Review

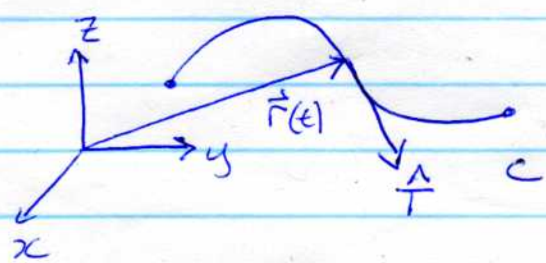
Things you must know:

Curves

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle \quad a \le t \le b$ parameterizes a curve.

$$\frac{ds}{dt} = |\vec{r}\,'(t)| \quad \Rightarrow \quad ds = |\vec{r}\,'(t)|\, dt$$

$$\frac{d\vec{r}}{ds} = \hat{T} \quad \Rightarrow \quad d\vec{r} = \hat{T}\, ds$$



Although $\hat{n}$ appears in formulas involving integrals over C, you don't need $\hat{n}$ to evaluate them!

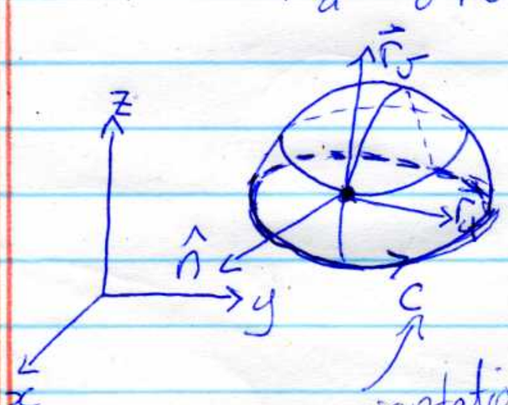
$$d\vec{r} = \frac{d\vec{r}}{dt}\, dt$$

Surfaces

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle \quad (u,v) \in D$ parameterizes a surface.

$$d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$



Closed surface has positive orientation if $\hat{n}$ point outward

orientation determined by right hand rule with $\hat{n}$.

del operators

$\vec{F} = \langle M, N, P \rangle$ vector field

f scalar

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$ del vector operator

$\text{grad} f = \nabla f = \langle f_x, f_y, f_z \rangle$

$\text{div} \vec{F} = \nabla \cdot \vec{F} = M_x + N_y + P_z$

$\text{curl} \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$

If $\vec{F} = \nabla f$, then $\vec{F}$ is conservative

F.T.L.I. $\displaystyle\int_A^B \nabla f \cdot d\vec{r} = f(B) - f(A)$

curve C: $\vec{r}(t)$, $t=a$ to $t=b$: $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

surface S: $\vec{r}(u,v)$ $(u,v) \in D$: Area of Surface $= \displaystyle\iint_S d\sigma = \iint_D |\vec{r}_u \times \vec{r}_v|\,du\,dv$

$$\iint_S G(x,y,z)\,d\sigma = \iint_D G(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,du\,dv.$$

Parameterizations

- Critical! Especially important are polar, spherical, cylindrical, and being able to create your own.

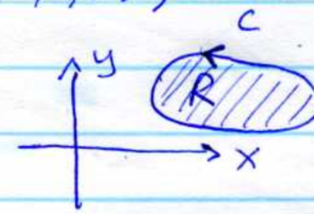
Circulation $= \oint_C \vec{F} \cdot d\vec{r}$

$\mathbb{R}^3$ (or $\mathbb{R}^2$) use $\vec{F} = \langle M, N, P \rangle$ and $C: \vec{r}(t)$ to evaluate as line integral:

$$\text{circ} = \oint_C M\,dx + N\,dy + P\,dz \qquad \text{(write everything in } t\text{)}$$

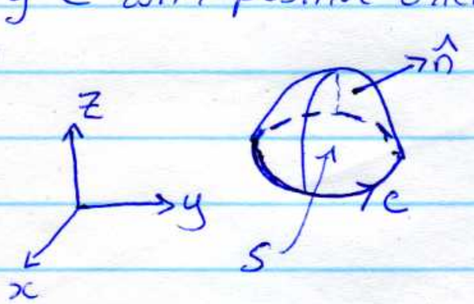
$\mathbb{R}^2$ use $\vec{F} = \langle M, N \rangle$ and R: region enclosed by curve C in xyplane (ccw) to evaluate using Green's theorem: (write $\vec{F} = \langle M, N, 0 \rangle$)

$$\text{circ} = \iint_R (\nabla \times \vec{F}) \cdot \hat{k}\,dx\,dy$$



$\mathbb{R}^3$ use $\vec{F} = \langle M, N, P \rangle$ and S: surface bounded by C with positive orientation to evaluate using Stokes' Theorem:

$$\text{circ} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\,d\sigma$$



Flux in $\mathbb{R}^2 = \oint_C \vec{F}\cdot\hat{n}\,ds$

Use $\vec{F} = \langle M,N\rangle$ and $C: \vec{r}(t)$ to evaluate as a line integral:

$$\text{flux} = \oint_C M\,dy - N\,dx$$ (write everything in t)

Use $\vec{F} = \langle M,N\rangle$ and R: region enclosed by curve C with ccw orientation in xy-plane to evaluate using Green's theorem:

$$\text{flux} = \iint_R \nabla\cdot\vec{F}\,dx\,dy$$ (double integral)

Flux in $\mathbb{R}^3 = \iint_S \vec{F}\cdot\hat{n}\,d\sigma$

Use $\vec{F} = \langle M,N,P\rangle$ and R: region over which u,v range to define S to evaluate as a surface integral:

$$\text{flux} = \iint_R \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$ (make sure $\vec{r}_u\times\vec{r}_v$ gives correct normal)

Use $\vec{F} = \langle M,N,P\rangle$ and E: region which is bounded by S with positive orientation to evaluate using Divergence theorem:

$$\text{flux} = \iiint_E \nabla\cdot\vec{F}\,dV$$ (triple integral)

Problems are from Stewart, Multivariable Calculus, 4th ed.

17.8.4 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma$

Ans: -4π

if $\vec{F} = \langle x + \tan^{-1}(yz),\, y^2 z,\, z\rangle$ and S is the part of the hemisphere $x = \sqrt{9-y^2-z^2}$ that lies inside the cylinder $y^2+z^2=4$ oriented in the direction of the positive x-axis.

17.8.6 Use Stokes' Theorem to evaluate $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma$

Ans: 0

if $\vec{F} = \langle xy,\, e^z,\, xy^2\rangle$ and S consists of the pyramid with vertices $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,0,1)$ and $(0,1,0)$ that lie to the right of the xz-plane, with positive orientation.

Hint: Use the fact that

$$\iint_{S_1} \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r} = \iint_{S_2} \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma$$

to come up with a simpler region S_2 that has the same boundary as S to integrate over.

17.9.22 Use the Divergence Theorem to evaluate

Ans: $\frac{4\pi}{3}$

$$\iint_S (2x+2y+z^2)\,d\sigma$$

Challenge Problem!

where S is the sphere $x^2+y^2+z^2=1$.

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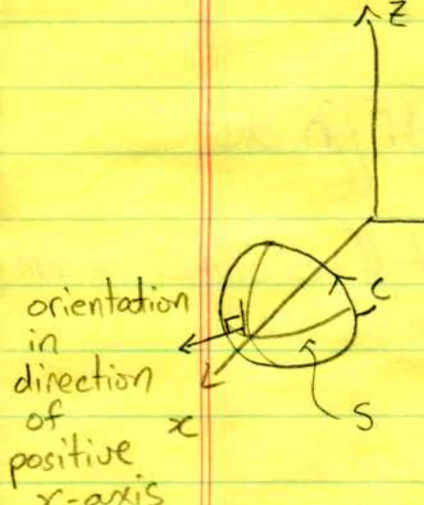
office hours Sunday 12^{00}-3^{00}.

$\boxed{17.8.4}$ $\vec{F}(x,y,z) = (x+\tan^{-1} yz)\,\hat{\imath} + y^2 z\,\hat{\jmath} + z\,\hat{k}$



Stokes' Theorem:

$$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,d\vec{s} = \oint_C \vec{F}\cdot d\vec{r}$$

where C is a space curve which is the boundary of S.

$x = \sqrt{9-y^2-z^2}$ with $y^2+z^2 \le 4$

C is the intersection of the cylinder and sphere:

$$x = \sqrt{9-4} \quad \& \quad y^2+z^2=4$$
$$= \sqrt{5}$$

Parameterization of C: $x = \sqrt{5}$, $y = 2\cos t$, $z = 2\sin t$, $0 \le t \le 2\pi$

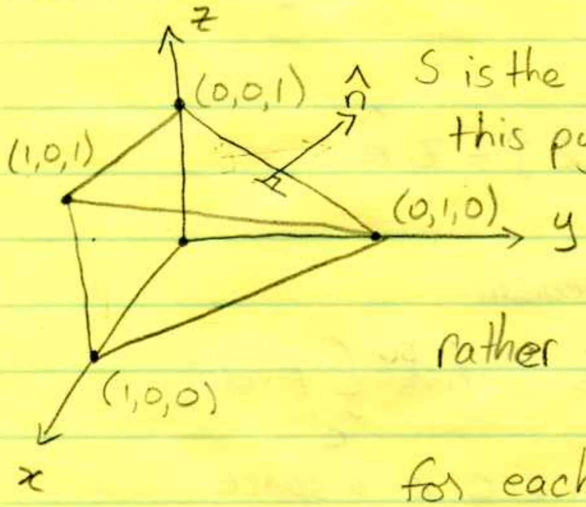
$\vec{r}(t) = \langle x, y, z\rangle$
$= \langle \sqrt{5}, 2\cos t, 2\sin t\rangle$

$\vec{F}(\vec{r}(t)) = \langle \sqrt{5} + \tan^{-1}(4\cos t\sin t),\ 8\cos^2 t\sin t,\ 2\sin t\rangle$

$\vec{r}\,'(t) = \langle 0, -2\sin t, 2\cos t\rangle$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\vec{r}\,'(t)\,dt$$
$$= \int_0^{2\pi} (-16\cos^2 t\sin^2 t + 4\cos t\sin t)\,dt$$
$$= -4\pi.$$

17.8.6



S is the 4 sides of this pyramid.

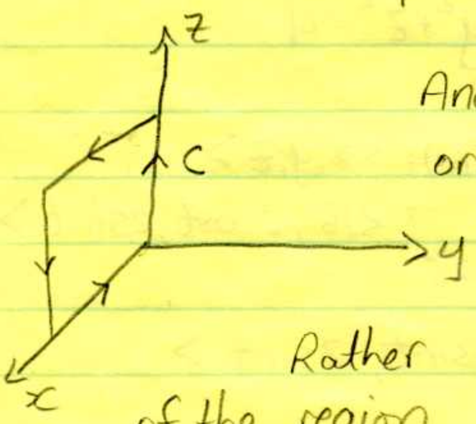
orientation: direction of positive y-axis.

rather than working out $\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma$

for each of the 4 sides, we shall use Stokes' Theorem.

$$\iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma = \oint_c \vec{F}\cdot d\vec{r}$$

where c is a boundary curve of S. This boundary curve is the square region:

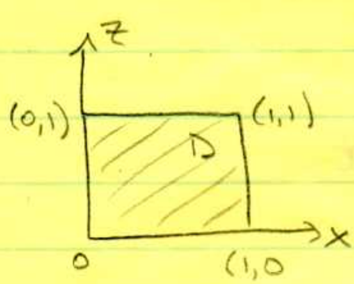


And the orientation is required by the orientation of the pyramid.

Rather than working out $\oint_c \vec{F}\cdot d\vec{r}$ on the 4 sides of the region, we can use Green's Theorem

$$\oint_c \vec{F}\cdot d\vec{r} = \iint_D \operatorname{curl}\vec{F}\cdot\hat{j}\,dA$$

(or, reuse Stokes' Theorem)



$$= \iint_D \langle e^z, 0, -x\rangle\cdot\langle 0,1,0\rangle\,dA$$

$$= \iint_D 0\,dA$$

$$= 0$$

$D: \{(x,y,z)\mid 0\le x\le 1,\ 0\le z\le 1,\ y=0\}$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & e^z & xy^2 \end{vmatrix} = \langle 2xy - e^z,\ -y^2+0,\ 0-x\rangle$$

$$= \langle e^z, 0, -x\rangle \text{ on } D\ [y=0]$$

17.9.22 $\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_E \operatorname{div}\vec{F}\,dV$ E is volume with surface boundary S.

$\iint_S (2x+2y+z^2)\,d\sigma$ where S is sphere $x^2+y^2+z^2=1$

We need to find $\vec{F}$ such that

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S (2x+2y+z^2)\,d\sigma$$

$\vec{F}\cdot\hat{n} = 2x+2y+z^2$

S: $x^2+y^2+z^2=1$ $\quad x = \sin\phi\cos\theta \quad 0\le\theta\le 2\pi$

$y = \sin\phi\sin\theta \quad 0\le\phi\le\pi.$

$z = \cos\phi$

$\vec{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

$\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$

$$\vec{r}_\theta\times\vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix} = \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\sin\phi\cos\phi\rangle$$

$$|\vec{r}_\theta\times\vec{r}_\phi| = \sqrt{\sin^4\phi\cos^2\theta + \sin^4\phi\sin^2\theta + \sin^2\phi\cos^2\phi}$$

$$= \sin\phi$$

$$\hat{n} = \frac{\vec{r}_\theta\times\vec{r}_\phi}{\sin\phi} = \langle -\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi\rangle$$

inward normal! Convention says closed surface has outward normal

$\vec{F}\cdot\hat{n} = 2\sin\phi\cos\theta + 2\sin\phi\sin\theta + \cos^2\phi$

$= \vec{F}\cdot\langle +\sin\phi\cos\theta, +\sin\phi\sin\theta, +\cos\phi\rangle$

$\Rightarrow \vec{F} = \langle +2, +2, +\cos\phi\rangle$

$= \langle +2, +2, +z\rangle$

Aside: We need $\nabla\cdot\vec{F}$, and we only know it in cartesian coordinates. You can work out expressions for ∇ and ∇^2 in other coordinate systems.

$\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(z)$

$= 1$

So $\iint_S (2x+2y+z^2)\,d\sigma = \iint_S \vec{F}\cdot\hat{n}\,d\sigma$

$= \iiint_E \nabla\cdot\vec{F}\,dV$

$= \iiint_E dV$

$=$ Volume of unit sphere

$= \frac{4\pi}{3}$