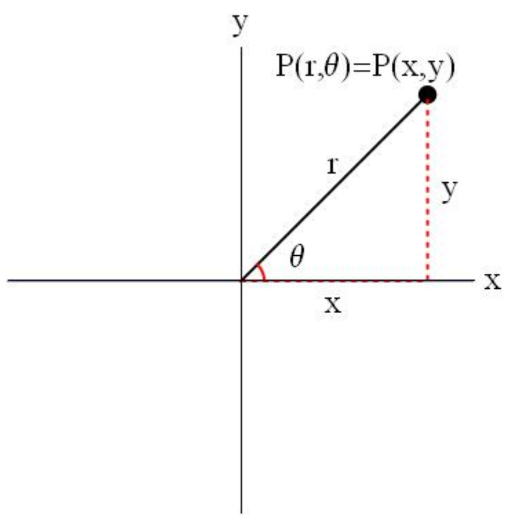
Concepts: Polar Coordinates, converting between polar and cartesian coordinates, distance in polar coordinates.

Until now, we have worked in one coordinate system, the Cartesian coordinate system. This is the xy-plane. However, we can use other coordinates to determine the location of a point. An important coordinate system is polar coordinates, which is useful if the function has rotational symmetry.

The Cartesian coordinates (x, y) and polar coordinates (r, θ) of a point P:



Using basic trigonometry, we get the relation between the two coordinate systems. To go from polar to Cartesian, we use:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

and to go from Cartesian to polar, we use

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \arctan(y/x). \end{aligned}$$

Note the following:

- the origin is called the pole,
- the polar axis is the axis off which the angle θ is measured,
- angles are positive if measured in counter clockwise direction from the polar axis, negative if measured clockwise,
- the value of r can be negative; say $r > 0$, then the points (r, θ) and $(-r, \theta)$ lie on the same line through the origin and at the same distance from the origin, but on opposite sides of the origin. The point (r, θ) lies in the same quadrant as θ; the point $(-r, \theta)$ lies in the quadrant on the opposite side of the origin.
- there is more than one representation of a point in polar coordinates.

Example The polar coordinates $(4, \pi/4)$, $(4, \pi/4 + 2\pi)$, and $(-4, \pi/4 + \pi)$ all refer to the same Cartesian point:

$(4, \pi/4)$:

$$\begin{aligned} x &= 4\cos\frac{\pi}{4} = 4\frac{\sqrt{2}}{2} = 2\sqrt{2} \\ y &= 4\sin\frac{\pi}{4} = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}. \end{aligned}$$

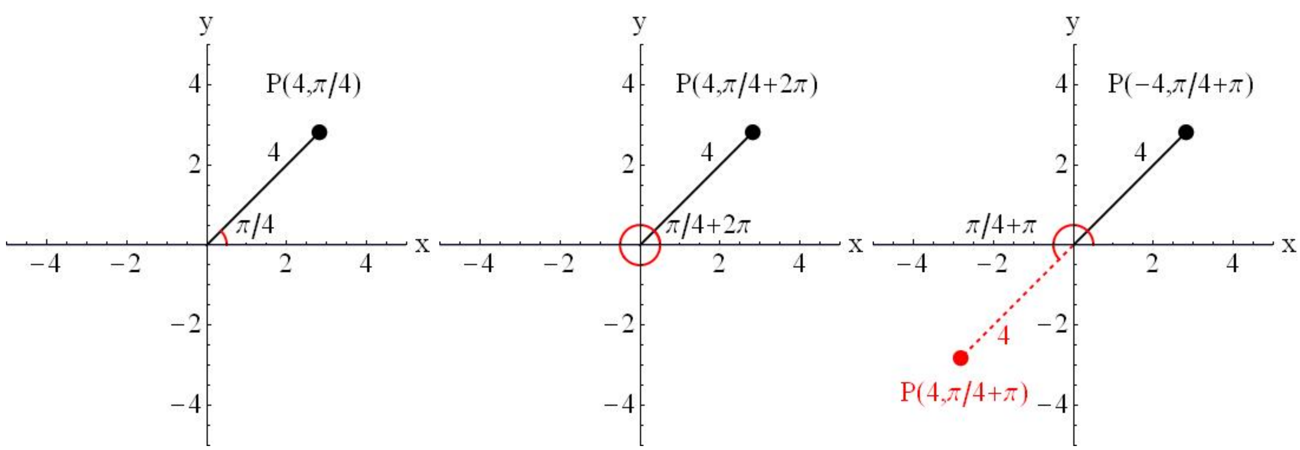
$(4, \pi/4 + 2\pi)$:

$$\begin{aligned} x &= 4\cos\left(\frac{\pi}{4} + 2\pi\right) = 4\cos\left(\frac{\pi}{4}\right) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2} \\ y &= 4\sin\left(\frac{\pi}{4} + 2\pi\right) = 4\sin\left(\frac{\pi}{4}\right) = 4\frac{\sqrt{2}}{2} = 2\sqrt{2}. \end{aligned}$$

$(-4, \pi/4 - \pi)$:

$$\begin{aligned} x &= -4\cos\left(\frac{\pi}{4} - \pi\right) = -4\left[\cos\left(\frac{\pi}{4}\right)\cos(\pi) + \sin\left(\frac{\pi}{4}\right)\sin(\pi)\right] = -4\left[\left(\frac{\sqrt{2}}{2}\right)(-1) + \left(\frac{\sqrt{2}}{2}\right)(0)\right] = 2\sqrt{2} \\ y &= -4\sin\left(\frac{\pi}{4} - \pi\right) = -4\left[\sin\left(\frac{\pi}{4}\right)\cos(\pi) - \cos\left(\frac{\pi}{4}\right)\sin(\pi)\right] = -4\left[\left(\frac{\sqrt{2}}{2}\right)(-1) + \left(\frac{\sqrt{2}}{2}\right)(0)\right] = 2\sqrt{2} \end{aligned}$$

Geometrically:



The following are all the same point in Cartesian space, where n is an integer:

$$P(r, \theta) = P(r, \theta + 2n\pi) = P(-r, \theta + (2n+1)\pi)$$

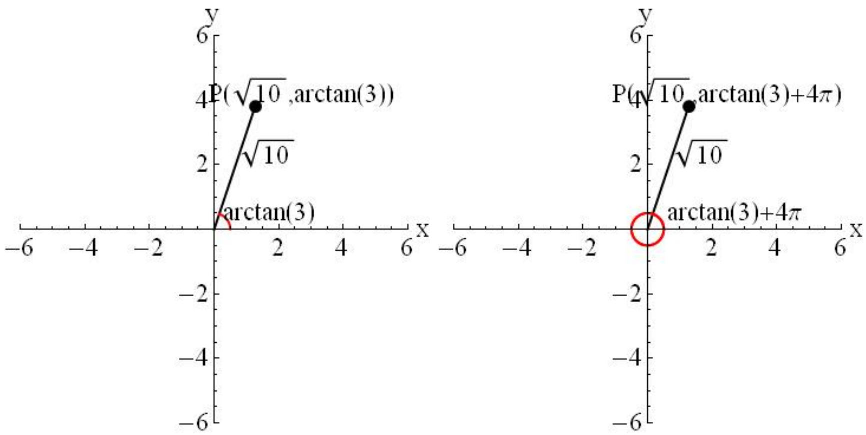
Example Find the polar coordinates of the point P given the cartesian coordinates $P(1,3)$.

We use the relation:

$$\begin{aligned} r^2 &= x^2+y^2 = 1^2+3^2 = 1+9 = 10 \longrightarrow r=\sqrt{10} \sim 3.16228 \\ \theta &= \arctan(3/1) = \arctan 3 \sim 1.24905. \end{aligned}$$

So $P(r,\theta) = (\sqrt{10}, \arctan 3)$. We could also write the point P as
$P(r, \theta + 2n\pi) = (\sqrt{10}, \arctan 3 + 2n\pi)$, or
$P(-r, \theta + (2n+1)\pi) = (-\sqrt{10}, \arctan 3 + (2n+1)\pi)$, for any integer n.

Geometrically:



Converting Equations Between Polar and Cartesian Coordinates

In cartesian coordinates, we had functional relations like $y = f(x)$.

You can have something similar in polar coordinates, $r = f(\theta)$.

We can convert equations from a polar representation to a cartesian representation, and vice versa.

Example Convert the rectangular equation $(x+3)^2 + (y+3)^2 = 18$ to a polar equation.

We simply use our relations:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, \end{aligned}$$

$$\begin{aligned} (x+3)^2 + (y+3)^2 &= 18 \\ (r\cos\theta+3)^2 + (r\sin\theta+3)^2 &= 18 \\ (r^2\cos^2\theta + 9 + 6r\cos\theta) + (r^2\sin^2\theta + 9 + 6r\sin\theta) &= 18 \\ r^2(\cos^2\theta+\sin^2\theta) + 18 + 6r\cos\theta + 6r\sin\theta &= 18 \\ r^2(1) + 6r\cos\theta + 6r\sin\theta &= 18-18 \\ r^2 + 6r\cos\theta + 6r\sin\theta &= 0 \\ r(r + 6\cos\theta + 6\sin\theta) &= 0 \\ r + 6\cos\theta + 6\sin\theta &= 0, \qquad r \neq 0 \\ r &= -6\cos\theta - 6\sin\theta \end{aligned}$$

Both these equations represent circles of radius $\sqrt{18}$ centered at $(x,y) = (3,3)$.

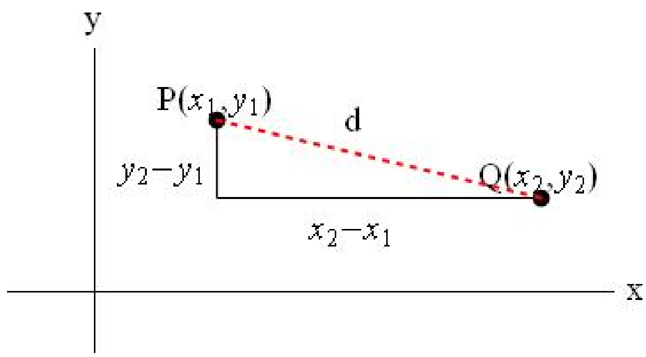
Example Convert the rectangular equation of a circle $x^2 + y^2 = c^2$ to a polar equation.

$$\begin{aligned} x^2 + y^2 &= c^2 \\ (r\cos\theta)^2 + (r\sin\theta)^2 &= c^2 \\ r^2\cos^2\theta + r^2\sin^2\theta &= c^2 \\ r^2(\cos^2\theta + \sin^2\theta) &= c^2 \\ r^2 &= c^2 \\ r &= \pm c \end{aligned}$$

So we see that a circle or radius c can be described by the polar equation $r = c$ (or, $r = -c$). This is much simpler than the cartesian equation $x^2 + y^2 = c^2$.

Finding Distance in Cartesian Coordinates

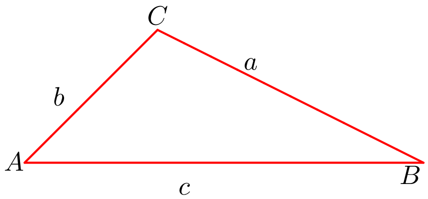
Recall the distance formula in cartesian coordinates, which is based on the Pythagorean theorem:



$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

Finding Distance in Polar Coordinates Using Law of Cosines

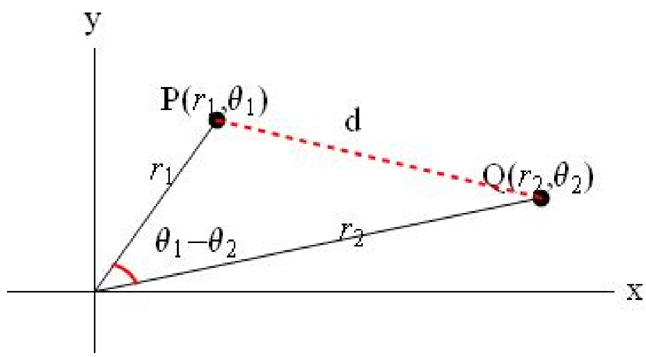
We can easily find the distance between two points given in polar coordinates using the Law of Cosines:



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$



Comapring to the general Law of Cosines result, we should should choose: $A = \theta_1 - \theta_2$, $a = d$, $b = r_1$, $c = r_2$, and use $a^2 = b^2 + c^2 - 2bc\cos A$.

$$\begin{aligned} a^2 &= b^2 + c^2 - 2ab\cos A \\ d^2 &= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2) \\ d &= \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)} \end{aligned}$$

Example Find the distance between the points $P(2, \pi/2)$ and $Q(3, \pi/4)$, which are given in polar coordinates.

$$\begin{aligned} d &= \sqrt{2^2 + 3^2 - 2(2)(3)\cos(\pi/2 - \pi/4)} \\ &= \sqrt{4 + 9 - 12\cos(\pi/4)} = \sqrt{4 + 9 - 12\frac{\sqrt{2}}{2}} = \sqrt{13 - 6\sqrt{2}} \end{aligned}$$

We can check this by converting the points to cartesian points, and then using the cartesian distance formula.

$P(2, \pi/2)$:

$$\begin{aligned} x_1 &= r\cos\theta = 2\cos\frac{\pi}{2} = 2(0) = 0 \\ y_1 &= r\sin\theta = 2\sin\frac{\pi}{2} = 2(1) = 2 \end{aligned}$$

$Q(3, \pi/4)$:

$$\begin{aligned} x_2 &= r\cos\theta = 3\cos\frac{\pi}{4} = 3\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \\ y_2 &= r\sin\theta = 3\sin\frac{\pi}{4} = 3\frac{\sqrt{2}}{2} = \frac{3\sqrt{2}}{2} \end{aligned}$$

Using the cartesian distance formula:

$$
\begin{aligned}
d &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \\
&= \sqrt{\left(0 - \frac{3\sqrt{2}}{2}\right)^2 + \left(2 - \frac{3\sqrt{2}}{2}\right)^2} \\
&= \sqrt{\frac{18}{4} + 4 - \frac{12\sqrt{2}}{2} + \frac{18}{4}} \\
&= \sqrt{9 + 4 - 6\sqrt{2}} \\
&= \sqrt{13 - 6\sqrt{2}}
\end{aligned}
$$