}\right)^2 && \text{Yuck! Start over and try something else.} \\
\tan^4 t + \tan^2 t &= (\tan^2 t)(\tan^2 t + 1) && \text{factor } \tan^2 x \\
&= (\sec^2 t - 1)(\sec^2 t), \quad 1 + \tan^2 t = \sec^2 t && \text{use (twice) } \tan^2 t + 1 = \sec^2 t \\
&= \sec^4 t - \sec^2 t
\end{aligned}$$

Example Prove the identity $\dfrac{1-\cos x}{\sin x} = \dfrac{\sin x}{1+\cos x}$.

$$\begin{aligned}
\frac{1-\cos x}{\sin x} &= \frac{1}{\sin x} + \frac{\cos x}{\sin x} \\
&= \csc x + \cot x && \text{Yuck! Try something else} \\
\frac{1-\cos x}{\sin x} &= \left(\frac{1-\cos x}{\sin x}\right) \cdot 1 \\
&= \left(\frac{1-\cos x}{\sin x}\right) \cdot \frac{1+\cos x}{1+\cos x} && \text{put the } 1+\cos x \text{ in the denominator} \\
&= \frac{(1-\cos x)(1+\cos x)}{\sin x(1+\cos x)} \\
&= \frac{(1-\cos^2 x)}{\sin x(1+\cos x)} && \text{use } \cos^2 x + \sin^2 x = 1 \\
&= \frac{\sin^2 x}{\sin x(1+\cos x)}, \\
&= \frac{\sin x}{1+\cos x}
\end{aligned}$$

Example Is $\dfrac{\sin(2x)}{2} = \sin x$ an identity?

We have a suspicion this is not an identity. Let's evaluate both sides at $x = 0$.

$$\begin{aligned}
\frac{\sin(2(0))}{2} &= \frac{0}{2} = 0 \\
\sin(0) &= 0
\end{aligned}$$

Note that this doesn't prove we have an identity! Try a different value of x, since something special might be happening at $x = 0$. Let's try $x = \pi/2$

$$\begin{aligned}
\frac{\sin(2(\pi/2))}{2} &= \frac{\sin(\pi)}{2} = \frac{0}{2} = 0 \\
\sin(\pi/2) &= 1
\end{aligned}$$

Since the two are not equal at $x = \pi/2$, this is not an identity.

Example Prove the following identity $\tan^2 y - \sin^2 y = \tan^2 y \sin^2 y$.

$$\begin{aligned}
\tan^2 y - \sin^2 y &= \frac{\sin^2 y}{\cos^2 y} - \sin^2 y && \text{convert to all sines and cosines} \\
&= (\sin^2 y)\left(\frac{1}{\cos^2 y} - 1\right) && \text{factor} \\
&= (\sin^2 y)\left(\frac{1}{\cos^2 y} - \frac{\cos^2 y}{\cos^2 y}\right) && \text{get common denominator} \\
&= (\sin^2 y)\left(\frac{1 - \cos^2 y}{\cos^2 y}\right) && \text{use } \cos^2 x + \sin^2 x = 1 \\
&= (\sin^2 y)\left(\frac{\sin^2 y}{\cos^2 y}\right) \\
&= (\sin^2 y)\left(\tan^2 y\right) \\
&= \tan^2 y \sin^2 y
\end{aligned}$$

Example Prove the identity $\dfrac{\sin t}{1-\cos t} + \dfrac{1+\cos t}{\sin t} = \dfrac{2(1+\cos t)}{\sin t}$.

$$\begin{aligned}
\frac{\sin t}{1-\cos t} + \frac{1+\cos t}{\sin t} &= \frac{\sin t}{1-\cos t}\left(\frac{\sin t}{\sin t}\right) + \frac{1+\cos t}{\sin t}\left(\frac{1-\cos t}{1-\cos t}\right) && \text{get common denominator} \\
&= \frac{\sin^2 t}{(1-\cos t)(\sin t)} + \frac{(1+\cos t)(1-\cos t)}{\sin t(1-\cos t)} \\
&= \frac{\sin^2 t}{(1-\cos t)(\sin t)} + \frac{1-\cos^2 t}{(1-\cos t)(\sin t)} \\
&= \frac{\sin^2 t + 1 - \cos^2 t}{(1-\cos t)(\sin t)} && \text{look at final form: get all cosines in numerator} \\
&= \frac{1 - \cos^2 t + 1 - \cos^2 t}{(1-\cos t)(\sin t)} && \text{use } \cos^2 t + \sin^2 t = 1 \\
&= \frac{2(1-\cos^2 t)}{(1-\cos t)(\sin t)} && \text{use difference squares } a^2 - b^2 = (a+b)(a-b) \\
&= \frac{2\cancel{(1-\cos t)}(1+\cos t)}{\cancel{(1-\cos t)}(\sin t)} \\
&= \frac{2(1+\cos t)}{\sin t}
\end{aligned}$$