14.5.18 $f(x,y) = x^2y + e^{xy}\sin y$, $P_0(1,0)$

Increase most rapidly in direction of ∇f.

$$\frac{\partial f}{\partial x} = 2xy + ye^{xy}\sin y \qquad \left.\frac{\partial f}{\partial x}\right|_{P_0} = 0$$

$$\frac{\partial f}{\partial y} = x^2 + xe^{xy}\sin y + e^{xy}\cos y \qquad \left.\frac{\partial f}{\partial y}\right|_{P_0} = 2$$

$$\nabla f\Big|_{P_0} = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle\Big|_{P_0} = \langle 0, 2 \rangle$$

Increase most rapidly at $P_0(1,0)$ in direction $\langle 0,2\rangle$. $\vec{u} = \langle 0,1\rangle$

Decrease most rapidy at $P_0(1,0)$ in direction $\langle 0,-2\rangle$. $\vec{u} = \langle 0,-1\rangle$

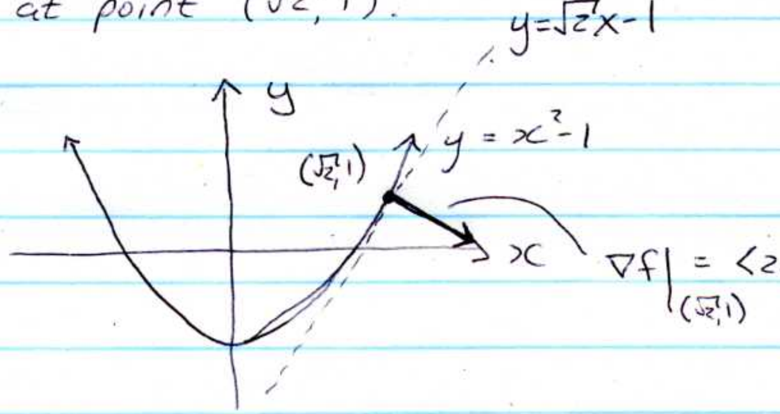
Derivative in these directions:

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \left(\nabla f\right)_{P_0} \cdot \langle 0,1\rangle = \langle 0,2\rangle \cdot \langle 0,1\rangle = 2$$

$$\left(\frac{df}{ds}\right)_{\vec{u},P_0} = \left(\nabla f\right)_{P_0} \cdot \langle 0,-1\rangle = \langle 0,2\rangle \cdot \langle 0,-1\rangle = -2.$$

14.5.24 $x^2 - y = 1$ at point $(\sqrt{2}, 1)$.

sketch



$\nabla f\big|_{(\sqrt{2},1)} = \langle 2\sqrt{2}, -1 \rangle$

Derivative:
$$2x - \frac{dy}{dx} = 0$$

Equation of tangent line: $y - y_0 = m(x - x_0)$

$x_0 = \sqrt{2}$, $y_0 = 1$ $\quad m = \frac{dy}{dx}\Big|_{\substack{x=\sqrt{2} \\ y=1}} = 2(\sqrt{2})$. $\quad \Rightarrow \quad y - 1 = \sqrt{2}(x - \sqrt{2})$

or $y = \sqrt{2}x - 1$

$$\begin{aligned}
\nabla f &= \nabla(x^2 - y) \\
&= \left\langle \frac{\partial}{\partial x}(x^2-y), \frac{\partial}{\partial y}(x^2-y) \right\rangle \\
&= \langle 2x, -1 \rangle
\end{aligned}$$

$$\nabla f\big|_{(\sqrt{2},1)} = \langle 2\sqrt{2}, -1 \rangle$$

14.5.32 $\left(D_{\vec{u}} f\right)_{P_0} = 2\sqrt{3}$ where $\vec{u} = \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle$

$$\Rightarrow \quad \left(\nabla f\right)_P \cdot \vec{u} = 2\sqrt{3}$$

$$\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle\Big|_P \cdot \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = 2\sqrt{3}$$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 6 \qquad \text{nothing } ①$$

$$\left(\nabla f\right)_P \cdot \vec{u} = |\nabla f| \cancel{\cos\theta}^{1} = 2\sqrt{3}$$

$$\text{so } |\nabla f| = 2\sqrt{3} = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2} \qquad \text{nothing } ②$$

$$\Rightarrow \quad 12 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$

$$\left(\nabla f\right)_P = k\langle 1,1,-1\rangle$$

$$\frac{\partial f}{\partial x} = k \qquad \frac{\partial f}{\partial y} = k \qquad \frac{\partial f}{\partial z} = -k$$

Put into ①: $k + k + k = 6 \Rightarrow k = 2 \Rightarrow \left(\nabla f\right)_P = \langle 2,2,-2\rangle$

Put into ②: $\sqrt{k^2 + k^2 + k^2} = 2\sqrt{3} \Rightarrow k = 2$

(either ① or ② gets the result).

$$\left(\nabla f\right)_P \cdot \vec{u}_1 = \langle 2,2,-2\rangle \cdot \frac{1}{\sqrt{2}}\langle 1,1,0\rangle$$
$$= \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

14.5.33

$$(\nabla f)_{P_0} \cdot \hat{u}$$

let

$\hat{u} = \langle u_1, u_2, u_3 \rangle$

$$= \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \Big|_{P_0} \cdot \langle u_1, u_2, u_3 \rangle$$

$$(\nabla f)_{P_0} \cdot \hat{u} = \left(u_1 \frac{\partial f}{\partial x} + u_2 \frac{\partial f}{\partial y} + u_3 \frac{\partial f}{\partial z} \right) \Big|_{P_0}$$

Scalar component of $(\nabla f)_{P_0}$ in direction of $\hat{u}$ is

$$= (\nabla f)_{P_0} \cdot \frac{\hat{u}}{\cancel{|\hat{u}|}_{1}} \quad (\hat{u} \text{ is a unit vector})$$

$$= (\nabla f)_{P_0} \cdot \hat{u}.$$

They are the same quantity since $\hat{u}$ is a <u>unit vector</u>.