4452 Mathematical Modeling Lecture 1: Single Variable Optimization and Sensitivity Analysis

Introduction

Many of the mathematical techniques we will use in this class you will have already seen. What you may not have seen is how these techniques are used to model real world situations, and the sensitivity analysis of these models.

Lectures will be split into two parts, from 12:00–12:45, a 10 minute break, and then from 12:55–1:40. Lectures will probably follow the following format.

- I will lecture on the mathematical technique that we are going to use to model some type of situation.

  This may either be a short refresher if it is a technique we are all familiar with, or it may be more involved if our experience with the technique is limited.
- I will lecture on the sensitivity analysis for the technique.
- We will talk about where the technique can be used, and what sorts of problems it is well suited to model.
- We will study an example of the technique in a model setting.
- We will discuss homework and any questions that have arisen.
- We will discuss further questions that arise, and how we can find the solution.
- The communication of ideas is very important in this class, so to begin we will be spending more time on the presentation of our solutions than we will later, and the mathematical techniques will be simpler.

I will be placing all my lecture notes on my web page for you download. Hopefully before the lecture, but since I am creating them shortly before the lecture, that probably won't happen!

The Five Step Method

The method that we will be using to model and solve real world problems is the five step method. The most important aspect of this method for us at the moment is how it will help us to present solutions to the model, how it can help us ask the right questions, and how it can help us make our model understandable to a person without our technical background.

Typically, people think solutions are arrived at in a continuous fashion: you start at the beginning and proceed to the end. This is rarely the case, as most of you know, for real problems! This idea can certainly frustrate freshman in calculus, who think what is needed to succeed at math is an ability to arrive at the correct solution. A far more important skill is to be able to talk about and around a problem, until the solution becomes apparent. Rarely is the first draft of a solution something you would be happy with.

The five step method consists of, well, five steps.

1. Ask the question.

   At this point we want to talk about the problem we want to solve in non-technical terms. We want to think about and state any assumptions that occur to us. We want to list the variables in the problem, as well as the constants! An important step here is to make sure we assign the correct units to all the variables and constants we will use. We also want to state the objective, that is, what is the actual question we are trying to answer (there can be more than one question for a given problem, so this is an important step!).

   Do not worry about being completely *correct* at this stage, simply start organizing you thoughts. You will invariably come back and make changes to this section as you proceed.

2. Select the modeling approach.

   At this stage you decide what modeling approach to use. This will become easier with experience. In this class you will typically be told how to model a problem (there are usually more than one way to model any problem!).

   If you like, you can describe in this part of the solution, in words, how you will proceed. This is actually good practice to help you explain techniques to others, so I recommend it.

3. Formulate the model.

   Convert the assumptions and variables from step one into a proper mathematical notation. At this step you may change the notation you picked in Step 1, and in fact may realize that you missed things that should be included in Step 1! This is typical, and not a sign of mathematical weakness. Rewrite Step 1 (and possibly Step 2) as needed.

4. Solve the model.

   Implement mathematically the process you described in words in Step 2. Step 4 is typically what students see as mathematics in most of their courses.

5. Answer the question.

   Answer the question using simple English, as if the person who is reading has no mathematical knowledge at all.

   Also, in this Step (although it could occur in Step 4, I believe), you should perform the sensitivity analysis. Without sensitivity analysis, you solution is virtually useless. You need to know that the solution is one that will not change dramatically with changes in the model.

   The sensitivity analysis can take a great deal of effort. You need to decide how much effort is enough.

General Thoughts

You are searching for a robust solution–one which may not be entirely correct, but one which is close enough to give answers that are useful in the real world.

A solution that is 80% right but took 20% of the time a solution that is 100% right takes may be exactly what is needed! That is, simpler models that ignore some aspects of the problem may be fine for the questions which are asked.

How accurate do you need to be? Is getting more decimals of accuracy in a solution (if you are doing it numerically) at all desirable? Dimensional analysis can be very helpful at this stage (small effects should be ignored in a first solution).

Time spent on the communication of your solutions is time well spent; if you can communicate to a non-technical audience technical techniques, you will become a better mathematician.

We rarely proceed to solution by doing the steps in order! Usually, there is considerable movement back and forth between Steps 1 to 3 before we get to the point where a solution can be found.

Any questions that come up should be added to you assumptions, and if possible, answered at the end.

How Solutions Now Are Different From Before

Typically, what we have is a big wrapper of English and discussion around what would pass for a solution in a lower level math course that would be included in Step 4.

For example, single variable optimization as presented in Calculus I would typically involve something like this:

Example Find the maximum of the function

$$f(x) = (1500 - 100x)(1 + \frac{3x}{20}).$$

The extrema of the function can be found by solving $f'(x_c) = 0$ for x_c.

The Second Derivative Test tells us that if $f''(x_c) > 0$, we have a local minimum at $x = x_c$, and if $f''(x_c) < 0$, we have a local maximum at $x = x_c$.

$$\begin{aligned} f(x) &= (1500 - 100x)(1 + \frac{3x}{20}) \\ &= 1500 + 125x - 15x^2 \\ f'(x) &= 125 - 30x \\ f''(x) &= -30 \end{aligned}$$

$$f'(x_c) = 125 - 30x_c = 0 \longrightarrow x_c = \frac{125}{30} = \frac{25}{6}$$

Since $f''(25/6) < 0$, we have a local maximum at $x_c = 25/6$. Since the function is a parabola that opens down, this maximum is the absolute maximum.

That solution, although a required part of the mathematical modeling process, is only a very small part of the final solution to any modeling problem.

Sensitivity Analysis

For each mathematical technique we study, we will have a different way of estimating the sensitivity of the solution to parameters in the solution.

Generally, what we do is leave some aspect of the equation we wish to solve unspecified, and then work out the solution which gives the optimal solution in terms of the unspecified quantity. We then investigate how our solution changes with respect to changes in the unspecified quantity.

Large changes in the solution due to small changes in the unspecified quantity means our solution is fragile, and if that is so we need to know it (this is bad)!

Small changes in the solution due to small changes in the unspecified quantity means our solution is robust, and that is what we hope is the case.

Example Is the value of x which maximizes the function

$$f(x) = (1500 - 100x)(1 + \frac{\alpha x}{20})$$

sensitive to the value of α when $\alpha \sim 3$?

Sensitivity is measured by the sensitivity function. This is a single variable optimization, so the sensitivity is defined simply as:

$$S(x_c, \alpha) = \frac{dx_c}{d\alpha} \cdot \frac{\alpha}{x_c}.$$

Notice that I am writing this as the sensitivity of x_c (the value of x which optimizes the function $f(x)$) to α.

We proceed by optimizing the function, which means taking the derivative, setting the derivative equal to zero, and solving for x_c:

$$\begin{aligned} f(x) &= (1500 - 100x)(1 + \frac{\alpha x}{20}) \\ &= 1500 + (75\alpha - 100)x - 5\alpha x^2 \\ f'(x) &= 75\alpha - 100 - 10\alpha x \\ f''(x) &= -10\alpha \end{aligned}$$

Let's stop and say a few words. If $\alpha > 0$, $f''(x_c) < 0$, and we know that we have a maximum at x_c. If $\alpha < 0$ we will get a minimum. If we are interested in a maximum, then we have a restriction on α. Typically these are where restrictions on different parameters in the solution become evident, so you are wise to look for them. OK, let's continue.

$$f'(x_c) = (75\alpha - 100) - 10\alpha x_c = 0 \longrightarrow x_c = \frac{75\alpha - 100}{10\alpha}$$

Now we have the value of x which maximizes the function $f(x)$ as a function of the parameter α. To get the sensitivity, we proceed to calculate the derivative, and get:

$$\frac{dx_c}{d\alpha} = \frac{10}{\alpha^2}, \qquad S(x_c, \alpha) = \frac{dx_c}{d\alpha} \cdot \frac{\alpha}{x_c} = \frac{10}{\alpha^2} \cdot \frac{\alpha}{\frac{75\alpha - 100}{10\alpha}} = \frac{4}{3\alpha - 4}.$$

Sensitivity is usually thought of in terms of percentage changes. This is because:

$$S(x_c, \alpha) = \frac{\left(\frac{dx_c}{x_c}\right)}{\left(\frac{d\alpha}{\alpha}\right)}.$$

At $\alpha = 3$, we have $S(x_c, 3) = \frac{4}{5}$, so a 5% increase in α results in a 4% increase in the optimal value of x, x_c.

I am always skeptical of these arguments, so let's look at it more closely:

$$x_c\big|_{\alpha=3} = \left.\frac{75\alpha - 100}{10\alpha}\right|_{\alpha=3} = \frac{25}{6}$$

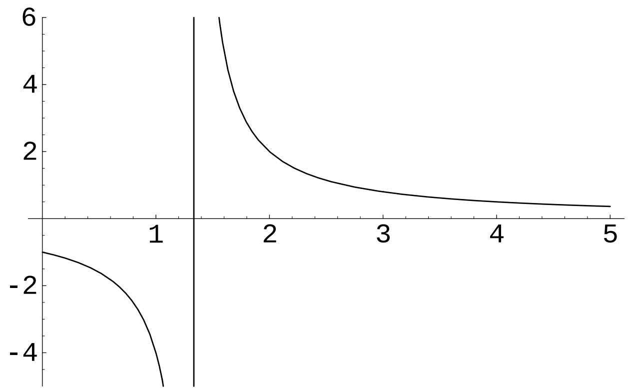
5% increase of $\alpha \longrightarrow \alpha = 3.15$.

$$x_c\big|_{\alpha=3.15} = \left.\frac{75\alpha - 100}{10\alpha}\right|_{\alpha=3.15} = \frac{545}{126}$$

$$\frac{25}{6} \sim 4.167; \qquad \frac{25}{6} \cdot 1.04 \sim 4.333, \frac{545}{126} \sim 4.325$$

This is a reasonable amount of change, so we would say that our method is robust when $\alpha = 3$.

Another, possibly better way of looking at this would be to plot the sensitivity of x_c to α vs α, and see if the function looks almost horizontal near $\alpha = 3$. This has the advantage of showing regions where the model is fragile with respect to α.



Notice that the sensitivity is very large for $\alpha \sim 4/3$. This is because the sensitivity changes from an increase:increase sensitivity to an increase:decrease sensitivity (the sensitivity changes sign).