16.1.13 Find the integral of $f(x,y,z) = x+y+z$ over straight line segment from $(1,2,3)$ to $(0,-1,1)$.

Vector equation of line: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

$\vec{v} = \langle 0-1, -1-2, 1-3 \rangle$
$= \langle -1, -3, -2 \rangle$

$\vec{r}_0 = \langle 1, 2, 3 \rangle$

$\Rightarrow \vec{r}(t) = \langle 1,2,3 \rangle + t\langle -1,-3,-2 \rangle$
$= \langle 1-t, 2-3t, 3-2t \rangle$ for $t=0$ to $t=1$

$\vec{r}\,'(t) = \langle -1,-3,-2 \rangle$

$|\vec{r}\,'(t)| = \sqrt{14}$

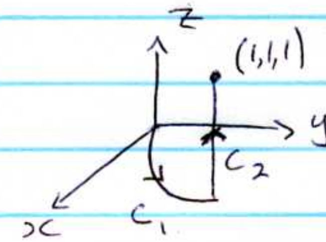
$ds = |\vec{r}\,'(t)|\,dt = \sqrt{14}\,dt$

$f(x,y,z) = x+y+z$
$= (1-t) + (2-3t) + (3-2t)$
$= 6-6t$

$$\int_C f(x,y,z)\,ds = \int_0^1 (6-6t)\sqrt{14}\,dt$$
$$= \sqrt{14}\left(6t-3t^2\right)\Big|_0^1$$
$$= \sqrt{14}\,(6-3)$$
$$= 3\sqrt{14}$$

16.1.15 Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over path from $(0,0,0)$ to $(1,1,1)$ given by $C_1 + C_2$:

$$C_1: \vec{r}(t) = t\hat{i} + t^2\hat{j}, \quad 0 \le t \le 1$$
$$C_2: \vec{r}(t) = \hat{i} + \hat{j} + t\hat{k} \quad 0 \le t \le 1.$$



Use the line integral formula:

$$\int_C f(x,y,z)\,ds = \int_a^b f(g(t),h(t),k(t))\,|\vec{v}(t)|\,dt$$

since $\vec{r}(t) = \langle g(t), h(t), k(t) \rangle$

$$ds = |\vec{v}(t)|\,dt = |\vec{r}\,'(t)|\,dt$$

For C_1: $\vec{r}(t) = \langle t, t^2, 0 \rangle$

$\vec{r}\,'(t) = \langle 1, 2t, 0 \rangle \quad |\vec{r}\,'(t)| = \sqrt{1+4t^2}$

$$f(x,y,z) = x + \sqrt{y} - z^2 = t + t - 0^2 = 2t$$

$$\int_{C_1} f(x,y,z)\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt$$

let $u = 1+4t^2$
$du = 8t\,dt$
when $t=1, u=5$
$t=0, u=1$

$$= \frac{1}{4}\int_1^5 u^{1/2}\,du = \frac{1}{4}\left.\frac{u^{3/2}}{3/2}\right|_1^5 = \frac{1}{6}\left(5\sqrt{5}-1\right)$$

For C_2: $\vec{r}(t) = \langle 1, 1, t \rangle$

$\vec{r}\,'(t) = \langle 0, 0, 1 \rangle$

$|\vec{r}\,'(t)| = 1$

$$f(x,y,z) = x + \sqrt{y} - z^2 = 1 + 1 - t^2 = 2 - t^2$$

$$\int_{C_2} f(x,y,z)\,ds = \int_0^1 (2-t^2)\,dt = 2t - \left.\frac{t^3}{3}\right|_0^1 = 2 - \frac{1}{3} = \frac{5}{3}$$

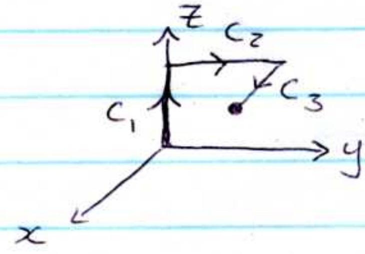
$$\Rightarrow \int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds = \frac{5\sqrt{5}}{6} - \frac{1}{6} + \frac{5}{3}$$
$$= \frac{5\sqrt{5}}{6} + \frac{3}{2}$$

16.1.16 Integrate $f(x,y,z) = x+\sqrt{y}-z^2$ over path from $(0,0,0)$ to $(1,1,1)$ given by

$C_1 : \vec{r}(t) = \langle 0,0,t\rangle \quad 0\le t\le 1$

$C_2 : \vec{r}(t) = \langle 0,t,1\rangle \quad 0\le t\le 1$

$C_3 : \vec{r}(t) = \langle t,1,1\rangle \quad 0\le t\le 1$



For C_1: $\vec{r}(t) = \langle 0,0,t\rangle$

$\vec{r}\,'(t) = \langle 0,0,1\rangle$

$|\vec{r}\,'(t)| = 1$

$ds = |\vec{r}\,'(t)|\,dt = dt$

$f(x,y,z) = x+\sqrt{y}-z^2$
$= 0+0-t^2$
$= -t^2$

$\int_{C_1} f(x,y,z)\,ds = \int_0^1 (-t^2)\,dt$
$= -\frac{t^3}{3}\Big|_0^1 = -\frac{1}{3}$

For C_2: $\vec{r}(t) = \langle 0,t,1\rangle$

$\vec{r}\,'(t) = \langle 0,1,0\rangle$

$|\vec{r}\,'(t)| = 1$

$ds = |\vec{r}\,'(t)|\,dt = dt$

$f(x,y,z) = x+\sqrt{y}-z^2$
$= 0+\sqrt{t}-1$
$= \sqrt{t}-1$

$\int_{C_2} f(x,y,z)\,ds = \int_0^1 (\sqrt{t}-1)\,dt$
$= -\frac{1}{3}$

For C_3: $\vec{r}(t) = \langle t,1,1\rangle$

$\vec{r}\,'(t) = \langle 1,0,0\rangle$

$|\vec{r}\,'(t)| = 1$

$ds = |\vec{r}\,'(t)|\,dt = dt$

$f(x,y,z) = x+\sqrt{y}-z^2$
$= t+1-1$
$= t$

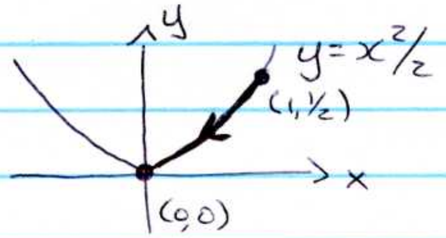
$\int_{C_3} f(x,y,z)\,ds = \int_0^1 t\,dt$
$= \frac{1}{2}$

Therefore, $\int_C = \int_{C_1} + \int_{C_2} + \int_{C_3}$

$\Rightarrow \int_C f(x,y,z)\,ds = -\frac{1}{3} - \frac{1}{3} + \frac{1}{2}$
$= -\frac{1}{6}.$

16.1.20 Integrate $f(x,y) = \dfrac{x+y^2}{\sqrt{1+x^2}}$ over $C: y = \dfrac{x^2}{2}$ from $(1,\frac{1}{2})$ to $(0,0)$.

C:



Parameterize C:

$$x = t$$
$$y = t^2/2$$

t goes from 1 to 0.

$$\Rightarrow \vec{r}(t) = \langle t, t^2/2 \rangle$$

$$\vec{r}\,'(t) = \langle 1, t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1+t^2}$$

$$ds = |\vec{r}\,'(t)|\,dt = \sqrt{1+t^2}\,dt$$

$$f(x,y) = \frac{x+y^2}{\sqrt{1+x^2}} = \frac{t + t^4/4}{\sqrt{1+t^2}}$$

$$\int_C f(x,y)\,ds = \int_1^0 \frac{t + t^4/4}{\cancel{\sqrt{1+t^2}}} \cancel{\sqrt{1+t^2}}\,dt$$

$$= \int_1^0 \left(t + \frac{t^4}{4}\right) dt$$

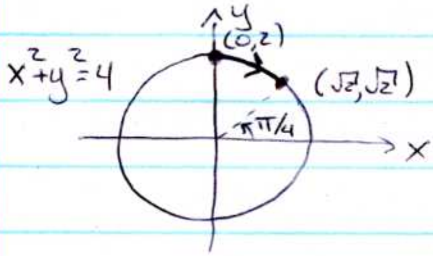
$$= \frac{t^2}{2} + \frac{t^5}{20} \Big|_1^0$$

$$= -\frac{1}{2} - \frac{1}{20}$$

$$= \frac{-11}{20}$$

16.1.22 Integrate $f(x,y) = x^2 - y$ over $C: x^2+y^2=4$ in the first quadrant from $(0,2)$ to $(\sqrt{2}, \sqrt{2})$.

Sketch & parameterize C:



$x = 2\cos t$ $\quad$ t goes from

$y = 2\sin t$ $\quad$ $\pi/2$ to $\pi/4$

$\Rightarrow \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$|\vec{r}\,'(t)| = 2$

$ds = |\vec{r}\,'(t)|\,dt = 2\,dt$

$f(x,y) = x^2 - y$

$= 4\cos^2 t - 2\sin t$

$$\int_C f(x,y)\,ds = \int_{\pi/2}^{\pi/4} (4\cos^2 t - 2\sin t)\, 2\,dt$$

$$= 8\int_{\pi/2}^{\pi/4} \cos^2 t\,dt - 4\int_{\pi/2}^{\pi/4} \sin t\,dt$$

$$= 8\int_{\pi/2}^{\pi/4} \tfrac{1}{2}(1+\cos 2t)\,dt + 4\cos t\,\Big|_{\pi/2}^{\pi/4}$$

$$= 4t\,\Big|_{\pi/2}^{\pi/4} + \frac{4\sin 2t}{2}\,\Big|_{\pi/2}^{\pi/4} + 4\left(\frac{\sqrt{2}}{2} - 0\right)$$

$$= -\pi + 2(1-0) + 2\sqrt{2}$$

$$= -\pi + 2 + 2\sqrt{2}$$