Concepts: Functions, Graphs, Domain & Range, Continuity, Increasing & Decreasing, Bounded, Extrema, Asymptotes, End Behaviour.

Functions and Their Properties

These are concepts you will get practice with throughout this course, as we study and learn properties for specific functions. When you get to calculus, the concepts continuity, increasing/decreasing, extrema, asymptotes, end behaviour will be discussed using the ideas of calculus (limits and derivatives).

A *function* f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set R.

The *range* R is the set of all possible values of $f(x)$, when x varies over the entire *domain* D.

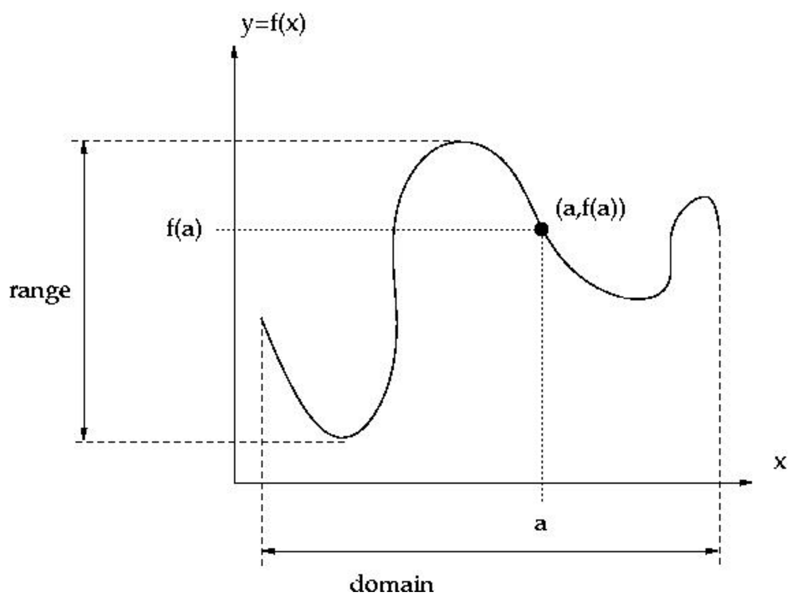
The functions we consider have the domain and range as subsets of the real numbers. The real numbers are denoted $(-\infty, \infty)$ or $\mathbb{R}$.

We often use $y = f(x)$ as dependent variable (it's called dependent because it depends on the value of x). This notation is called *Euler's function notation*. This is read as "y equals f of x". Note that this is not multiplication, that is, $f(x)$ does not mean f times x.

Graph of $y = f(x)$: A graph of $y = f(x)$ pictorially represents the relationship between ordered pairs, where the first element in the pair is the domain, the second element the range:

$$\{(x, f(x))|x \in D\}$$

read: "the set of ordered pairs $(x, f(x))$ such that x is an element of D which is the domain." The graph contains more information than the other descriptions. One of the goals in this class is to be able to sketch the graphs of functions by hand.



More on Domain and Range Given $y = f(x)$, the values of x that can go into $f(x)$ and yield an output which is a real number form the domain. All the possible y's that come out form the range.

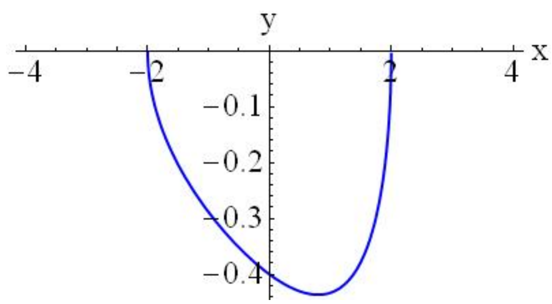
Example Find the domain and range of $h(x) = \dfrac{\sqrt{4-x^2}}{x-5}$.
We cannot have division by zero, so we want to see where the denominator is zero and exclude that value of x from the domain:

$$\begin{aligned} x - 5 &= 0 \\ x &= 5 \end{aligned}$$

so $x = 5$ is not in the domain.
We also cannot take the square root of a negative number and get a result which is a real number. So we must have values of x for which

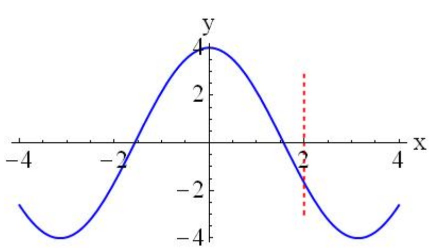
$$\begin{aligned} 4 - x^2 &\geq 0 \text{ (We'll see a cool way of simplifying inequalities later–for now, use our number sense.)} \\ 4 &\geq x^2 \\ x^2 &\leq 4 \\ -2 \leq x &\leq 2 \end{aligned}$$

This means the domain of the function $h(x)$ is $-2 \leq x \leq 2$, or $x \in [-2, 2]$. The point $x = 5$ is excluded, but that is already contained in the restriction based on the square root.
The range is all possible output values. This is usually more complicated to figure out than the domain, but easy to find if we plot a graph using a computer of calculator:

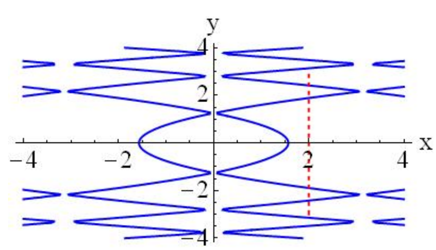


From the graph, we estimate the range to be $y \in [-0.44, 0]$. To get the range precisely we could use ideas from calculus, or other ideas we will develop later.

Vertical Line Test A graph represents a function if every vertical line you can draw intersects the graph only once (this ensures we have exactly one element $f(x)$ for each x).



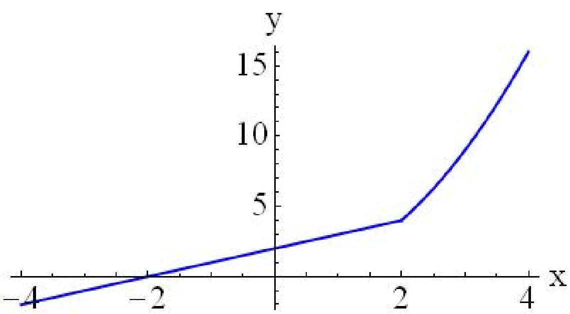
this graph represents a function



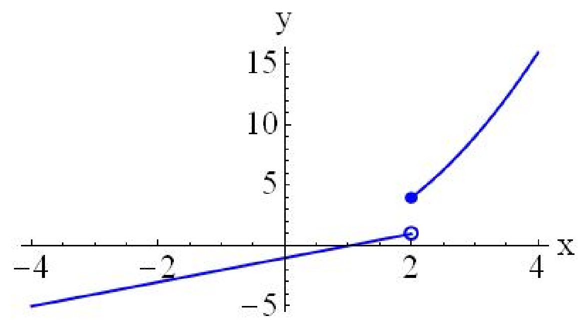
this graph does not represent a function

Continuity

Graphically, a *continuous* function can be drawn without lifting your pen.



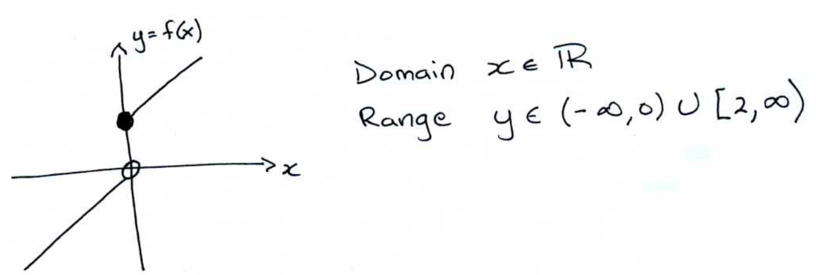
this graph represents a continuous function



this graph does not represent a continuous function

If there is a closed circle corresponding to the value $x = a$, then the point $(a, f(a))$ is part of the function definition. An open circle at the point $(a, f(a))$ means that point is not part of the function definition.

Example Find the domain and range of $f(x) = \begin{cases} x \text{ if } x < 0 \\ x + 2 \text{ if } x \geq 0 \end{cases}$



Read the section in the text on continuity; this concept is developed in greater detail in calculus. The precise definition of continuous requires a precise definition of the *limit of a function*,

$$\lim_{x \to a} f(x).$$

For now, think of this limit as the words "the value of $f(x)$ as x approaches a". The concept of limit is where a calculus course begins. We will be talking more about limits this semester, to help you get used to the notation.

Boundedness

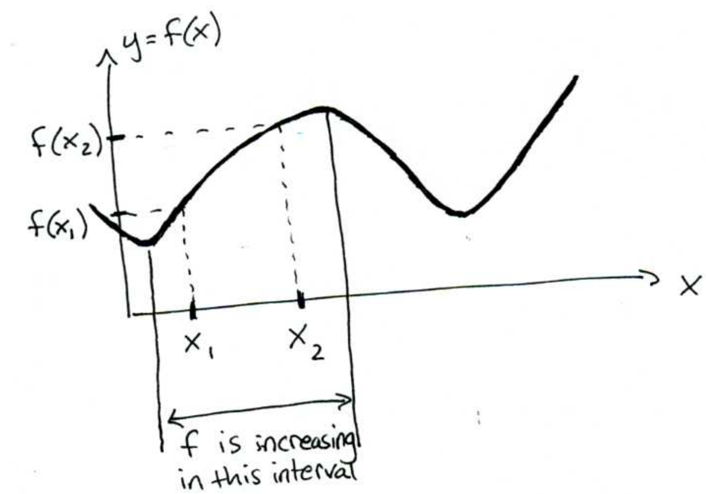
A function f is

- bounded below if there is a number b that is less than or equal to every other number in the range of f.
- bounded above if there is a number B that is greater than or equal to every other number in the range of f.
- bounded if it is bounded above and below.

Increasing and Decreasing Functions

A function is *increasing* on an Interval I if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

A function is *decreasing* on an Interval I if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in I.



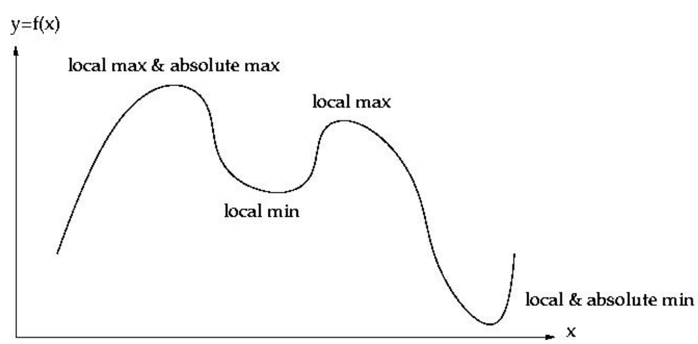
The first derivative of a function, which you will learn about in calculus, gives you information on whether a function is increasing or decreasing without having to look at a sketch.

Local and Absolute Extrema

A function can have peaks and valleys; the value of the function at the peak is called a maximum, the value of the function at a valley is called a minimum. These extreme values of the function are called extrema.

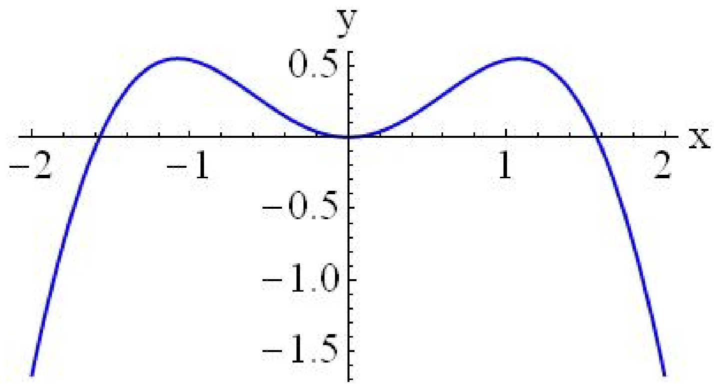
Local extrema are peaks and valleys only in a local area; absolute extrema are the maximum and minimum value of the function over the entire range of f.

These are easy to spot from a graph. In calculus, you will learn a way to determine extrema without looking at a graph of a function.

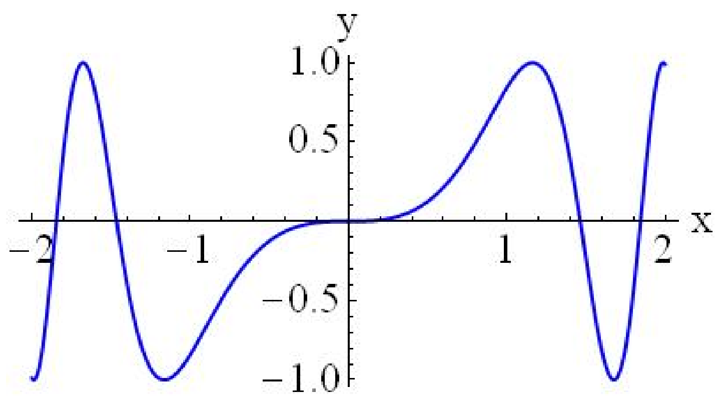


Symmetry

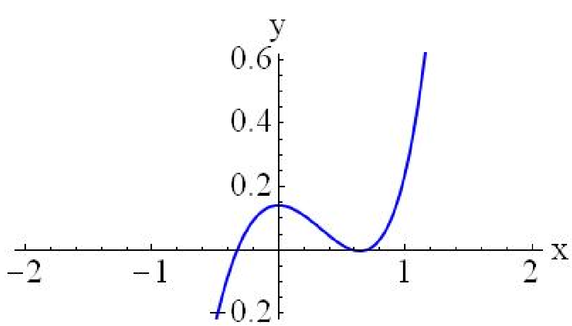
Even functions satisfy $f(-x) = f(x)$. Geometrically this means the function is symmetric about the y-axis.



Odd functions satisfy $f(-x) = -f(x)$. Geometrically this means the function is symmetric if we rotate 180 degrees about the origin.



NOTE: A function can be either even, or odd, or neither!

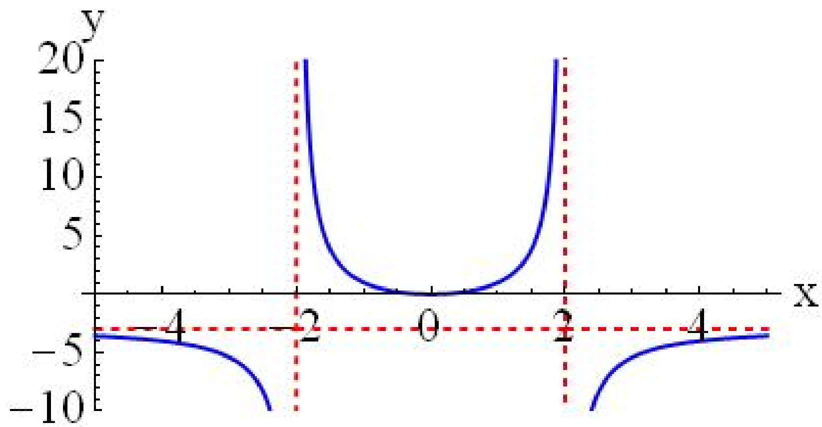


Asymptotes

An *asymptote* is a straight line that a function approaches. There are three types of asymptotes, *vertical*, *horizontal*, and *slant*. Determining asymptotes algebraically is best done using the ideas of limits, which are developed in calculus.

However, we should know what asymptotes are before we get to calculus! Basically, vertical asymptotes are places where the function is not defined, and horizontal and slant asymptotes are straight lines the function approaches as x gets very, very large. We will make this more precise in coming weeks.

Example Consider the function $g(x) = \dfrac{3x^2}{4 - x^2}$. Identify the horizontal and vertical asymptotes from a sketch of this function, which is provided (we will learn how to sketch this by hand in the coming weeks).



Vertical Asymptotes: $x = -2$ and $x = 2$. Horizontal Asymptotes: $y = -3$.

End Behaviour

Knowing what happens to a function if x is very large (positive or negative) can help help you manipulate and understand functions. Analysing end behaviour is an important skill to develop.

We have a notation for end behaviour, using the concept of *limits*. Limits will be discussed in much greater detail in a calculus course, but we want to use the correct notation even in precalculus. There are two limits to evaluate when considering end behaviour, one as $x \to \infty$ and one as $x \to -\infty$.

We write *the limit of the function $f(x)$ as x approaches infinity* mathematically as: $\lim_{x \to \infty} f(x)$.

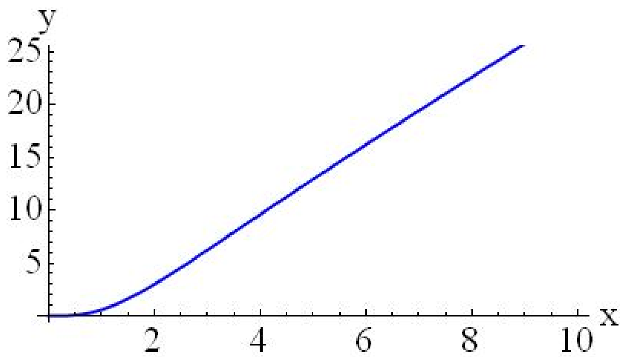
We write *the limit of the function $f(x)$ as x approaches minus infinity* mathematically as: $\lim_{x \to -\infty} f(x)$.

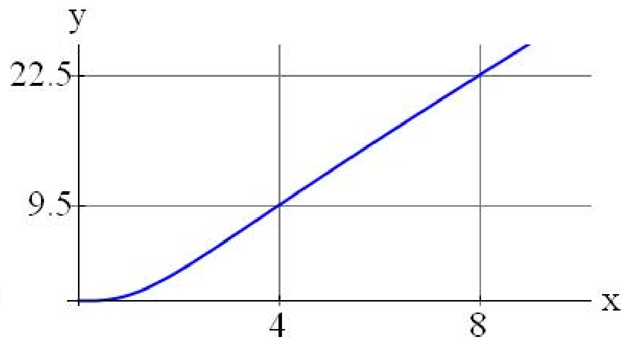
To get horizontal and slant asymptotes <u>algebraically</u> we need to know about end behaviour.

Example What does the function $g(x) = \dfrac{3x^3}{4 + x^2}$ look like if x is very large?

Graphical Solution

Let's get a sketch using a computer or calculator, and see what we can learn. We will learn how to sketch this by hand in the coming weeks.





From the sketch, the graph appears to be linear for large x. We could get points on the line and determine the slope, which might be useful to know.

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{22.5 - 9.5}{8 - 4} = \frac{13}{4} = 3.25.$$

So when x is large, the function $g(x)$ is essentially linear with slope 3.25.

Algebraic Solution

Usually, an algebraic solution is more accurate than a graphical solution, and hence preferable.
When x is large, we can look for ways to simplify the function. In this case, $4 + x^2 \sim x^2$ if x is very large. Therefore, if x is large,

$$g(x) = \frac{3x^3}{4 + x^2} \sim \frac{3x^3}{x^2} = 3x$$

We have found that for x large, the function $g(x)$ approaches the line $y = 3x$. Here's a sketch, with the slant asymptote shown as the dashed red line.