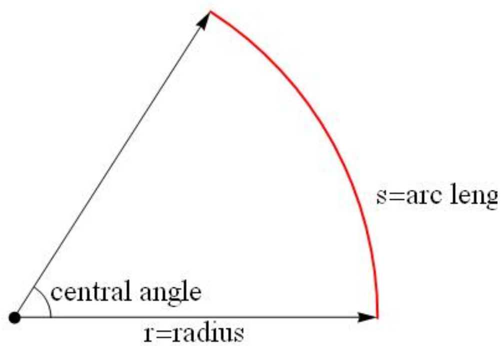
Concepts: Angular Measure (degrees, radians), arc length, π, angular speed.

Angular Measure

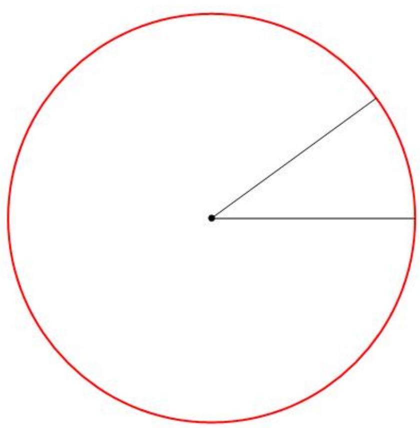
There are actually three types of angular measure, radians (rad), degrees (deg), and grads. They are kind of like the difference between feet, inches, and meters. They are just different units.

Angle Measure Basics

If we fix a line of length r at one end, and rotate it about this point, we sweep out a curve. This curve will have a length, called the (circular) *arc length*, s. The *central angle* is a measure of how much we have rotated the line, or you can think of it as the angle between the original straight line and the final straight line.



If we rotate enough, we sweep out a circle. The length of the line we used to sweep out the circle is the *radius*, r.

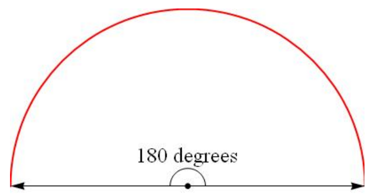


The arc length here is the perimeter of the circle.

Degrees

The degree measurement is based on an ancient love for the number 60. This ancient love is also why we have 60 seconds in a minute, and 60 minutes in an hour. Time and angles measured in degrees share many similarities.

It was decided by the Babylonians that 360 of these things called degrees would be needed to complete one full rotation of the circle. This means there are $360°$ in a circle. Another way of saying this is that there are 180 degrees required to create a straight angle.



This means if our line swept out half a circle, the central angle would be $180°$. This is extremely arbitrary, and does not relate the angular length to a linear length at all.

Degrees can be split up into minutes and seconds, but that is not very important for calculus. I will not be covering that material in this course.

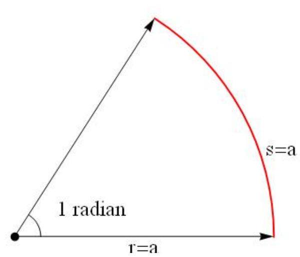
Grads

It was decided that 400 of these things called grads would be needed to complete one full rotation of the circle. I have never used grad measurement in my life, and you probably won't either.

Radians

Radians are a dimensionless measure of angle. They are the preferred unit of measure for angles in math because of this. They also relate angular length to linear length, which is important for calculus.

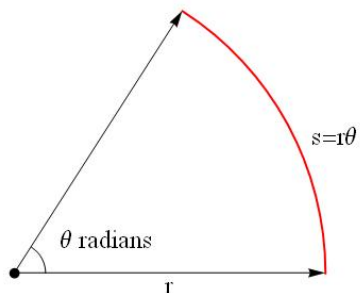
Definition: A central angle has a measure of *1 radian* if it intercepts an arc with the same length as the radius.



From this definition, we can get a formula for arc length

Arc Length Formula

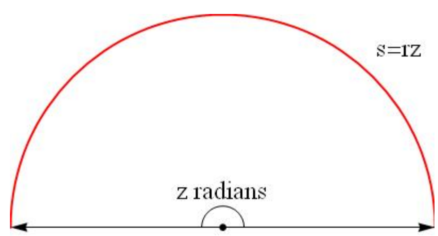
A central angle of θ radians will always intersect a circle of radius r with an arc length of $s = r\theta$.



Note that from this expression we can see that the angular measure radian has no units, since $\theta = \dfrac{s}{r}$ and the units cancel out. This is not true for angular measure in degrees, where the units are degrees.

How Many Radians to Sweep Out a Circle?

From the arc length formula, we can determine how many radians it takes to sweep out a half circle. Let's say that z is the number of radians that are required. We need to figure out what the number z is!



The perimeter of the half circle will be the arc length $s = rz$, using the arc length formula.

The perimeter of the full circle will be $p = 2s = 2rz$.

$$z = \frac{p}{2r} = \frac{p}{d} = \frac{\text{perimeter of the circle}}{\text{diameter of circle}}.$$

and we see that, of course, $z = \pi$ (this ratio is the definition of π).

It takes 2π radians to sweep out a circle.

Notice that here we did not define how many radians it took to sweep out the circle, we discovered it once we had defined what a radian was. This is in contrast to the degree (or grads), which was defined in terms of how many degrees (or grads) were required to sweep out a circle.

Converting Between Radians and Degrees

Basic relation: 180 degrees $= \pi$ radians .

Use ratios to make the conversion.

Example Convert 37° to radians.

$$\begin{aligned}
37 \text{ degrees} &= x \text{ radians} \\
\frac{37 \text{ degrees}}{180 \text{ degrees}} &= \frac{x \text{ radians}}{\pi \text{ radians}} \\
\pi \frac{37}{180} &= x \\
x &= \pi \frac{37}{180} \sim 0.645772 \\
\\
37 \text{ degrees} &= 0.645772 \text{ radians}
\end{aligned}$$

Example Convert 3 radians to degrees.

$$\begin{aligned}
x \text{ degrees} &= 3 \text{ radians} \\
\frac{x \text{ degrees}}{180 \text{ degrees}} &= \frac{3 \text{ radians}}{\pi \text{ radians}} \\
x &= 180 \cdot \frac{3}{\pi} = \frac{540}{\pi} \sim 171.887 \\
3 \text{ radians} &\sim 171.887 \text{ degrees}
\end{aligned}$$

Angular Speed

Linear speed has units like meters per second.

Angular speed has units like revolutions per second.

The connection between the two is provided by the arc length formula, or unit conversion factors.

Example A bicycle has 13 inch radius wheels. When the bike is traveling at a linear speed of 44 ft/sec, how many revolutions per minute is the wheel making?

The perimeter of the bike tire (circumference) is $p = 2\pi r = 2\pi(13) = 26\pi$ inches.
There are 12 inches in a foot.
The perimeter of the bike tire is $p = 26\pi$ inches $\cdot \dfrac{1 \text{ ft}}{12 \text{ inches}} = \dfrac{13\pi}{6}$ ft.
The bike travels $\dfrac{13\pi}{6}$ feet per revolution $= \dfrac{13\pi}{6} \dfrac{\text{ft}}{\text{rev}}$.
The bike tire is therefore making $\dfrac{44\text{ft/s}}{\left(\frac{13\pi}{6} \frac{\text{ft}}{\text{rev}}\right)} = \dfrac{44 \cdot 6}{13\pi} \dfrac{\text{rev}}{\text{sec}} \sim 6.46414 \dfrac{\text{rev}}{\text{sec}} \sim 387.848 \dfrac{\text{rev}}{\text{min}}$.