16.3.28 Find potential function for $\vec{F} = \langle e^x \ln y, \frac{e^x}{y} + \sin z, y\cos z \rangle$

$y > 0$.

Looking for f such that $\vec{F} = \nabla f$

$$\Rightarrow \frac{\partial f}{\partial x} = e^x \ln y \quad \Rightarrow \quad f = e^x \ln y + g(y,z)$$

$$\frac{\partial f}{\partial y} = \frac{e^x}{y} + \sin z \quad \Rightarrow \quad f = e^x \ln y + y\sin z + h(x,z)$$

$$\frac{\partial f}{\partial z} = y\cos z \quad \Rightarrow \quad f = \qquad\qquad + y\sin z + \omega(x,y)$$

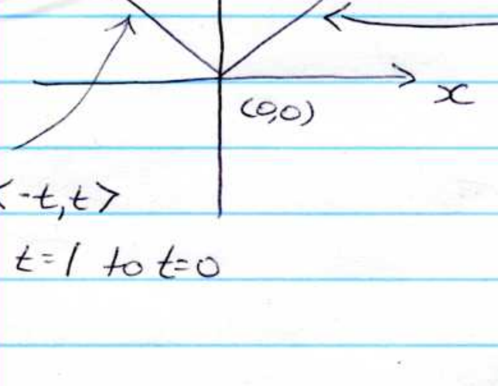
Comparing, $f = e^x \ln y + y\sin z + C$.

16.3.31 Let $\vec{F} = \nabla(x^3 y^2)$ and let C be path in xy-plane from $(-1,1)$ to $(1,1)$ that consists of line segment from $(-1,1)$ to $(0,0)$ followed by line segment from $(0,0)$ to $(1,1)$.

Evaluate $\int_C \vec{F} \cdot d\vec{r}$ in two ways:

a) parameterize the line segments.

b) Use the potential function.



$C_2 : \vec{r}(t) = \langle t, t \rangle$

$t = 0$ to $t = 1$.

$C_1 : \vec{r}(t) = \langle -t, t \rangle$

$t = 1$ to $t = 0$

Note: other parameterizations are possible.

a) $\displaystyle\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt + \int_{C_2} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt$

C_1: $\vec{r}(t) = \langle -t, t\rangle$

$\vec{r}\,'(t) = \langle -1, 1\rangle$

$\vec{F} = \langle \frac{\partial}{\partial x}(x^3y^2), \frac{\partial}{\partial y}(x^3y^2)\rangle$

$= \langle 3x^2y^2, 2x^3y\rangle$

$= \langle 3(-t)^2(t^2), 2(-t)^3t\rangle$

$= \langle +3t^4, -2t^4\rangle$

$\vec{F}\cdot\frac{d\vec{r}}{dt} = -3t^4 - 2t^4 = -5t^4$

C_2: $\vec{r}(t) = \langle t, t\rangle$

$\vec{r}\,'(t) = \langle 1, 1\rangle$

$\vec{F} = \langle 3x^2y^2, 2x^3y\rangle$

$= \langle 3t^4, 2t^4\rangle$

$\vec{F}\cdot\frac{d\vec{r}}{dt} = 3t^4 + 2t^4 = 5t^4$

$$= \int_1^0 (-5t^4)\,dt + \int_0^1 5t^4\,dt = -t^5\Big|_1^0 + t^5\Big|_0^1 = +1 + 1 = 2.$$

b) We already know $\nabla f = \vec{F}$ for $f = x^3y^2$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r}$$

$$= f(B) - f(A)$$

$$= x^3y^2\Big|_{(-1,1)}^{(1,1)}$$

$$= (+1)^3(1)^2 - (-1)^3(1^2)$$

$$= 2.$$

16.3.38 a) Find a potential function for the gravitational field

$$\vec{F} = -GmM \frac{\langle x,y,z \rangle}{(x^2+y^2+z^2)^{3/2}} \qquad G, m, M \text{ constants.}$$

b) Let P_1 & P_2 be points at distances s_1 & s_2 from origin. Show work done by gravitational field in moving a particle from P_1 to P_2 is $GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$

a) Looking for f such that $\vec{F} = \nabla f$

$$\Rightarrow \frac{\partial f}{\partial x} = -\frac{GmMx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + g(y,z)$$

$$\frac{\partial f}{\partial y} = -\frac{GmMy}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + h(x,z)$$

$$\frac{\partial f}{\partial z} = -\frac{GmMz}{(x^2+y^2+z^2)^{3/2}} \Rightarrow f = \frac{GmM}{\sqrt{x^2+y^2+z^2}} + \omega(x,y)$$

Comparing, we see $f(x,y,z) = \frac{GmM}{\sqrt{x^2+y^2+z^2}}$ is potential function.

b) $$\text{Work} = \int_C \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} \vec{\nabla} f \cdot d\vec{r}$$

$$= f(P_2) - f(P_1)$$

$$= GmM\left(\frac{1}{s_2} - \frac{1}{s_1}\right)$$

since $\sqrt{x^2+y^2+z^2}$ is distance.