4452 Mathematical Modeling Lecture 11: Linear Approximation, Eigenvalues, Stability (Discrete Dynamical Systems)

Introduction

We saw how to determine stability of continuous dynamical systems based on determining the fixed points of the system, then linearizing the system around the fixed points and determining the eigenvalues of the matrix which drove the dynamics.

We can perform a similar procedure for a discrete dynamical system. However, since there is a built in time delay in the discrete system, we can expect things to be different than the continuous case.

In *Mathematica*, RSolve can be used to solve recurrence relations. However, for systems which are nonlinear, RSolve may be unable to solve the system. You are often forced to work with the fixed points, the sequence solution itself, and the linearization procedure.

Linear Approximation

Consider the nonlinear autonomous system of discrete difference equations

$$\begin{aligned} x_1(n) - x_1(n-1) = f_1(x_1(n-1), x_2(n-1)) &= -5x_1(n-1) + 2x_2(n-1) + x_1(n-1)x_2(n-1) \\ x_2(n) - x_2(n-1) = f_2(x_1(n-1), x_2(n-1)) &= x_1(n-1) + x_2(n-1) - 4x_1(n-1)x_2(n-1), \\ x_1(0) = \alpha_1, x_2(0) = \alpha_2, & \\ n = 1, 2, 3, \ldots . & \end{aligned} \tag{1}$$

The above form was chosen to agree with the continuous model we studied in Lecture 10. This system can be written in terms of recurrence relations (which is how you would solve it in *Mathematica*) as

$$\begin{aligned} x_1(n) = g_1(x_1(n-1), x_2(n-1)) &= -4x_1(n-1) + 2x_2(n-1) + x_1(n-1)x_2(n-1) \\ x_2(n) = g_2(x_1(n-1), x_2(n-1)) &= x_1(n-1) + 2x_2(n-1) - 4x_1(n-1)x_2(n-1), \\ x_1(0) = \alpha_1, x_2(0) = \alpha_2, & \\ n = 1, 2, 3, \ldots . & \end{aligned} \tag{2}$$

The relation $g_i = f_i + x_i$ relates the two representations. It is nonlinear because of the terms $x_1(n-1)x_2(n-1)$. The fixed points for the system are found by solving the equations:

$$\begin{aligned} x_1(n+1) &= g_1(x_1(n), x_2(n)) = x_1(n) \\ x_2(n+1) &= g_2(x_1(n), x_2(n)) = x_2(n), \end{aligned}$$

for $(x_1(n), x_2(n))$. For this system, the fixed points are $(0, 0)$ and $(7/19, 7/9)$.

The linearization can be achieved in a manner similar to the linearization of continuous systems, however, we will work with the recurrence relation Eq. (2) and not the difference equation Eq. (1). Equation (2) can be linearized about the point (x_1^0, x_2^0) by using the fact that

$$\begin{aligned} g_1(x_1,x_2) &\sim g_1(x_1^0,x_2^0) + \frac{\partial g_1(x_1^0,x_2^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial g_1(x_1^0,x_2^0)}{\partial x_2}(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim -4x_1^0 + 2x_2^0 + x_1^0 x_2^0 + (-4 + x_2^0)(x_1 - x_1^0) + (2 + x_1^0)(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim -4x_1 + 2x_2, \quad x_1 \sim 0, x_2 \sim 0 \end{aligned}$$

where in the last step we choose the point $(x_1^0, x_2^0) = (0, 0)$, since this is one of the fixed points of the system. To determine the stability about the other fixed point $(7/19, 7/9)$ you would need to first linearize about that point.

Similarly, we find

$$\begin{aligned} g_2(x_1,x_2) &\sim g_2(x_1^0,x_2^0) + \frac{\partial g_2(x_1^0,x_2^0)}{\partial x_1}(x_1 - x_1^0) + \frac{\partial g_2(x_1^0,x_2^0)}{\partial x_2}(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim x_1^0 + 2x_2^0 - 4x_1^0 x_2^0 + (1 - 4x_2^0)(x_1 - x_1^0) + (2 - 4x_1^0)(x_2 - x_2^0), \quad x_1 \sim x_1^0, x_2 \sim x_2^0 \\ &\sim x_1 + 2x_2, \quad x_1 \sim 0, x_2 \sim 0. \end{aligned}$$

And so the nonlinear discrete dynamical system in Eq. (1) can be approximated by the linear system about the point $(0, 0)$ as

$$\begin{aligned} x_1(n) &= -4x_1(n-1) + 2x_2(n-1) \\ x_2(n) &= x_1(n-1) + 2x_2(n-1), \\ &\quad x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots. \end{aligned} \tag{3}$$

and about the point $(7/19, 7/9)$ as

$$\begin{aligned} x_1(n) &= -\frac{29}{9}x_1(n-1) + \frac{405}{171}x_2(n-1) - \frac{49}{171} \\ x_2(n) &= -\frac{361}{171}x_1(n-1) + \frac{10}{19}x_2(n-1) + \frac{196}{171}, \\ &\quad x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots. \end{aligned} \tag{4}$$

Both of these equations are linear in x_1, x_2. Once we have linearized about the fixed point, we can translate the fixed point from (x_1^0, x_2^0) to $(0, 0)$ and work around the origin. Therefore, for the stability analysis of the fixed point at $(7/19, 7/9)$, we would actually work with the system

$$\begin{aligned} x_1(n) &= -\frac{29}{9}x_1(n-1) + \frac{405}{171}x_2(n-1) \\ x_2(n) &= -\frac{361}{171}x_1(n-1) + \frac{10}{19}x_2(n-1), \\ &\quad x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots, \end{aligned}$$

and examine this system near the fixed point $(0, 0)$.

Notice that for the discrete dynamical system we linearize by using continuous calculus procedures (partial derivatives). All of the above can be written using matrix notation.

$$\begin{aligned}
\mathbf{x}(n) - \mathbf{x}(n-1) &= \mathbf{F}(\mathbf{x}(n-1)) \quad \text{(difference equation)} \\
\mathbf{x}(n) &= \mathbf{F}(\mathbf{x}(n-1)) + \mathbf{x}(n-1) = \mathbf{G}(\mathbf{x}(n-1)) \quad \text{(recurrence relation)} \\
\mathbf{x}(n) &\sim \mathbf{A} \cdot (\mathbf{x}(n-1) - \mathbf{x}^0) \quad \text{(linear approximation about } \mathbf{x}^0\text{.)} \\
\mathbf{x}(n) &\sim \mathbf{A} \cdot \mathbf{x}(n-1) \quad \text{(stability analysis performed around the origin } \mathbf{0}\text{.)}
\end{aligned}$$

where

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \mathbf{G} = \begin{pmatrix} f_1 + x_1 \\ f_2 + x_2 \end{pmatrix}, \mathbf{A} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_1}{\partial x_2}(x_1^0, x_2^0) \\ \frac{\partial g_2}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_2}{\partial x_2}(x_1^0, x_2^0) \end{pmatrix}.$$

This is easy to extend to higher dimensions.

Eigenvalues

For a continuous dynamical system, the solution was stable as long as the real part of the eigenvalues of the matrix were less than zero. This was apparent through a study of the differential equations which were solved, and their relation to a linear algebra eigensystem.

For the discrete case, we get something different. If the eigenvalues of the matrix $\mathbf{A}$ have absolute value less than one, then the fixed point is a stable point. For discrete systems, we don't have spiral points in the same way that continuous systems had, so if the eigenvalue is complex $a \pm ib$ we mean the eigenvalue has complex absolute value $\sqrt{a^2 + b^2}$ less than one for the fixed point to be stable.

The reason for this is apparent if we think in terms of a $\mathbf{A}$ as a *linear contraction* when the fixed point is stable. A linear contraction has the property that

$$\lim_{n \to \infty} \mathbf{A}^n \mathbf{x} \to \mathbf{0}$$

for an integer n. So, if λ is an eigenvalue of $\mathbf{A}$, we have $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$ and we get the sequence solution

$$\begin{aligned}
\mathbf{x}(0) &= \mathbf{a} \\
\mathbf{x}(1) &= \mathbf{A}\mathbf{x}(0) = \lambda \mathbf{a} \\
\mathbf{x}(2) &= \mathbf{A}\mathbf{x}(1) = \lambda \mathbf{x}(1) = \lambda^2 \mathbf{a} \\
&\vdots \\
\mathbf{x}(n+1) &= \mathbf{A}\mathbf{x}(n) = \lambda \mathbf{x}(n) = \lambda^{n+1} \mathbf{a}
\end{aligned}$$

and if $|\lambda| < 1$ we have

$$\lim_{n \to \infty} \mathbf{x}(n+1) = \lim_{n \to \infty} \lambda^{n+1} \mathbf{a} \to \mathbf{0}.$$

For a one dimensional case, we can compare the discrete and continuous models for a simple linear system. We will use a difference equation for the discrete case since we want to compare it to the continuous. The relation of importance is $\tilde{\lambda} = 1 + \lambda$.

$$\begin{aligned}
\text{Continuous:} \quad & \frac{dx}{dt} = \lambda x(t) \longrightarrow x = Ce^{\lambda t} \\
& \lim_{t\to\infty} x(t) \to 0 \text{ if } \lambda < 0 \\
\text{Discrete:} \quad & x(n) - x(n-1) = \lambda x(n-1) \longrightarrow x(n) = C(1+\lambda)^{1+n} \\
\text{(compare to continuous)} \quad & \lim_{n\to\infty} x(n) \to 0 \text{ if } |1+\lambda| < 1 \\
\\
\text{Discrete:} \quad & x(n) = \tilde{\lambda} x(n-1) \longrightarrow x(n) = C(\tilde{\lambda})^{1+n} \\
\text{(previous analysis)} \quad & \lim_{n\to\infty} x(n) \to 0 \text{ if } |\tilde{\lambda}| < 1
\end{aligned}$$

Example Returning to our example, we had linearized and found that about the point $(0, 0)$ Eq. (1) could be approximated by

$$\begin{aligned}
x_1(n) &= -4x_1(n-1) + 2x_2(n-1) \\
x_2(n) &= x_1(n-1) + 2x_2(n-1), \\
& \quad x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots.
\end{aligned}$$

The matrix $\mathbf{A}$ here is

$$\mathbf{A} = \begin{pmatrix} -4 & 2 \\ 1 & 2 \end{pmatrix}$$

which has eigenvalues that are -4.3 and 2.3, both with absolute value greater than one, so the point $(0, 0)$ is unstable. Recall that in the continuous case, we were able to say that $(0, 0)$ was unstable in one direction, but stable in another. The behaviour of the discrete system is very different!

About the point $(7/19, 7/9)$, the stability of Eq. (1) would be determined by examining

$$\begin{aligned}
x_1(n) &= -\frac{29}{9} x_1(n-1) + \frac{405}{171} x_2(n-1) \\
x_2(n) &= -\frac{361}{171} x_1(n-1) + \frac{10}{19} x_2(n-1), \\
& \quad x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots,
\end{aligned}$$

about the point $\mathbf{0} = (0, 0)$. The matrix $\mathbf{A}$ here is:

$$\mathbf{A} = \begin{pmatrix} -29/9 & 405/171 \\ -361/171 & +10/19 \end{pmatrix}$$

which has eigenvalues which are complex, $-1.35 \pm 1.22i$. The complex absolute value of these eigenvalues is $1.82 > 1$, so the point $(7/19, 7/19)$ is unstable. Figure 1 shows the behaviour of the solution near the fixed point $(7/19, 7/9)$.

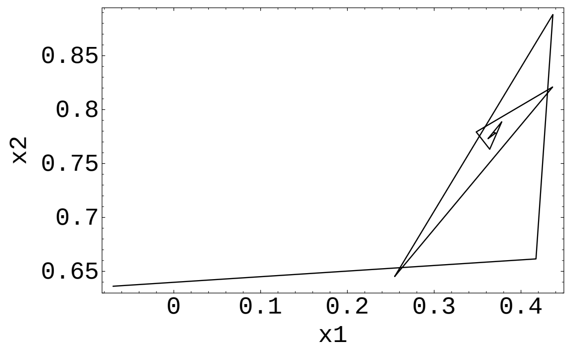

Figure 1: The solution of Eq. (1) near the fixed point $(7/19, 7/9)$. The initial conditions are $\alpha_1 = 7/19$, $\alpha_2 = 7/9 + 0.001$. The solution moves away from the fixed point.

Example Let's investigate another system, one for which there is a stable fixed point:

$$
\begin{aligned}
x_1(n) = g_1(x_1(n-1), x_2(n-1)) &= -0.4x_1(n-1) + 0.2x_2(n-1) + x_1(n-1)x_2(n-1) \\
x_2(n) = g_2(x_1(n-1), x_2(n-1)) &= 0.1x_1(n-1) + 0.2x_2(n-1) - 4x_1(n-1)x_2(n-1), \\
x_1(0) = \alpha_1, x_2(0) = \alpha_2&, \\
n = 1, 2, 3, \ldots&. \qquad (5)
\end{aligned}
$$

Notice that in this system, the nonlinear effects are large. Solving the equations

$$
\begin{aligned}
x_1 &= -0.4x_1 + 0.2x_2 + x_1x_2 \\
x_2 &= 0.1x_1 + 0.2x_2 - 4x_1x_2,
\end{aligned}
$$

for (x_1, x_2), we find only one fixed point, $(0, 0)$. To determine the stability of this fixed point, we calculate the eigenvalues of the matrix

$$
\mathbf{A} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_1}{\partial x_2}(x_1^0, x_2^0) \\ \frac{\partial g_2}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_2}{\partial x_2}(x_1^0, x_2^0) \end{pmatrix} = \begin{pmatrix} (-0.4 + x_2)_{0,0} & (0.2 + x_1)_{0,0} \\ (0.1 - 4x_2)_{0,0} & (0.2 - 4x_1)_{0,0} \end{pmatrix} = \begin{pmatrix} -0.4 & 0.2 \\ 0.1 & 0.2 \end{pmatrix},
$$

which are -0.43 and 0.23. The absolute value of both of these eigenvalues is less than one, so the point $(0, 0)$ is a stable equilibrium point. Figure 3 shows the behaviour of a solution to this system which begins near the fixed point.

If we wish to plot a vector field for the system, we may, but we must remember to plot the vector field for the difference equation and not the recurrence equation. The associated difference equation would be

$$
\begin{aligned}
x_1(n) - x_1(n-1) = f_1(x_1(n-1), x_2(n-1)) &= -1.4x_1(n-1) + 0.2x_2(n-1) + x_1(n-1)x_2(n-1) \\
x_2(n) - x_2(n-1) = f_2(x_1(n-1), x_2(n-1)) &= 0.1x_1(n-1) - 0.8x_2(n-1) - 4x_1(n-1)x_2(n-1), \\
& x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots. \qquad (6)
\end{aligned}
$$

The vector field for Eq. (6) is shown in Fig. 2.

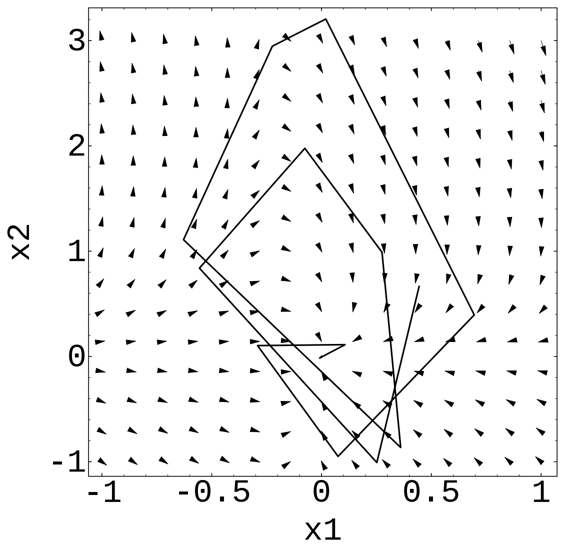

Figure 2: The solution of Eq. (6) near the fixed point $(0, 0)$ with the vector field. The initial condition for the solution is $\alpha_1 = 4/9$, $\alpha_2 = 2/3$.

Of course, we know that the fixed point is stable only in a region near the fixed point. Starting further away will lead to a solution which is unbounded (Fig. 4). We can get a stable fixed point with an unbounded solution for some initial conditions due to the nonlinearity of the system. For a linear system with a stable fixed point, all the solutions would approach the fixed point.

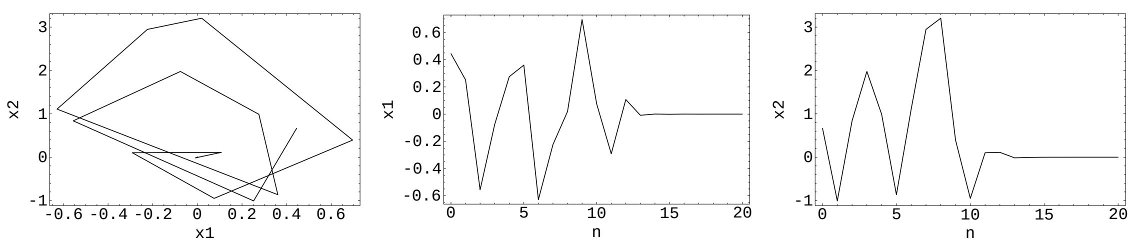

Figure 3: The solution of Eq. (5) near the stable fixed point $(0, 0)$. Initial conditions are $\alpha_1 = 4/9$, $\alpha_2 = 2/3$.

Sensitivity Analysis

The sensitivity analysis focuses on the eigenvalues, and what changes in the parameters will lead to a change in the stability structure of the fixed points. You could also do some simulation (which we will look at soon) to determine how important the nonlinear effects are (it may be important to know what initial conditions will send the solution to the fixed point, and what initial conditions will send the solution off to infinity). Let's analyze our previous problem to determine how the stability of $(0, 0)$ is affected by changes in a parameter.

$$\begin{aligned}
x_1(n) = g_1(x_1(n-1), x_2(n-1)) &= -0.4x_1(n-1) + 0.2x_2(n-1) + x_1(n-1)x_2(n-1) \\
x_2(n) = g_2(x_1(n-1), x_2(n-1)) &= \alpha x_1(n-1) + 0.2x_2(n-1) - 4x_1(n-1)x_2(n-1), \\
& x_1(0) = \alpha_1, x_2(0) = \alpha_2, \quad n = 1, 2, 3, \ldots \qquad (7)
\end{aligned}$$

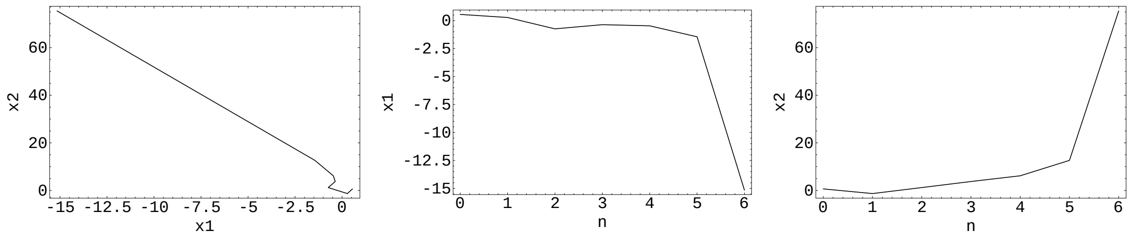

Figure 4: The solution of Eq. (5) near the stable fixed point $(0, 0)$. The initial conditions are $\alpha_1 = 5/9$, $\alpha_2 = 2/3$. This solution does not approach the stable fixed point since we begin too far away for our linear approximation to be valid. Nonlinear effects cause the solution to move away from $(0, 0)$.

To determine the stability of the fixed point, we calculate the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_1}{\partial x_2}(x_1^0, x_2^0) \\ \frac{\partial g_2}{\partial x_1}(x_1^0, x_2^0) & \frac{\partial g_2}{\partial x_2}(x_1^0, x_2^0) \end{pmatrix} = \begin{pmatrix} (-0.4 + x_2)_{0,0} & (0.2 + x_1)_{0,0} \\ (\alpha - 4x_2)_{0,0} & (0.2 - 4x_1)_{0,0} \end{pmatrix} = \begin{pmatrix} -0.4 & 0.2 \\ \alpha & 0.2 \end{pmatrix},$$

which are $\frac{-0.2 \pm 0.89\sqrt{0.45+\alpha}}{2}$. We want the absolute value of the eigenvalues to be less than one for the point at $(0, 0)$ to be stable, so we will want $\alpha < 3.64$. See Figure 5 for a graph of the eigenvalues.

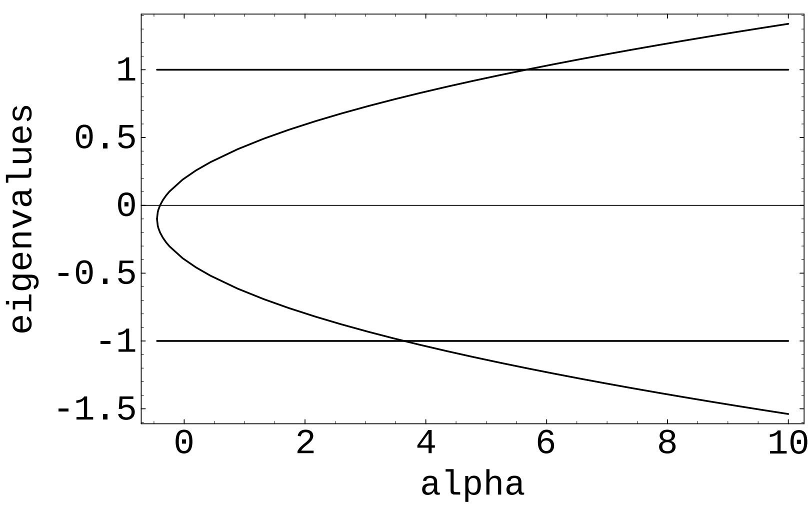

Figure 5: The eigenvalues of Eq. (7) which will determine the stability of the fixed point at $(0, 0)$. For the solution to be stable, we must have $\alpha < 3.64$.