4452 Mathematical Modeling Lecture 13: Chaos and Fractals

Introduction

In our textbook, the discussion on chaos and fractals covers less than 20 pages. Contrast that with the book *Chaos and Fractals: New Frontiers of Science* [1] which clocks in at over 900 pages. You can imagine that this field is a large one, and we will only be able to give it a cursory glance in this course. I want to give you a flavour of what chaos is all about, but I certainly am not going to be anywhere near complete!

The importance of chaos in modeling is that chaotic systems can have limit sets (the limit set being the long time behaviour of the solution, which until now we have generally thought of as either an equilibrium fixed point or unbounded divergence) which are fractal in nature.

Linear systems do not have limit cycles (closed loops), but it is surprising that simple nonlinear dynamical systems will have limit cycles, and can also exhibit chaos. It is important to note that the presence of a limit cycle does not mean the system is chaotic–the presence of a fractal limit cycle, however, does mean that the system is chaotic.

Since chaos deals with the long term behaviour of solutions (typically, chaos is thought of as sensitivity to initial conditions), systems can exhibit degrees of chaos. What I mean is, if we have two systems, they both may be chaotic, but one may be more chaotic than the other. There are ways to measure the degree of chaos in the system (Lyapunov exponent), which I will discuss, but not expect you to calculate.

Continuous Models: The Hénon–Heiles Model

There is a continuous dynamical system which was constructed to model the motion of a star moving in the field of its galaxy, known as the Hénon–Heiles model [3]. It is a two dimensional dynamical system based on Hamilton's equations of motion, which incorporates the position and momentum of the particle, making it a four dimensional model in the variables x, y, p_x, p_y.

$$\begin{aligned} x' &= p_x \\ y' &= p_y \\ p_x' &= -x - 2xy \\ p_y' &= -y - x^2 + y \end{aligned} \qquad (1)$$

This is system of two coupled nonlinear oscillators, and is unbounded. That means that for some initial conditions, the system will run off to infinity.

This can be solved using a numerical technique such as Euler's method, although we must be very careful that we choose the step size to be small enough to ensure our results are accurate. Although this is a four dimensional problem, we will be plotting only the xy portion of the phase space and $x(t)$ to get an idea about how the system is behaving.

I have plotted the solution to the Hénon-Heiles system (found using Euler's method, although Runge Kutta would be far more efficient) in Fig. 1. Looks like a spirograph, doesn't it? These seem to be solutions which are very regular, and not very chaotic. How can we estimate how chaotic a trajectory is? By doing this, we will be performing a sensitivity analysis of this trajectory to its initial conditions.

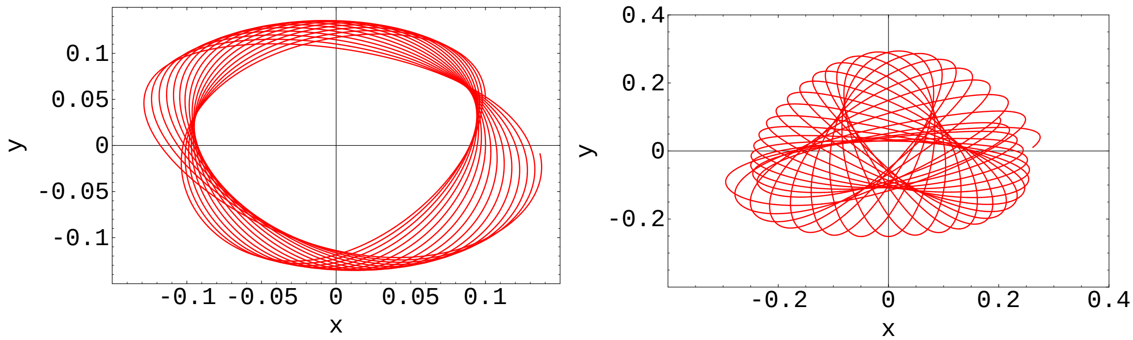

Figure 1: The xy phase space solution to the continuous system in Eq. (1). On the left the inital point is $x(0) = 0.1, y(0) = 0.05, p_x(0) = 0.0, p_y(0) = 0.11$, and on the right the initial point is $x(0) = 0.2, y(0) = 0.1, p_x(0) = 0.04, p_y(0) = -0.05$.

Lyapunov Exponents

The measure of the degree of chaos in a continuous system is expressed in the form of the Lyapunov exponent, which represents the long term behaviour of the nonlinear system. What follows is easily understood, but may be a bit much for us to implement in this class.

One way of deriving the Lyapunov exponent is in terms of the long time behaviour of the *monodromy matrix* (sometimes this matrix is called the Jacobi field matrix). For example, if you have a 2 dimensional system

$$\begin{aligned} x_1'(t) &= f_1(x_1(t), x_2(t)) \\ x_2'(t) &= f_2(x_1(t), x_2(t)). \end{aligned}$$

Then the sensitivity to initial conditions (which is one way of thinking of chaos) means we should take the derivative of the system with respect to the initial conditions. This would give us the rate of change of the variables along the solution curve as a function of time.

$$\begin{aligned} \left(\frac{\partial x_1(t)}{\partial x_1(0)}\right)' &= \frac{\partial}{\partial x_1(0)} f_1(x_1, x_2) = \frac{\partial}{\partial x_1} f_1(x_1, x_2) \cdot \frac{\partial x_1(t)}{\partial x_1(0)} + \frac{\partial}{\partial x_2} f_1(x_1, x_2) \cdot \frac{\partial x_2(t)}{\partial x_1(0)} \\ \left(\frac{\partial x_1(t)}{\partial x_2(0)}\right)' &= \frac{\partial}{\partial x_2(0)} f_1(x_1, x_2) = \frac{\partial}{\partial x_1} f_1(x_1, x_2) \cdot \frac{\partial x_1(t)}{\partial x_2(0)} + \frac{\partial}{\partial x_2} f_1(x_1, x_2) \cdot \frac{\partial x_2(t)}{\partial x_2(0)} \\ \left(\frac{\partial x_2(t)}{\partial x_1(0)}\right)' &= \frac{\partial}{\partial x_1(0)} f_2(x_1, x_2) = \frac{\partial}{\partial x_1} f_2(x_1, x_2) \cdot \frac{\partial x_1(t)}{\partial x_1(0)} + \frac{\partial}{\partial x_2} f_2(x_1, x_2) \cdot \frac{\partial x_2(t)}{\partial x_1(0)} \\ \left(\frac{\partial x_2(t)}{\partial x_2(0)}\right)' &= \frac{\partial}{\partial x_2(0)} f_2(x_1, x_2) = \frac{\partial}{\partial x_1} f_2(x_1, x_2) \cdot \frac{\partial x_1(t)}{\partial x_2(0)} + \frac{\partial}{\partial x_2} f_2(x_1, x_2) \cdot \frac{\partial x_2(t)}{\partial x_2(0)} \qquad (2) \end{aligned}$$

We've had to use the chain rule to expand the partial derivative. The monodromy matrix has elements which are called Jacobi fields, $J_{ij}(t) = \partial x_i(t)/\partial x_j(0)$, and it is given by

$$\mathbf{M} = \begin{pmatrix} \dfrac{\partial x_1(t)}{\partial x_1(0)} & \dfrac{\partial x_1(t)}{\partial x_2(0)} \\ \\ \dfrac{\partial x_2(t)}{\partial x_1(0)} & \dfrac{\partial x_2(t)}{\partial x_2(0)} \end{pmatrix}.$$

The Lyapunov exponents are defined as

$$\lambda_{ij} = \lim_{t\to\infty} \frac{1}{t} \ln |J_{ij}|,$$

which is the same as assuming the Jacobi fields grow as $|J_{ij}| = |\exp(\lambda_{ij} t)|$. Typically, the largest of the Lyapunov exponents is used to determine the degree of chaos present in the trajectory (different trajectories have different Lyapunov exponents). Sometimes, a coarse run of a great many solutions of the dynamical system and monodromy matrix is performed and the Lyapunov exponents averaged to estimate the choaticity of the entire system. There are other ways of determining the Lyapunov exponents as well, in terms of integrals.

All well and good, but to get λ_{ij} we need to solve the 4 dimensional system in Eq. (2)! The system is initialized using the fact that at time zero the monodromy matrix is the identity matrix. For the Henon Heiles system, we would solve for the phase space trajectory and monodromy matrix together, which would entail solving for 20 (4 phase space variables, 16 Jacobi fields) time evolved quantities at once.

In my experience, with molecular collisoins, I am forced to calculate the monodromy matrix anyway, so it is easy to get Lyapunov exponents. However, you do not want to calculate the monodromy simply as a means to get the Lyapunov exponents. Simply rerunning the analysis with a small change in the initial conditions and eyeballing the two solutions to see if there is a big difference may be good enough.

The Hénon–Heiles models actually exhibit a very low degree of chaos.

The Rössler Attractor

The Rössler attractor (found in 1976 by Otto Rössler) is a beautiful, elementary, example of chaos in continuous systems. Rössler's dynamical system is:

$$\begin{aligned} x' &= -(y + x) \\ y' &= x + 0.2y \\ z' &= 0.2 + xz - 5.7z \end{aligned} \tag{3}$$

I solved this system in *Mathematica* using an interpolating function using the following commands. The result is output in Fig. 2.

```
tf = 210;
sol = NDSolve[{
      x'[t] == -y[t] - z[t], x[0] == -0.04,
```

```
      y'[t] == x[t] + 0.2 y[t], y[0] == -0.3,
      z'[t] == 0.2 + x [t]z[t] - 5.7 z[t], z[0] == 0.2}, {x, y, z}, {t, 0, tf},
    Method -> RungeKutta, MaxSteps -> 2000]

plot1 = ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. sol], {t, 0, tf},
    PlotPoints -> 6000, PlotRange -> All]
```

Equation (3) is called an attractor since for different initial condition which are in the region of the orbit shown in Fig. 1, the long term behaviour of the solution will look very similar. The trajectory is not approaching an orbit in the sense of approaching a circle or some other stable, fixed, curve. The trajectory in phase space is bound to follow a pattern like the one in Fig 2, but the specific trajectory is chaotic over long time periods. The trajectory is bound, yet chaotic since where it is within that boundary is sensitive to initial conditions. Figure 3 shows this, in the sense that two trajectories which begin close together remain in the same general region, but move in different parts of the region after a period of time.

Hence, an attractor pulls in neighbouring points, and for a chaotic attractor orbits of nearby points must diverge from each other due to the sensitive dependence on initial conditions. To be an attractor, one of the Lyapunov exponents must be negative; to be chaotic, one of the Lyapunov exponents must be positive.

The *Lorenz* attractor is another beautiful chaotic system which is a model of thermal convection. Our textbook discusses it.

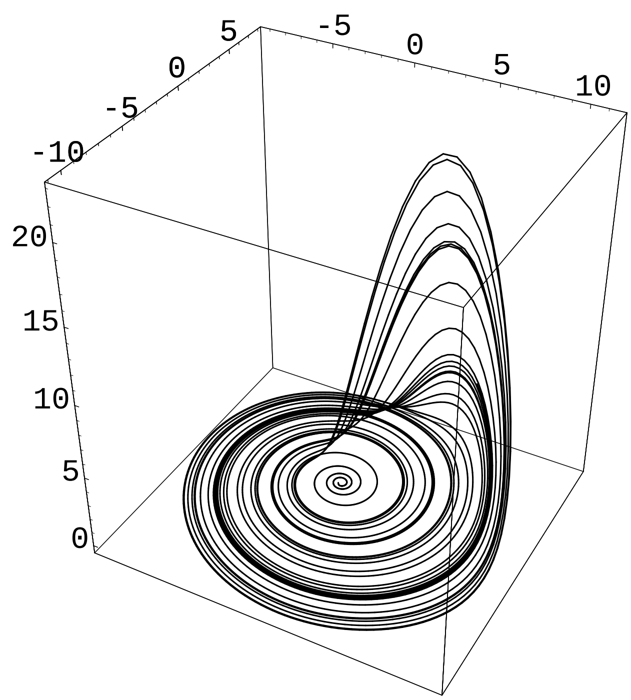

Figure 2: The phase space solution to the continuous system in Eq. (3), the Rössler attractor. The initial conditions are $x(0) = -0.04, y(0) = -0.3, z(0) = 0.2$.

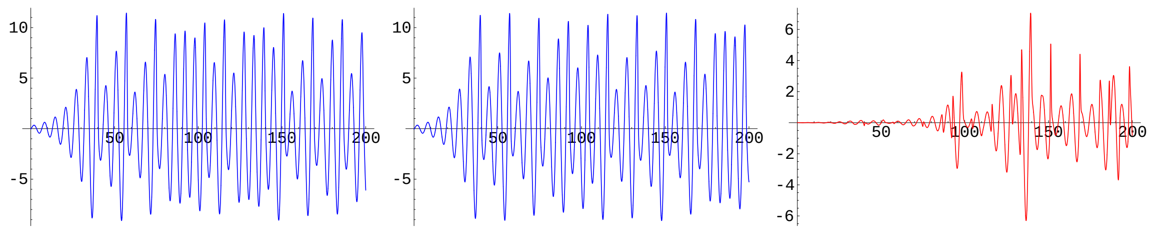

Figure 3: The $x(t)$ vs t solution to the continuous system in Eq. (3) for two trajectories: on the left we have $x(0) = -0.04, y(0) = -0.3, z(0) = 0.2$; in the middle we have $x(0) = -0.045, y(0) = -0.3, z(0) = 0.2$; on the right we have the difference in $x(t)$ between the two trajectories. Although the value of $x(t)$ remains bounded by the attractor, two trajectories which begin close together become far apart for $t > 100$.

Henon Map: Discrete Attractor

The Henon map is an example of a discrete system which exhibits an attraction to a limit cycle. The Henon map is given by

$$\begin{aligned} x_n &= y_{n-1} + 1 - 1.4x_{n-1} \\ y_n &= 0.3x_{n-1}, \quad n = 1, 2, 3, \ldots \end{aligned} \tag{4}$$

Figure 4 shows the Henon map phase space for two different initial conditions. We can see that in both cases, the same limit cycle is approached. So we have an attractor, but it is chaotic (since where we are on the limit cycle is dependent on initial conditions).

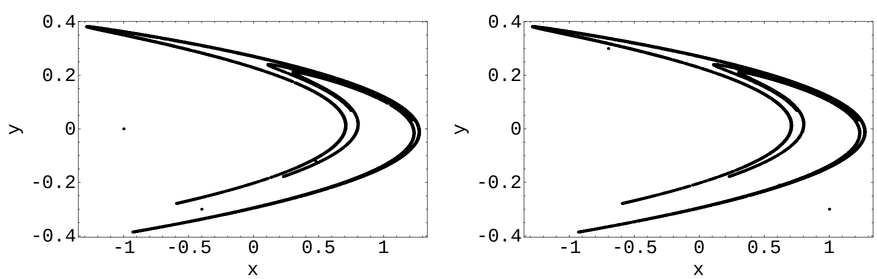

Figure 4: The phase space solution to the discrete system in Eq. (4), the Henon Map. The initial conditions for the graph on the left are $x(0) = -1, y(0) = 0$; for the graph on the right $x(0) = 1, y(0) = -0.3$. The number of points is 12000.

In economics, models of Brownian motion (the motion of a small particle suspended in water) are used to predict the stock market. This is a chaotic system, since the behaviour is random but the particle stays in some localized volume. The banking industry snatched up many physicists and engineers about 10 years ago since they had experience with the models the banking industry wanted to use. Now there are schools who train economists in the theory directly, so the crossover doesn't happen as much today.

You may have heard the term *strange attractor*. Strange attractors are the point where chaos and fractals meet. As dynamical systems, strange attractors are chaotic. As geometric shapes, strange attractors are fractals. I guess we should talk about fractals, eh?

Fractals

There are certain symmetries in plane mathematical geometry. You can reflect an object, you could rotate an object, or you could translate an object. Objects may possess no symmetry, some of these symmetries, or all of these symmetries. A circle looks same under all these symmetries; the letter "A" does not.

There is another symmetry, the symmetry of magnification. An object which possesses this type of symmetry is called a *fractal*. Fractals are objects which look the same on any scale. Thus, fractals are self similar. Their are many famous examples of fractal geometry, and with the explosion of the internet fractal art has also exploded. It is very easy to search the internet and find a fractal landscape that someone has created to stir our souls. People also create fractal music, although I haven't heard any myself.

The reason fractals pops up when people talk about discrete dynamical systems is because fractal concepts like the Julia set, the Mandelbrot set, continued fractions, are all computed using an iterative technique.

For example, here is how the Julia set is constructed by an iteration technique:

$$z_n = \pm\sqrt{z_{n-1} - c} \tag{5}$$

where z_n is a complex number, and c is a complex constant. At each step we randomly decide if we want the positive or negative square root.

If we plot z in the complex plane, we get a fractal image. In Fig. 5 I have plotted the Julia set on two different scales. The self-similarity of the Julia set is evident.

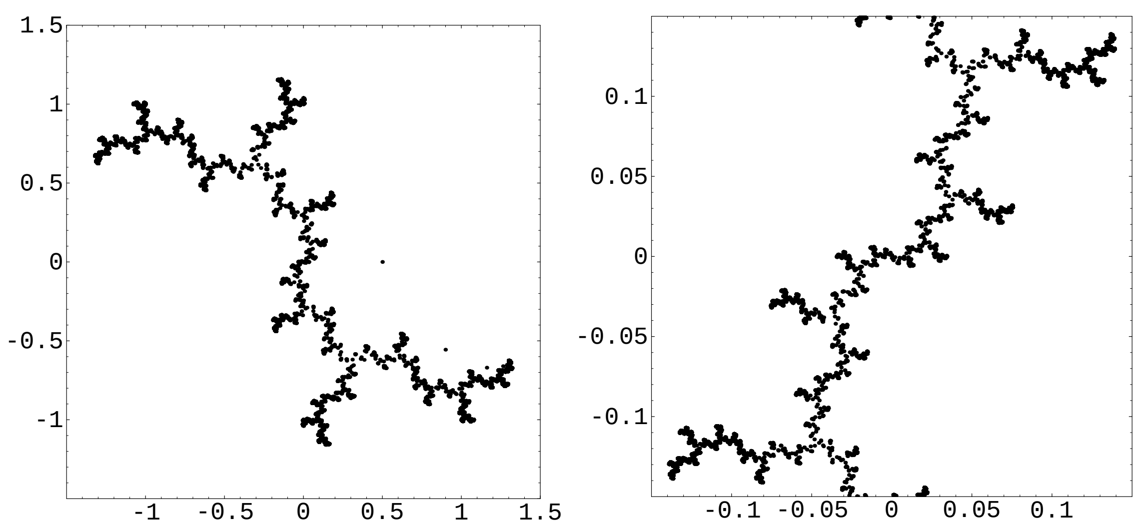

Figure 5: The complex plane representation of the iteration of the Julia set in Eq. (5). The initial point was $z(0) = 1/2$, and $c = i$. The number of points used was 500000. Notice that changing the scale by an order of magnitude the geometry looks the same. This is an indication of self similarity, and fractal behaviour.

Notice that the initial conditions do not affect the final shape of the Julia set (Figs. 5 and 6). The initial conditions affect the transient solution, which is the first few iterations the system goes through before it "fall into" the limit cycle of the Julia set; once this transient solution is passed, the points will be constrained to fill out the Julia set. Once the transient solution has passed, the solution is chaotic (it fills out the Julia set, but *how* it fills it out is sensitive to the initial condition).

Fractals can be used to model objects in reality that possess magnification symmetry. Say, the bottom of the ocean, or a coastline. They are also used for data compression.

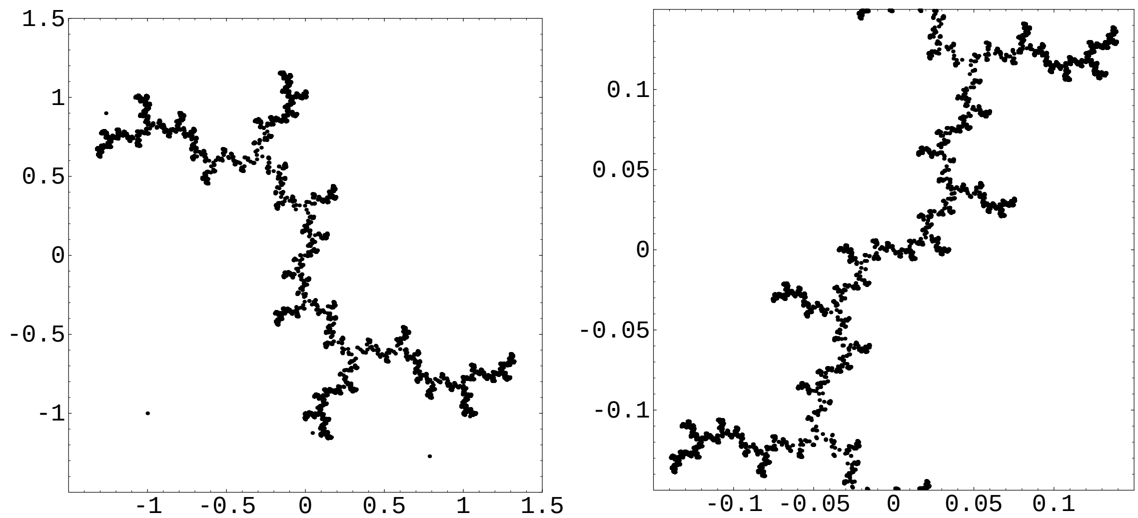

Figure 6: The complex plane representation of the iteration of the Julia set in Eq. (5). The initial point was $z(0) = -1 - i$, and $c = i$. The number of points used was 500000. This looks the same as Fig. 5, except for the initial transient solution.

References

[1] H. Peitgen, H. Jürgens & D. Saupe, Chaos and Fractals: New Frontiers of Science, Springer–Verlag (New York) 1992.

[2] The Nonlinear Lab: http://www.apmaths.uwo.ca/ bfraser/version1/nonlinearlab.html. Blair Fraser, 1997.

[3] M. Hénon & C. Heiles, Astron. J. **69**, 73 (1964).

[4] M. Meerschaert, Mathematical Modelling, 2nd ed., Academic Press (San Diego) 1999 (p200).