Concepts: Double Angle Identities, Power Reducing Identities, Half Angle Identities.

Memorized:

$$\begin{aligned}\cos^2 x + \sin^2 x &= 1\\ \cos(u-v) &= \cos u \cos v + \sin u \sin v\\ \sin(u+v) &= \sin u \cos v + \cos u \sin v\end{aligned}$$

Derive other identities you need from these as you solve problems.

Double Angle Identities

The double angle identities are found from letting $u = v$ in the sum identities. Notice how we continually build on our previous results.

$$\begin{aligned}\cos(u+v) &= \cos u \cos v - \sin u \sin v && \text{start with known result}\\ \cos(2u) = \cos(u+u) &= \cos u \cos u - \sin u \sin u\\ &= \cos^2 u - \sin^2 u\\ &= \cos^2 u - (1-\cos^2 u)\\ &= 2\cos^2 u - 1\\ &= 2(1-\sin^2 u) - 1\\ &= 1 - 2\sin^2 u\end{aligned}$$

$$\begin{aligned}\sin(u+v) &= \sin u \cos v + \cos u \sin v && \text{start with known result}\\ \sin(2u) = \sin(u+u) &= \sin u \cos u + \cos u \sin u\\ &= 2\sin u \cos u\end{aligned}$$

$$\begin{aligned}\tan(2u) = \frac{\sin(2u)}{\cos(2u)} &= \frac{2\sin u \cos u}{\cos^2 u - \sin^2 u}\\ &= \frac{2\sin u \cos u}{\cos^2 u - \sin^2 u} \cdot \left(\frac{\left(\frac{1}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right)\\ &= \frac{2\frac{\sin u}{\cos u}}{\frac{\cos^2 u}{\cos^2 u} - \frac{\sin^2 u}{\cos^2 u}}\\ &= \frac{2\tan u}{1-\tan^2 u}\end{aligned}$$

Power Reducing Identities

The power reducing identities are found by rearranging the double angle identities.

$$\begin{aligned} \cos(2u) &= 2\cos^2 u - 1 && \text{start with known result} \\ \cos^2 u &= \frac{1+\cos 2u}{2} \end{aligned}$$

$$\begin{aligned} \cos(2u) &= 1 - 2\sin^2 u && \text{start with known result} \\ \sin^2 u &= \frac{1-\cos 2u}{2} \end{aligned}$$

$$\begin{aligned} \tan^2 u = \frac{\sin^2 u}{\cos^2 u} &= \frac{\left(\frac{1-\cos 2u}{2}\right)}{\left(\frac{1+\cos 2u}{2}\right)} && \text{start with known result} \\ &= \frac{\left(\frac{1-\cos 2u}{2}\right)}{\left(\frac{1+\cos 2u}{2}\right)} \cdot \left(\frac{2}{2}\right) \\ &= \frac{1-\cos 2u}{1+\cos 2u} \end{aligned}$$

Half Angle Identities

The half angle identities are found from the power reducing identities. They have an inherent ambiguity in the sign of the square root, and this ambiguity can only be removed by checking which quadrant $u/2$ lies in on a case-by-case basis.

$$\begin{aligned} \cos^2 u &= \frac{1+\cos 2u}{2} && \text{start with known result} \\ \cos^2(u/2) &= \frac{1+\cos u}{2} \\ \cos(u/2) &= \pm\sqrt{\frac{1+\cos u}{2}} \end{aligned}$$

$$\begin{aligned} \sin^2 u &= \frac{1-\cos 2u}{2} && \text{start with known result} \\ \sin^2(u/2) &= \frac{1-\cos u}{2} \\ \sin(u/2) &= \pm\sqrt{\frac{1-\cos u}{2}} \end{aligned}$$

$$\begin{aligned} \tan^2 u &= \frac{1-\cos 2u}{1+\cos 2u} && \text{start with known result} \\ \tan^2(u/2) &= \frac{1-\cos u}{1+\cos u} \\ \tan(u/2) &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}} \end{aligned}$$

For the half angle tangent identities, we can write two additional identities that do not have the ambiguity of the sign of the square root since the sine and tangent are both negative in the same intervals.

$$\begin{aligned}
\tan(u/2) &= \pm\sqrt{\frac{1-\cos u}{1+\cos u}} \\
&= \pm\sqrt{\frac{(1-\cos u)(1-\cos u)}{(1+\cos u)(1-\cos u)}} \\
&= \pm\sqrt{\frac{(1-\cos u)^2}{(1-\cos^2 u)}} \\
&= \pm\sqrt{\frac{(1-\cos u)^2}{\sin^2 u}} \\
\tan(u/2) &= \frac{1-\cos u}{\sin u}
\end{aligned}$$

$$\begin{aligned}
\tan(u/2) &= \frac{1-\cos u}{\sin u}\cdot\left(\frac{1+\cos u}{1+\cos u}\right) \\
&= \frac{(1-\cos u)(1+\cos u)}{\sin u(1+\cos u)} \\
&= \frac{1-\cos^2 u}{\sin u(1+\cos u)} \\
&= \frac{\sin^2 u}{\sin u(1+\cos u)} \\
\tan(u/2) &= \frac{\sin u}{1+\cos u}
\end{aligned}$$

The Trig Identities: A Look Back

We have now seen all the special identities you should be able to reproduce, and use to prove other, less basic identities.

Other than the identities that followed essentially from the definition of the trig functions and their sketches, all the identities follow from previously derived identities, except for the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ and the cosine of a difference identity $\cos(u-v) = \cos u \cos v + \sin u \sin v$.

These identities are listed on the front flap of your text for easy reference.

The reciprocal and quotient identities follow from the definition of the trig function.

The even and odd identities follow from the sketches of the trig functions.

The Pythagorean identity $\cos^2 x + \sin^2 x = 1$ was found from the unit circle. The other Pythagorean identities can be derived from it.

The cosine of a difference identity $\cos(u-v) = \cos u \cos v + \sin u \sin v$ was found by drawing the unit circle, and moving the angle $\theta = u - v$ into standard position. The other sum and difference identities can be derived from it.

The cofunction identities were introduced by considering the complementary angles in a right triangle, but they are most easily obtained from the difference identities and knowing the value of the trig functions at the angle $\pi/2$.

The double angle identities are found by letting $u = v$ in the sum identities.

The power reducing identities are found by rearranging the double angle identities.

The half angle identities are found by evaluating the power reducing identities at $u/2$, and then taking a square root.

Memorize

$$\begin{aligned}
\cos^2 x + \sin^2 x &= 1 \\
\cos(u-v) &= \cos u \cos v + \sin u \sin v \\
\sin(u+v) &= \sin u \cos v + \cos u \sin v
\end{aligned}$$

Derive other identities you need from these as you solve problems.

There are three standard problems we can now solve:

$$\begin{aligned}
\cos(2x) + \cos x = 0 \\
\sin(2x) + \sin x = 0 \\
\tan(2x) + \tan x = 0
\end{aligned}$$

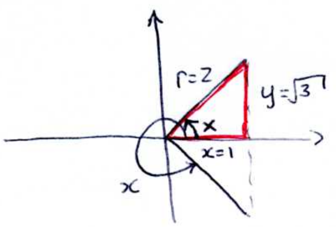
Example Solve $\cos(2x) + \cos x = 0$ for $x \in [0, 2\pi)$.

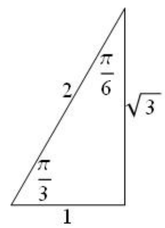
$$\begin{aligned}
\cos(2x) + \cos x &= 0 \\
\cos^2 x - \sin^2 x + \cos x &= 0 \qquad \text{use } \cos(2x) = \cos^2 x - \sin^2 x \\
\cos^2 x - (1 - \cos^2 x) + \cos x &= 0 \qquad \text{use } \cos^2 x + \sin^2 x = 1 \\
2\cos^2 x + \cos x - 1 &= 0 \qquad \text{quadratic in } \cos x
\end{aligned}$$

$$\cos x = \frac{-1 \pm \sqrt{1 - 4(2)(-1)}}{2(2)} = -1 \text{ or } \frac{1}{2}$$

Solve $\cos x = -1$: This gives $x = \pi$, since this is one of our quadrantal angles.

Solve $\cos x = \frac{1}{2}$: This gives $x = \pi/3$ since this is one of the special angles.





Other angles in the interval $[0, 2\pi)$: $2\pi - \pi/3 = 5\pi/3$.

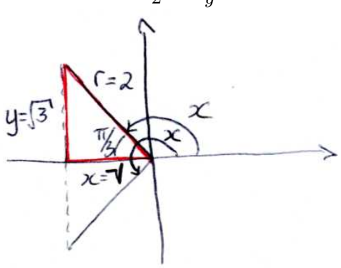
Solution: $x = \pi/3, \pi, 5\pi/3$.

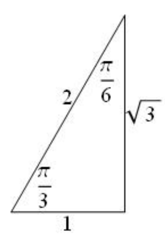
Example Solve $\sin(2x) + \sin x = 0$ for $x \in [0, 2\pi)$.

$$\begin{aligned}
\sin(2x) + \sin x &= 0 \\
2\sin x \cos x + \sin x &= 0 \qquad \text{use } \sin(2x) = 2\sin x \cos x \\
\sin x(2\cos x + 1) &= 0 \qquad \text{factor}
\end{aligned}$$

Solve $\sin x = 0$: This gives $x = 0, \pi$, since this is one of our quadrantal angles.

Solve $\cos x = -\frac{1}{2} = \frac{x}{y}$: We must have $x = -1$ and $r = 2$. Sketch:





Comparing, we see the angle in the red triangle must be $\pi/3$.

Therefore, $x = \pi - \pi/3 = 2\pi/3$ or $\pi + \pi/3 = 4\pi/3$.

Solution: $x = 0, 2\pi/3, \pi, 4\pi/3$.

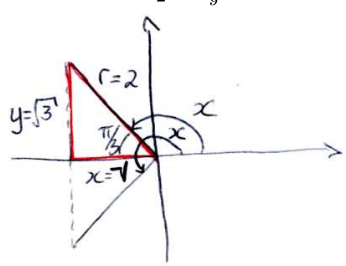
Example Solve $\tan(2x) + \tan x = 0$ for $x \in [0, 2\pi)$.

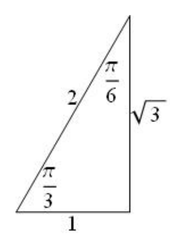
$$
\begin{aligned}
\tan(2x) + \tan x &= 0 && \text{convert to sine and cosine} \\
\frac{\sin(2x)}{\cos(2x)} + \frac{\sin x}{\cos x} &= 0 \\
& && \text{use } \sin(2x) = 2\sin x \cos x \\
& && \text{use } \cos(2x) = 2\cos^2 x - 1 \\
\frac{2\sin x \cos x}{2\cos^2 x - 1} + \frac{\sin x}{\cos x} &= 0 && \text{get common denominator} \\
\frac{2\sin x \cos^2 x}{(2\cos^2 x - 1)\cos x} + \frac{\sin x(2\cos^2 x - 1)}{(2\cos^2 x - 1)\cos x} &= 0 \\
\frac{2\sin x \cos^2 x + \sin x(2\cos^2 x - 1)}{(2\cos^2 x - 1)\cos x} &= 0 \\
2\sin x \cos^2 x + \sin x(2\cos^2 x - 1) &= 0 && \text{factor} \\
\sin x(2\cos^2 x + 2\cos^2 x - 1) &= 0 \\
\sin x(4\cos^2 x - 1) &= 0
\end{aligned}
$$

Solve $\sin x = 0$: This gives $x = 0, \pi$, since this is one of our quadrantal angles.

The other factor, $4\cos^2 x - 1 = 0$, has two cases: $\cos x = \pm\frac{1}{2}$.

Solve $\cos x = -\frac{1}{2} = \frac{x}{y}$: We must have $x = -1$ and $r = 2$. Sketch:

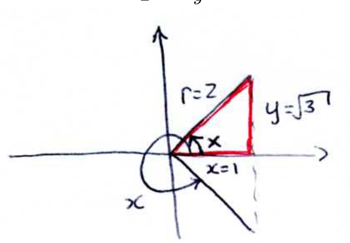


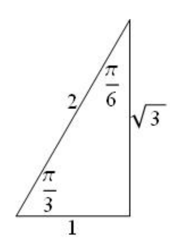


Comparing, we see the angle in the red triangle must be $\pi/3$.

Therefore, $x = \pi - \pi/3 = 2\pi/3$ or $\pi + \pi/3 = 4\pi/3$.

Solve $\cos x = \frac{1}{2} = \frac{x}{y}$: We must have $x = 1$ and $r = 2$. Sketch:





Comparing, we see the angle x in the red triangle must be $\pi/3$.

Therefore, $x = \pi/3$ or $2\pi - \pi/3 = 5\pi/3$.

Solution: $x = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3$.