Concepts: Higher degree polynomials, local extrema of polynomials, zeros of polynomials, end behaviour, sketching polynomials.

Polynomials of degree 1 and 2 are so important because we can do so much with them; get x-intercepts, sketch graphs quickly, and basically understand them fully.

Higher degree polynomials are more difficult to describe. There is no beautiful technique like completing the square we can use easily.

However, we can still describe the graphs of higher degree polynomials, and describe general behaviour.

Vocabulary

The standard form for writing a polynomial of degree n is $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$.

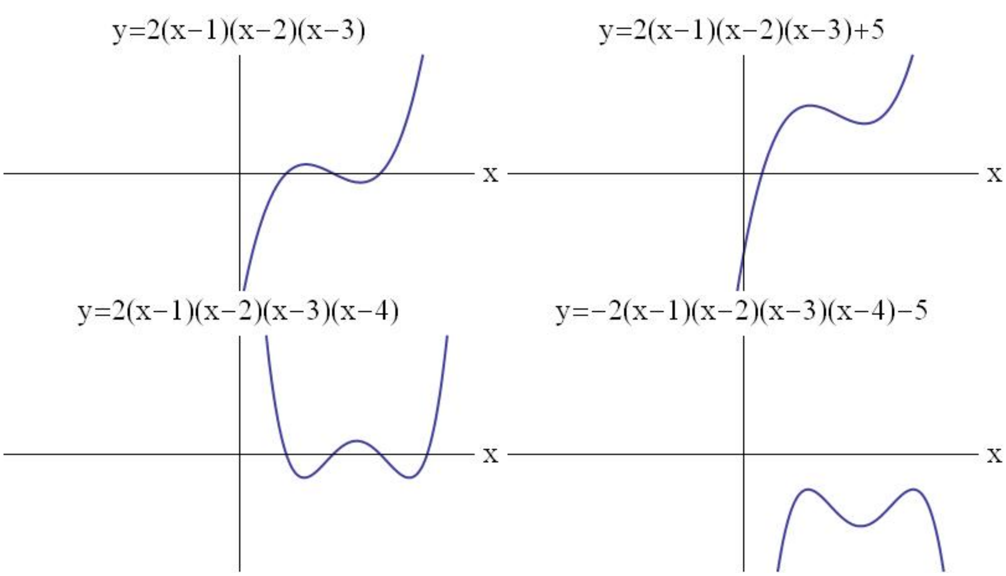
Each monomial ($a_n x^n$, $a_{n-1}x^{n-1}$, etc.) is a *term* of the polynomial.
The a_i for $i = 0, 1, 2, 3, \ldots, n$ are the *coefficients* of the polynomial.
The term $a_n x^n$ is the *leading term* of the polynomial. The leading term is dominant, and is used to determine the end behaviour of the polynomial.

Local Extrema and Zeros of Polynomials

Polynomials are continuous, or *smooth* functions.

A polynomial function of degree n has at most $n - 1$ local extrema and n zeros. This is certainly true for the quadratic case (degree 2), where you get two real roots, one real root, or no real roots, and there is only one extrema.

Here's some examples of higher degree polynomials, with varying numbers of roots and extrema.



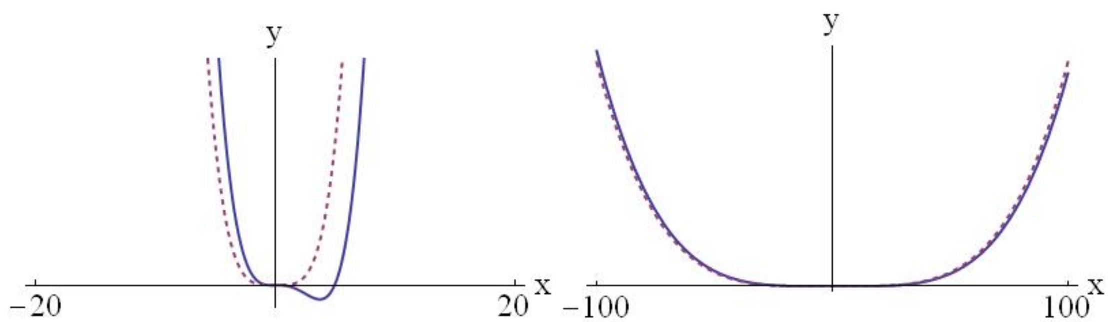
End Behaviour of Polynomials

Example What is the end behaviour of the polynomial $f(x) = x^4 - 5x^3 + x - 1$?

We can determine the end behaviour by noting that the leading term is the dominant part of the polynomial for large values of x.

$$f(x) = x^4 - 5x^3 + x - 1 \sim x^4 \text{ for } x \text{ large, and } x \text{ large negative.}$$

This means the graph of $f(x)$ will approach the graph of its leading term (a monomial) for extreme values of x. Notice that if we zoom out to really see this (the graph on the right), it is impossible to see the interesting behaviour of $f(x)$ near the origin (left). This is one of the reasons why sketching by hand is so important.



Zeros of Polynomials

Finding the zeros of a function f is equivalent to finding the x-intercepts of the graph $y = f(x)$. Finding the x-intercepts algebraically relies on being to able to *factor* a polynomial.

Factoring a polynomial of degree higher than a quadratic is typically difficult, although it can be done for low degree polynomials.

Recall: Factoring Rules

$$\begin{aligned} x^2 + 2xy + y^2 &= (x+y)^2 \\ x^2 - y^2 &= (x+y)(x-y) \\ x^3 + y^3 &= (x+y)(x^2 - xy + y^2) \\ x^3 - y^3 &= (x-y)(x^2 + xy + y^2) \end{aligned}$$

If you cannot factor a quadratic by inspection, you can use the quadratic formula to factor.

Long Division of Polynomials is also useful for factoring polynomials, and we will look at that more later.

Example Find the zeros of the polynomial $f(x) = 5x^3 - 5x^2 - 10x$.

We do this by factoring:

$$\begin{aligned} f(x) &= 5x^3 - 5x^2 - 10x \\ &= 5x(x^2 - x - 2) \\ &= 5x(x+1)(x-2) \end{aligned}$$

The zeros are $x = 0, -1, 2$. Done!

If we can't factor a quadratic by inspection, we can use the quadratic formula to find the roots.

Definition (multiplicity): If the polynomial f has $(x-c)^m$ as a factor but not $(x-c)^{m+1}$, then c is a *zero of f of multiplicity m*.

If c is a zero of the polynomial f with odd multiplicity, then the graph of f crosses the x axis at $x = c$. This is because the function f will change sign at $x = c$.

If c is a zero of the polynomial f with even multiplicity, then the graph of f does not cross the x axis at $x = c$. This is because the function f will not change sign at $x = c$.

If the multiplicity is greater than or equal to 2, the graph will be horizontal where it touches the x-axis.

Example Find the zeros with multiplicity for the polynomial $f(x) = x(3x-5)^4(2+x)^3$. What do you know about the behaviour of f near the zeros from this?

- zero at $x = 0$ has multiplicity 1 (odd) so f changes sign at $x = 0$,
- zero at $x = 5/3$ has multiplicity 4 (even) so f does not change sign at $x = 5/3$,
- zero at $x = -2$ has multiplicity 3 (odd) so f changes sign at $x = -2$, and since multiplicity was greater than 2, f will be horizontal at $x = -2$.

Sketching Higher Degree Polynomials

To sketch a polynomial, you must determine at least two things for the polynomial:

- Zeros with multiplicity of the polynomial (by factoring, which may involve long division of polynomials)
  - Use multiplicity of each zero to determine if polynomial crosses x-axis at the zero (i.e., changes sign).
- End behaviour (by examining leading term)
  - Are there horizontal or slant asymptotes? (the answer is no, except for polynomial $f(x) = mx + b$)

You may also choose to label points of interest, such as y-intercepts. Finding the exact location of extrema for higher degree polynomials will require calculus.

Example Sketch the graph of the polynomial $g(x) = x(9 - 6x + x^2)(9x^2 - 24x + 16)$ by hand.

This is a fifth degree polynomial, so it will have at most 5 real valued roots.
To determine all the roots, we need to see if we can factor the two quadratics any further.

$$9 - 6x + x^2 = x^2 - 6x + 9 = (x-3)(x-3) = (x-3)^2 \qquad \text{factored by inspection}$$
$$9x^2 - 24x + 16 = (3x-4)(3x-4) = (3x-4)^2 \qquad \text{factored by inspection}$$

$$g(x) = x(x-3)^2(3x-4)^2$$

The polynomial will have three zeros, at $x = 0, 4/3, 3$. Multiplicities:

- zero at $x = 0$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 4/3$ has multiplicity 2 (even) so g does not change sign,
- zero at $x = 3$ has multiplicity 2 (even) so g does not changes sign.

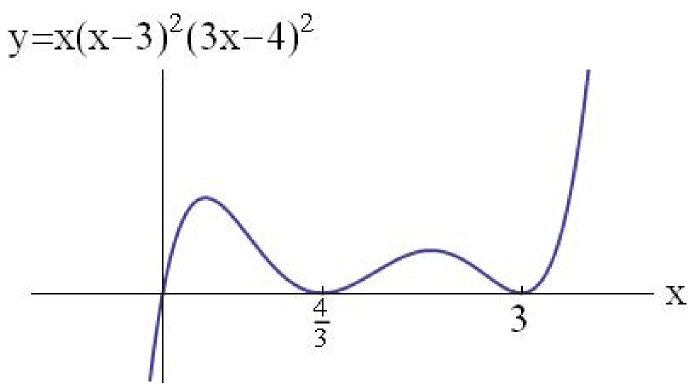
The end behaviour of the polynomial is found by determining the leading term, which is

$$x(x-3)^2(3x-4)^2 \sim xx^2(3x)^2 = 9x^5 \text{ for large } |x| \ .$$

Note "for large $|x|$" here is shorthand for saying "for x large positive or x large negative".
The end behaviour of the monomial $9x^5$ is

$$\lim_{x \to -\infty} 9x^5 = -\infty \qquad \lim_{x \to \infty} 9x^5 = \infty$$

From all this, we can sketch the graph of $g(x)$ without turning on a machine.



Example Sketch the graph of the polynomial $g(x) = x(x+3)^3(x-3)^2(2-x)$ by hand.

This is a seventh degree polynomial, so it will have at most 7 real valued roots.
The polynomial is already factored, which saves us a lot of work!

The polynomial will have four zeros, at $x = -3, 0, 2, 3$. Multiplicities:

- zero at $x = -3$ has multiplicity 3 (odd) so g changes sign (since multiplicity is greater than 2, g will be horizontal at $x = -3$),
- zero at $x = 0$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 2$ has multiplicity 1 (odd) so g changes sign,
- zero at $x = 3$ has multiplicity 2 (even) so g does not changes sign.

The end behaviour of the polynomial is found by determining the leading term, which is

$$x(x+3)^3(x-3)^2(2-x) \sim x(x)^3(x)^2(-x) = -x^7 \text{ for large } |x|.$$

The end behaviour of the monomial $-x^7$ is

$$\lim_{x\to-\infty} -x^7 = \infty \qquad \lim_{x\to\infty} -x^7 = -\infty$$