Concepts: Laws of Exponents, Laws of Logarithms, Change of Base, Inverse Functions.

Laws of Exponents (memorize) If x and y are real numbers, and $b > 0$ is real, then

1. $b^x \cdot b^y = b^{x+y}$
2. $\dfrac{b^x}{b^y} = b^{x-y}$
3. $(b^x)^y = b^{xy}$

Laws of Logarithms (memorize) If x and y are positive numbers, and $b > 0, b \neq 1$ is real, then

1. $\log_b(xy) = \log_b x + \log_b y$
2. $\log_b \left(\dfrac{x}{y}\right) = \log_b x - \log_b y$
3. $\log_b(x^r) = r \log_b x$ where r is any real number

Example of Laws of Logarithms Evaluate $\log_2 80 - \log_2 5$.
Solution: $\log_2 80 - \log_2 5 = \log_2 \left(\dfrac{80}{5}\right) = \log_2 16 = 4$ (since $2^4 = 16$).

Example Express $\ln a + \frac{1}{2} \ln b$ as a single logarithm.
Solution: $\ln a + \dfrac{1}{2} \ln b = \ln a + \ln b^{1/2} = \ln(ab^{1/2})$.

Example Given $f(x) = \ln(x^3)$, $g(x) = e^{4x}$, and $h(x) = x^2$. Find the following functions, and simplify as much as possible.

$$
\begin{aligned}
(f \circ g \circ h)(x) &= f\big(g(h(x))\big) \\
&= f\big(g(x^2)\big) \\
&= f\big(e^{4x^2}\big) \\
&= \ln\left[\left(e^{4x^2}\right)^3\right] = \ln\left[e^{3 \cdot 4x^2}\right] \\
&= \ln\left[e^{12x^2}\right] = 12x^2 \\
(h \circ g \circ f)(x) &= h\big(g(f(x))\big) \\
&= h\big(g(\ln(x^3))\big) \\
&= h\big(e^{4\ln(x^3)}\big) \\
&= h\big(e^{\ln((x^3)^4)}\big) = h\big(e^{\ln(x^{4\cdot 3})}\big) \\
&= h\big(e^{\ln(x^{12})}\big) \\
&= h\big(x^{12}\big) = \big(x^{12}\big)^2 = x^{24}
\end{aligned}
$$

Change of Base

Logarithms with any base can be expressed in terms of the natural logarithm:

Theorem For any positive number a ($a \neq 1$), we have

$$\log_a x = \frac{\ln x}{\ln a}.$$

Proof:

$$\begin{aligned} y &= \log_a x \\ \text{which we write as: } a^y &= x \\ \ln(a^y) &= \ln x \\ y \ln a &= \ln x \\ y &= \frac{\ln x}{\ln a} \end{aligned}$$

We have shown $\log_a x = \dfrac{\ln x}{\ln a}$. Every logarithmic function is a constant multiple of the natural logarithmic function, i.e., a vertical stretch or shrink.

I encourage you to understand the process of the proof rather than trying to memorize the formula.

Example Convert $100 \log_3 \left(\frac{10}{3}\right)$ into natural logarithms. Most calculators do not do base 3, but will do base e, so you need to do this if you want a decimal approximation to the number.

$$\begin{aligned} y &= 100 \log_3 \left(\frac{10}{3}\right) \\ y/100 &= \log_3 \left(\frac{10}{3}\right) \\ 3^{y/100} &= \left(\frac{10}{3}\right) \\ \ln\left(3^{y/100}\right) &= \ln\left(\frac{10}{3}\right) \\ \frac{y}{100} \ln 3 &= \ln\left(\frac{10}{3}\right) \\ y &= 100 \frac{\ln\left(\frac{10}{3}\right)}{\ln 3} \end{aligned}$$

Example Find the inverse function $f^{-1}(x)$ if $f(x) = -3\ln(x-5)$.
Finding the inverse function is a three step process.

$$\begin{aligned}
\text{Step 1. } f(x) = y &= -3\ln(x-5) \\
\text{Step 2. (interchange) } x &= -3\ln(y-5) \\
\text{Step 3. (solve for } y\text{) } x &= -3\ln(y-5) \\
x &= \ln\left((y-5)^{-3}\right) \quad (\text{use } \log_b(x^r) = r\log_b x) \\
e^x &= \exp\left(\ln\left((y-5)^{-3}\right)\right) \quad (\text{take exponential of both sides of equation}) \\
e^x &= (y-5)^{-3} \\
e^{-x/3} &= y-5 \\
y &= 5 + e^{-x/3} \\
f^{-1}(x) &= 5 + e^{-x/3}
\end{aligned}$$

Check cancellation equations:

$$\begin{aligned}
f(f^{-1}(x)) &= f(5 + e^{-x/3}) \\
&= -3\ln(5 + e^{-x/3} - 5) \\
&= -3\ln(e^{-x/3}) \\
&= \ln\left((e^{-x/3})^{-3}\right) \\
&= \ln(e^x) \\
&= x
\end{aligned}$$

$$\begin{aligned}
f^{-1}(f(x)) &= f^{-1}(-3\ln(x-5)) \\
&= 5 + \exp\left(-\frac{1}{3}(-3\ln(x-5))\right) \\
&= 5 + \exp(\ln(x-5)) \\
&= 5 + (x-5) \\
&= x
\end{aligned}$$

Example Find the inverse function $f^{-1}(x)$ if $f(x) = -2e^{-x}$.

$$\begin{aligned}
\text{Step 1. } f(x) = y &= -2e^{-x} \\
\text{Step 2. (interchange) } x &= -2e^{-y} \\
\text{Step 3. (solve for } y\text{) } x &= -2e^{-y} \\
-\frac{x}{2} &= e^{-y} \\
\ln\left(-\frac{x}{2}\right) &= \ln e^{-y} \\
\ln\left(-\frac{x}{2}\right) &= -y \\
y &= -\ln\left(-\frac{x}{2}\right) \\
f^{-1}(x) &= -\ln\left(-\frac{x}{2}\right)
\end{aligned}$$