Concepts: population models, constructing exponential population growth models from data, instantaneous exponential growth rate models, logistic growth rate models.

Population can mean anything from

- bacteria in a petri dish,
- amount of radioactive material,
- amount of money in an investment,
- fish in a pond,
- the spread of a disease,

and more. There are two main types of population growth models:

- population changes at a constant percentage rate r each time period, or
- instantaneous growth rate models.

A skill we start to develop here is looking for patterns, which we will use to solve the first type of model. It will prove useful in other areas down the road (especially if you study Taylor Series in calculus).

There are also formulas we can use for the two types of models, although we need calculus to understand where the solution to the instantaneous growth rate model comes from, so I will will just give you the formula.

Population Changes at a Constant Percentage Rate r Each Time Period

Example Consider the population of fish in a suburban pond. The pond was initially seeded with 400 fish. If the population of fish is increasing by 45% each year, by how many fish has the population of fish increased by after 18 months?

We need to introduce some notation to solve this problem. Let $P(t)$ be the population of fish after t years. Then $P_0 = P(0) = 400$ is the initial population of fish in the pond. Let $r = 0.45$ be the percentage increase each year.

$$\begin{aligned}
P(0) &= P_0 \\
P(1) &= P_0 + P_0 r = P_0(1+r) \\
P(2) &= P(1) + P(1)r = P(1)(1+r) = P_0(1+r)^2 \\
P(3) &= P(2) + P(2)r = P(2)(1+r) = P_0(1+r)^3 \\
&\vdots \quad \text{generalize from the pattern} \\
P(t) &= P(t-1) + P(t-1)r = P(t-1)(1+r) = P_0(1+r)^t = 400(1.45)^t
\end{aligned}$$

Now we can answer the question. The population of fish in the pond after 18 months = 3/2 years is given by

$$\begin{aligned}
P(3/2) &= 400(1.45)^{3/2} \\
&= 400(1.74603) = 698.412 = 698
\end{aligned}$$

We truncate the population to an integer since we cannot have .412 fish.
The population of fish in the pond has increased by $698 - 400 = 298$ fish after 18 months.

Example There are initially 300 rats in a field. The population triples every 100 days. Find an expression for the population present after t days. When will the population be greater than 1000?

$$\begin{aligned} p(0) &= p_0 = 300 \\ p(100) &= 3p_0 \\ p(200) &= 3p(1) = 3^2 p_0 \\ p(300) &= 3p(2) = 3^3 p_0 \\ &\vdots \quad \text{generalize from the pattern} \\ p(t) &= 3^{t/100} p_0 = 300 \cdot 3^{t/100} \end{aligned}$$

To find out when the population of rats is 1000, we need to solve the equation

$$p(t) = 1000 = 300 \cdot 3^{t/100}$$

for t. This will require logarithms.

$$\begin{aligned} 1000 &= 300 \cdot 3^{t/100} \\ \frac{1000}{300} &= 3^{t/100} \\ \frac{10}{3} &= 3^{t/100} \\ \log_3\left(\frac{10}{3}\right) &= \log_3(3^{t/100}) \\ \log_3\left(\frac{10}{3}\right) &= \frac{t}{100} \\ t &= 100 \log_3\left(\frac{10}{3}\right) \end{aligned}$$

To get a decimal, we will probably have to change the base. Let's change to the natural logarithm.

$$\begin{aligned} t &= 100 \log_3\left(\frac{10}{3}\right) \\ &= 100 \frac{\ln\left(\frac{10}{3}\right)}{\ln 3} \sim 109.59 \end{aligned}$$

It takes about 110 days for the rat population to exceed 1000, based on our model.

We could have arrived at this answer automatically by using the natural logarithm above instead of the logarithm with base 3.

Formula for This Model in General

If a population is changing at a constant percentage rate r each year, $r \in (-1, 0) \cup (0, \infty)$, then

$$P(t) = P_0(1 + r)^t$$

where P_0 is the initial population, and t is time in years.

If $r > 0$ then the base is $(1 + r) > 1$, this is a growth function, and the population is increasing.

$$P(t) = P_0(1 + r)^t = P_0 b^t, \quad b = 1 + r > 1.$$

If $-1 < r < 0$ then the base is $0 < (1 + r) < 1$, then this is a decay function, and the population is decreasing.

$$P(t) = P_0(1 + r)^t = P_0 b^t, \quad b = 1 + r < 1.$$

If $r < -1$, then the base is $(1 + r) < 0$ which is not allowed for an exponential function.

Note: Although we have just written down a formula we could use, I will want you to be able to construct solutions using the process of looking for a pattern we used in the first example, for the fish in the pond. Make sure you can do that!

Bacteria Growth

Suppose that we know a population of bacteria doubles every hour. Say the population is $p(t)$, and t is the time in hours. If our initial population is p_0, then we have:

$$\begin{aligned} p(0) &= p_0 \\ p(1) &= 2p_0 \\ p(2) &= 2p(1) = 2^2 p_0 \\ p(3) &= 2p(2) = 2^3 p_0 \\ &\vdots \quad \text{generalize from the pattern} \\ p(t) &= 2^t p_0 \end{aligned}$$

We see that "doubling every hour" is the same as saying we have a 100% growth rate every hour.

$$p(t) = p_0 2^t = p_0(1 + 1)^t = p_0(1 + r)^t.$$

Q: What if the population tripled every hour, how would that change our final answer? A: $p(t) = 3^t p_0$**.**

Radioactive Decay and Half-Life

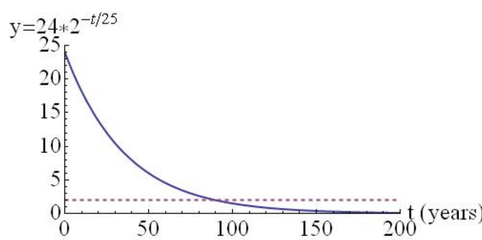
The *half-life* is defined for radioactive compounds to be the time it takes for any amount to decay into half that amount. The fact that we have a decay rather than a growth with automatically be taken care of as we solve the problem.

Example The half-life for strontium-90 (^{90}Sr) is 25 years. If we have 24mg of ^{90}Sr, find an expression for amount left after t years. How long will it take for there to be 2 mg of strontium-90 left?

We shall let $m(t)$ be the amount of ^{90}Sr (in mg) at time t in years. The initial amount is $m(0) = m_0 = 24$ mg.

$$\begin{aligned}
m(0) &= m_0 = 24 \\
m(25) &= \frac{1}{2}m(0) = \frac{1}{2}m_0 \\
m(50) &= \frac{1}{2}m(25) = \frac{1}{2^2}m_0 \\
m(75) &= \frac{1}{2}m(50) = \frac{1}{2^3}m_0 \\
m(100) &= \frac{1}{2}m(75) = \frac{1}{2^4}m_0 \\
&\vdots \quad \text{generalize from the pattern} \\
m(t) &= \frac{1}{2^{t/25}}m_0 = m_0 \cdot 2^{-t/25} = 24 \cdot 2^{-t/25}
\end{aligned}$$

Graphical estimate of how long it takes to get to 2mg (not very accurate, and requires an exact sketch):



From the graph, it looks like it will take about 90 years to decay to only 2mg left.
Here is how you would do it algebraically: Take the natural logarithm of both sides of our equation:

$$\begin{aligned}
\ln m &= \ln(m_0 \cdot 2^{-t/25}) \\
\ln m &= \ln m_0 + \ln 2^{-t/25} \\
\ln m &= \ln m_0 - \frac{t}{25}\ln 2 \\
\frac{t}{25}\ln 2 &= \ln m_0 - \ln m = \ln\left(\frac{m_0}{m}\right) \\
t &= \frac{25}{\ln 2}\ln\left(\frac{m_0}{m}\right)
\end{aligned}$$

So the inverse function is

$$t = f^{-1}(m) = \frac{25}{\ln 2}\ln\left(\frac{m_0}{m}\right)$$

which gives the time it takes for the mass to decay to m milligrams. For our numbers, to get to 2mg would take $t = f^{-1}(2) = \frac{25}{\ln 2}\ln\left(\frac{24}{2}\right) = 89.62$ years.
Notice that in finding the inverse in this case we did not interchange variables, since the variables m and t have physical meaning in the problem.

Instantaneous Growth Rate Models

The other type of model is one you will see when you study calculus, since you need calculus to derive it. It is a very important model, however, so I want you to see it here, since you may use it in your other science classes.

The main idea for instantaneous growth rate models is that we assume:

The instantaneous rate of change in population is equal $= k \times$ current population

for some constant k. More mathematically, this can be written as

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = kP$$

for P the population, t the time. Notice that the quantity $\frac{\Delta P}{\Delta t}$ would be the average rate of change of population over an interval, and the limit is what makes it an instantaneous rate of change.

As I mentioned above, actually solving this requires some knowledge of calculus. Here is the solution:

$$P(t) = P_0 e^{kt}$$

where $P_0 = P(0)$ is the initial population.

Let's revisit the radioactive decay problem and use this model to solve.

Example The half-life for strontium-90 (^{90}Sr) is 25 years. If we have 24mg of ^{90}Sr, find an expression for amount left after t years. How long will it take for there to be 2 mg of strontium-90 left (use the instantaneous decay model)?

The amount of strontium-90 m is given by the model:

$$m(t) = m_0 e^{kt}$$

We need to figure out the value of k, using the information given. Since $m_0 = m(0) = 24$, and half-life is 25 years, $m(25) = \frac{1}{2}m_0$, we have using the model equation:

$$\begin{aligned}
\frac{1}{2}m_0 &= m_0 e^{k(25)} \\
\frac{1}{2} &= e^{25k} \\
\ln\left(\frac{1}{2}\right) &= \ln e^{25k} \\
\ln\left(\frac{1}{2}\right) &= 25k \\
\frac{\ln\left(\frac{1}{2}\right)}{25} &= k \\
k = \frac{\ln\left(\frac{1}{2}\right)}{25} &= \frac{\ln\left(2^{-1}\right)}{25} = -\frac{\ln 2}{25}
\end{aligned}$$

The model looks like

$$m(t) = 24e^{-\frac{\ln 2}{25}t}$$

To find out algebraically how long it will take for there to be 2mg left, we need to use logarithms.

$$\begin{aligned}
2 &= 24e^{-\frac{\ln 2}{25}t} \\
\frac{1}{12} &= e^{-\frac{\ln 2}{25}t} \\
\ln\left(\frac{1}{12}\right) &= \ln e^{-\frac{\ln 2}{25}t} \\
-\ln(12) &= -\frac{\ln 2}{25}t \\
t &= \frac{25\ln 12}{\ln 2} \sim 89.62
\end{aligned}$$

It would take 89.62 years.

The two models we constructed for the radioactive decay problem are similar, and in fact ended up giving the same answer for the time to decay to 2mg. Are they actually the same? Let's check!

$$\begin{aligned}
m(t) &= 24 \cdot 2^{-t/25} \text{ first model} \\
&= 24 \cdot e^{\ln(2^{-t/25})} \text{ change base, using the fact that } b = e^{\ln b} \\
&= 24 \cdot e^{-\frac{\ln 2}{25}t} \text{ second model}
\end{aligned}$$

They are exactly the same model! Our two seemingly different formulas are actually mathematically equivalent.

Proof two Exponential Growth Models are the Same

$$\begin{aligned} P(t) &= P_0(1+r)^t \\ &= P_0 e^{\ln(1+r)^t} \\ &= P_0 e^{t\ln(1+r)} \\ &= P_0 e^{kt} \end{aligned}$$

where $k = \ln(1+r)$. Notice that you should use whichever model has the information provided, so if you are given

- a growth rate per time period k, use $P(t) = P_0(1+r)^t$,
- a growth constant k, use $P(t) = P_0 e^{kt}$.

I will not try to trick you about which one to use. Remember, I will want you to be able to construct solutions using the process of looking for a pattern we used in the first example, for the fish in the pond.

Logistic Models

Exponential models have a flaw since they assume population can grow without bound.

In many situations, growth begins exponentially but then slows and approaches zero. The population approaches a maximum sustainable population. Logistic models incorporate an upper bound on the population.

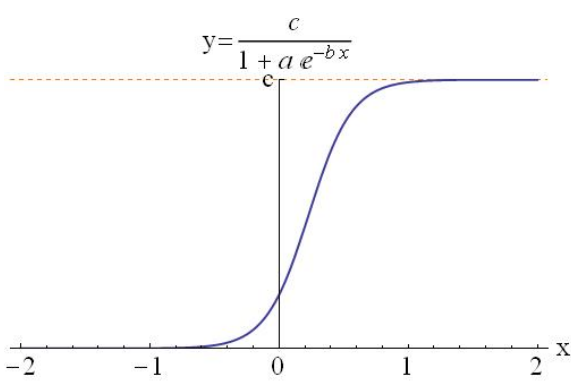
To derive the logistic model again requires use of calculus, and that is one of the reasons why it has been included on the list of 12 basic functions–it is one of the most important population growth models, and we want you to have some experience with it before taking calculus.

If a, b, c, and k are positive constants, and $b < 1$, then a *logistic* growth function can be written as

$$f(x) = \frac{c}{1 + a \cdot b^x} = \frac{c}{1 + a \cdot e^{-kx}}$$

Logistic growth is bounded above.

Sketch of $y = f(x) = \dfrac{c}{1 - ae^{-bx}}$:



Example The number of students infected with flu at a high school after t days is modeled by the function:

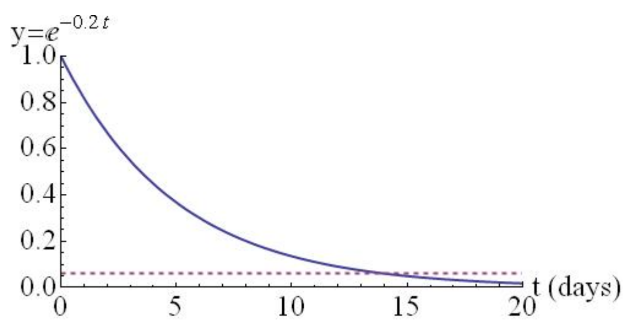
$$P(t) = \frac{800}{1 + 49e^{-0.2t}}.$$

(a) What is the initial number of infected students?
(b) When will the number of infected students be 200?
(c) The school will close when 300 students are infected. When does the school close?

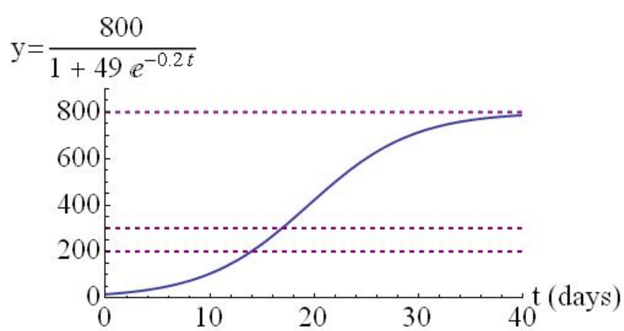
Note that this model has serious flaws. It does not allow for an infected student getting better! As $t \to \infty$, the entire 800 students will be infected. This is unlikely.
(a) At $t = 0$, we have $P(0) = \frac{800}{1 + 49e^0} = \frac{800}{1 + 49} = \frac{800}{50} = 16$. There are initially 16 students infected.
(b) The number of infected students is equal to 200 when

$$\begin{aligned} P(t) = 200 &= \frac{800}{1 + 49e^{-0.2t}} \\ \frac{1}{200} &= \frac{1 + 49e^{-0.2t}}{800} \\ \frac{800}{200} = 4 &= 1 + 49e^{-0.2t} \\ \frac{3}{49} &= e^{-0.2t} \end{aligned}$$

To go any further algebraically we need to use logarithms. We can get an estimate of the solution using a graph:



From the sketch, it looks like 200 students will be infected just short of 15 days. We could also have sketched the function, and got an estimate of the solution that way:



(c) To find out when 300 students are infected, we can use the graph above to see that this will occur after about 17 days. You should verify this is correct using algebra (we will do that in the section on Equation Solving).