15.7.1

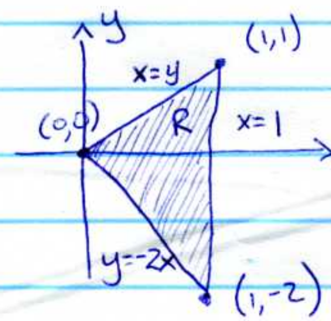
a) $u = x - y$ Since this is linear, we can solve

$v = 2x + y$ using Cramer's Rule:

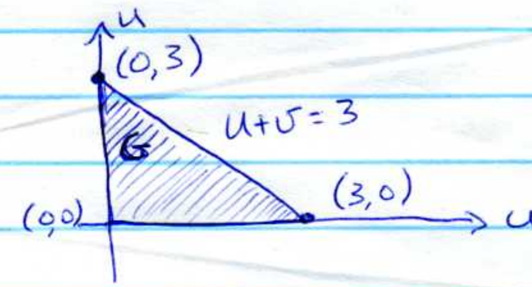
$$x = \frac{\begin{vmatrix} u & -1 \\ v & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}} = \frac{u+v}{3} \qquad y = \frac{\begin{vmatrix} 1 & u \\ 2 & v \end{vmatrix}}{3} = \frac{v-2u}{3}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/3 & 1/3 \\ -2/3 & 1/3 \end{vmatrix} = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$$

b)



$\Rightarrow$ create table



$\Uparrow$ create sketch in uv-plane

xy-equations for boundary of R	Corresponding uv-equations for boundary of G	Simplified uv-equations
$x = 1$	$\frac{u+v}{3} = 1$	$u+v=3$
$x = y$	$\frac{u+v}{3} = \frac{v-2u}{3}$	$u=0$
$y = -2x$	$-2\left(\frac{u+v}{3}\right) = \left(\frac{v-2u}{3}\right)$	$v=0$

15.7.3

a) $u = 3x + 2y$

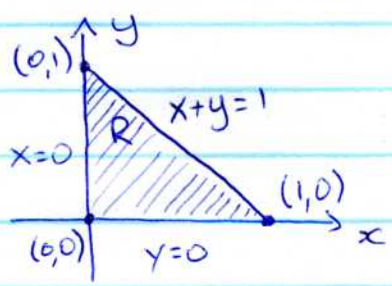
$v = x + 4y$

Since this is linear, we can use Cramer's Rule.

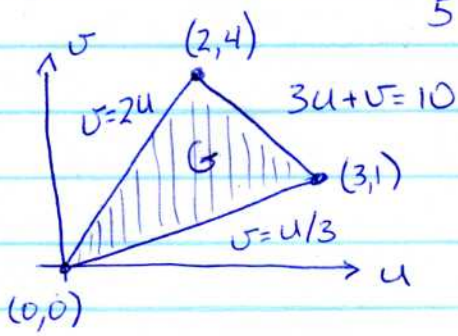
$$x = \frac{\begin{vmatrix} u & 2 \\ v & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}} = \frac{4u - 2v}{10} = \frac{2u - v}{5} \qquad y = \frac{\begin{vmatrix} 3 & u \\ 1 & v \end{vmatrix}}{10} = \frac{3v - u}{10}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2/5 & -1/5 \\ -1/10 & 3/10 \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}$$

b)

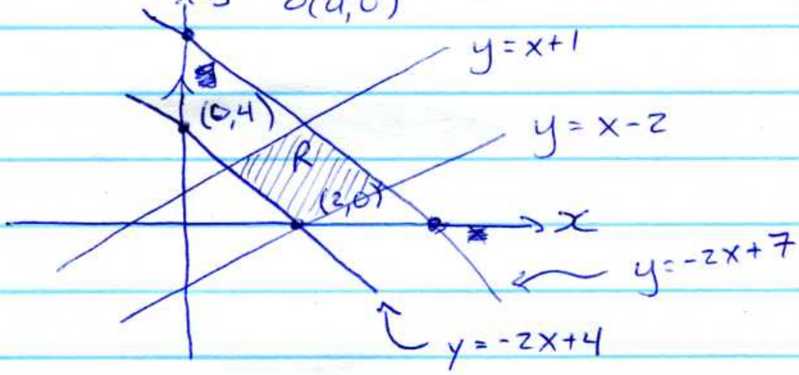


xy-equations for boundary of R	Corresponding uv-equations for boundary of G	simplified uv-equations
$x = 0$	$\frac{2u - v}{5} = 0$	$v = 2u$
$y = 0$	$\frac{3v - u}{10} = 0$	$u = 3v$
$x + y = 1$	$\frac{2u - v}{5} + \frac{3v - u}{10} = 1$	$3u + v = 10$

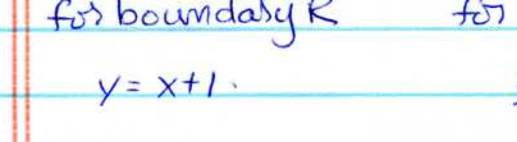


15.7.6) From 15.7.1, $\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{3}$ and $x = \frac{u+v}{3}$ $y = \frac{v-2u}{3}$

R:



xy-equation for boundary R	Corresponding uv-equations for Boundary G	simplified uv-equations
$y = x+1$	$\frac{v-2u}{3} = \frac{u+v}{3} + 1$	$u = -1$
$y = x-2$	$\frac{v-2u}{3} = \frac{u+v}{3} - 2$	$u = 2$
$y = -2x+4$	$\frac{v-2u}{3} = -2\left(\frac{u+v}{3}\right) + 4$	$v = 4$
$y = -2x+7$	$\frac{v-2u}{3} = -2\left(\frac{u+v}{3}\right) + 7$	$v = 7$



Integrand: $2x^2 - xy - y^2 = 2\left(\frac{u+v}{3}\right)^2 - \left(\frac{u+v}{3}\right)\left(\frac{v-2u}{3}\right) - \left(\frac{v-2u}{3}\right)^2$

$= uv$

$$\iint_R (2x^2 - xy - y^2)\,dx\,dy$$

$$= \iint_G uv \frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$

$$= \frac{1}{3}\int_4^7\int_{-1}^2 uv\,du\,dv$$

$$= \frac{1}{3}\int_4^7 \left.\frac{u^2 v}{2}\right|_{u=-1}^{u=2} dv$$

$$= \frac{1}{6}\int_4^7 (4v - v)\,dv$$

$$= \frac{3}{6}\left(\frac{v^2}{2}\right)_4^7 = \frac{1}{2}\left(\frac{49}{2} - \frac{16}{2}\right)$$

$$= 33/4.$$

15.7.7

From 15.7.3, $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{10}$ and $x = \dfrac{2u-v}{5}$ $y = \dfrac{3v-u}{10}$

Note: I used MMA to simplify. File online.

xy-equation for boundary in xy-plane	uv-equation for boundary in uv-plane.
$y = \frac{-3}{2}x+1$	$u=2$
$y = -\frac{3}{2}x+3$	$u=6$
$y = \frac{-1}{4}x$	$v=0$
$y = -\frac{1}{4}x+1$	$v=4$

Simplify integrand: $3x^2+14xy+8y^2 = uv$.

$$\iint_R (3x^2+14xy+8y^2)\,dx\,dy = \frac{1}{10}\iint_G uv\,du\,dv$$

$$= \frac{1}{10}\int_0^4\int_2^6 uv\,du\,dv$$

$$= \frac{64}{5}.$$

15.7.10

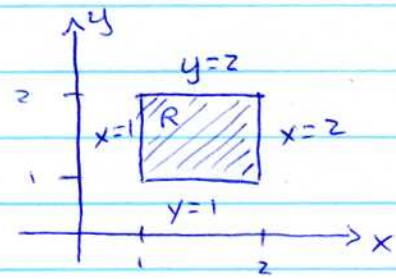
$x = u$ nonlinear → can't use Cramer's Rule to solve.

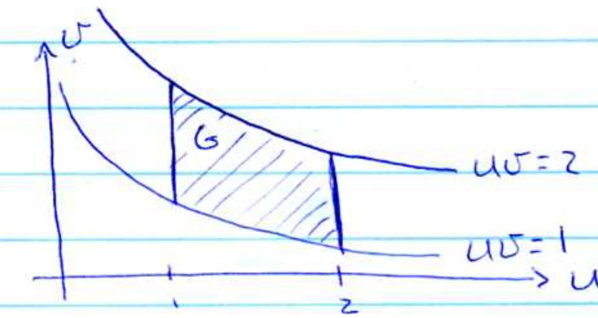
$y = uv$ It's pretty easy, however.

$$u = x$$

$$v = \frac{y}{u} = \frac{y}{x}$$

$$\int_1^2 \int_1^2 \frac{y}{x}\, dy\, dx \Rightarrow R = \{(x,y) \mid 1 \le y \le 2,\ 1 \le x \le 2\}.$$





xy-equations for boundary of R	corresponding uv-equations for boundary of G.
$x = 1$	$u = 1$
$x = 2$	$u = 2$
$y = 1$	$uv = 1 \Rightarrow v = 1/u$
$y = 2$	$uv = 2 \Rightarrow v = 2/u$

Jacobian: $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ v & u \end{vmatrix} = u$

Integrand $\dfrac{y}{x} = v$

$$I = \int_1^2 \int_1^2 \frac{y}{x}\,dy\,dx = \int_1^2 \left.\frac{y^2}{2x}\right|_{y=1}^{y=2} dx$$

$$= \int_1^2 \frac{4}{2x} - \frac{1}{x}\,dx$$

$$= \int_1^2 \frac{3}{2x}\,dx$$

$$= \frac{3}{2} \ln x \Big|_1^2$$

$$= \frac{3}{2}\left(\ln 2 - \cancelto{0}{\ln 1}\right) = \frac{3 \ln 2}{2}$$

$$I = \int_1^2 \int_{1/u}^{2/u} uv\,dv\,du$$

$$= \int_1^2 \left.\frac{uv^2}{2}\right|_{v=1/u}^{v=2/u} du$$

$$= \int_1^2 \frac{u}{2}\left(\frac{4}{u^2} - \frac{1}{u^2}\right) du$$

$$= \frac{3}{2}\int_1^2 \frac{du}{u}$$

$$= \frac{3}{2} \ln u \Big|_1^2 = \frac{3}{2}\ln 2.$$