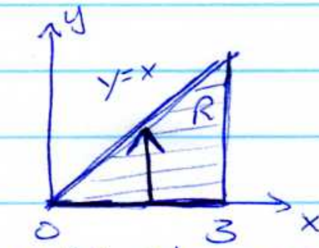
16.4.7 Find counterclockwise circulation & outward flux for $\vec{F} = \langle y^2 - x^2,\ x^2 + y^2 \rangle$

C: triangle bounded by $y = 0$, $x = 3$, $y = x$.



$R = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le x\}$

$$\text{Flux} = \oint_C \vec{F} \cdot \hat{n}\, ds$$

$M = y^2 - x^2 \qquad N = x^2 + y^2$

$\frac{\partial M}{\partial x} = -2x \qquad \frac{\partial N}{\partial y} = 2y$

$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

$$= \int_0^3 \int_0^x (-2x + 2y)\, dy\, dx$$

$$= \int_0^3 (-2yx + y^2)\Big|_{y=0}^{y=x} dx = \int_0^3 (-2x^2 + x^2)\, dx = -x^2 \Big|_0^3 = -9$$

$$\text{Circulation} = \oint_C \vec{F} \cdot \vec{T}\, ds$$

$M = y^2 - x^2 \qquad N = x^2 + y^2$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial x} = 2x$

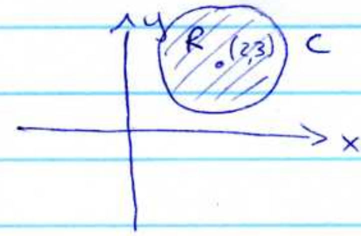
$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\, dy$$

$$= \int_0^3 \int_0^x (2x - 2y)\, dy\, dx$$

$$= \int_0^3 (2xy - y^2)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^3 (2x^2 - x^2)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\Big|_0^3 = 9.$$

6.4.19 $\oint_C (6y+x)\,dx + (y+2x)\,dy \qquad C: (x-2)^2+(y-3)^2=4$



everything is defined in C
(smooth, derivatives fine).

Use $\oint_C \vec{F}\cdot\vec{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$M = 6y+x \qquad N = y+2x$

$\frac{\partial M}{\partial y} = 6 \qquad \frac{\partial N}{\partial x} = 2$

$$\oint_C (6y+x)\,dx + (y+2x)\,dy = -4\iint_R dx\,dy$$

$$= -4\ (\text{Area of circle radius } 2)$$

$$= -4\left(\pi (2)^2\right)$$

$$= -16\pi.$$

16.4.29

R is region in xy plane bounded by simple closed curve C.

Show Area of $R = \oint_C x\,dy = -\oint_C y\,dx$

Use Green's Theorems

$$\text{Area} = \iint_R dx\,dy$$

$$= \iint_R \frac{\partial}{\partial x}(x)\,dx\,dy \qquad \text{ie) } M = x,\ N = 0$$

$$= \iint_R \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(0)\ dx\,dy \qquad \text{use } \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

$$= \oint_C x\,dy$$

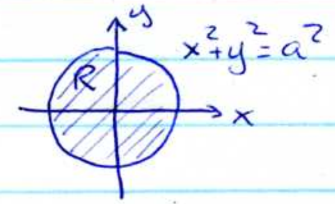
We can also write $\text{Area} = \iint_R dx\,dy$

$$= +\iint_R \frac{\partial}{\partial y}(+y)\,dx\,dy \qquad \text{ie } M = 0,\ N = y$$

$$= \iint_R \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(y)\ dx\,dy$$

$$= -\oint_C y\,dx$$

16.4.35 Let $f(x,y) = \ln(x^2+y^2)$ $\quad C: x^2+y^2=a^2$



a) Evaluate $\oint_C \nabla f \cdot \hat{n}\, ds$

$$\vec{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \left\langle \frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2} \right\rangle$$

$$\Rightarrow M = \frac{2x}{x^2+y^2}, \quad N = \frac{2y}{x^2+y^2}$$

Use Green's Theorem, $\oint_C \vec{F} \cdot \hat{n}\, ds = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\,dy$

$$\frac{\partial M}{\partial x} = \frac{(x^2+y^2)(2) - 2x(2x)}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial y} = \frac{(x^2+y^2)(2) - 2y(2y)}{(x^2+y^2)^2} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$$

$$\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = 0 \qquad \text{So } \oint_C \nabla f \cdot \hat{n}\, ds = 0$$

There's a problem with what we did above.
M, N are not continuous at $(0,0)$, so Green's Theorem doesn't hold in any region that contains $(0,0)$. The region $C: x^2+y^2=a^2$ contains $(0,0)$, so we can't use Green's Theorem!

What we've been able to show is $\oint_K \nabla f \cdot \hat{n}\, ds = 0$

if K does not contain the origin.

16.4.35
continued

lets evaluate $\oint_C \vec{F} \cdot \hat{n}\, ds$ using

$$\oint_C \vec{F} \cdot \hat{n}\, ds = \oint_C M\,dy - N\,dx$$

$C: \vec{r}(t) = \langle a\cos t, a\sin t \rangle$

$\Rightarrow x = a\cos t \quad 0 \le t \le 2\pi$

$y = a\sin t$

$$M = \frac{2x}{x^2+y^2} \qquad N = \frac{2y}{x^2+y^2}$$

$$= \frac{2a\cos t}{a^2} \qquad = \frac{2a\sin t}{a^2}$$

$$M = \frac{2}{a}\cos t \qquad N = \frac{2}{a}\sin t$$

$$dy = a\cos t\, dt \qquad dx = -a\sin t\, dt$$

$$= \int_0^{2\pi} (2\cos^2 t + 2\sin^2 t)\, dt$$

$$= 2\int_0^{2\pi} dt$$

$$= 4\pi$$